On the Saturation Spectrum of Odd Cycles

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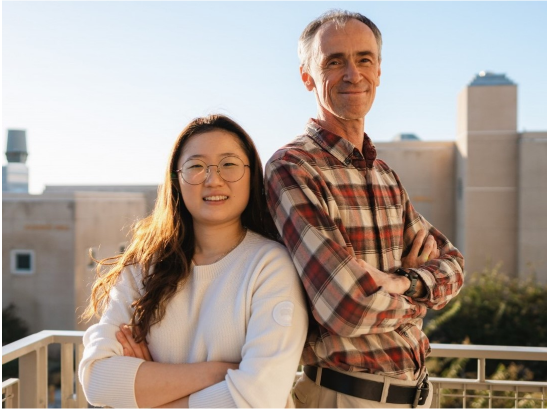
Joint work with Minjung "Michelle" Kang and Andre Kundgen



Definitions -

Definition

Given a graph H, a graph G of order n is said to be **H-saturated** if G contains no copy of H as a subgraph, but the addition of any missing edge to G creates a copy of H in G.

More Definitions

Definition

For the graph H, the **extremal number of H**, denoted $ext(H, n)$, is the *maximum* number of edges in an H-saturated graph of order n.

Definition

Given a graph H, the **saturation number for H**, denoted $sat(H, n)$ is the **minimum** number of edges in an H-saturated graph of order n.

Saturation graph for K_3 shows $sat(n, K_3) = n - 1$,

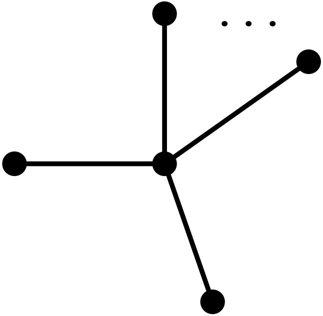


Figure: Saturation graph for K_3 with $n - 1$ edges.

Saturation number for C_5,

The situation for C_5 is much more involved.

Theorem (Y. Chen, 2009)

If $n \geq 5$, *then*

$$sat(n, C_5) = \left\lceil \frac{10(n-1)}{7} \right\rceil - \epsilon,$$

where $\epsilon = 1$ *for* $n \in \{11, 12, 13, 14, 16, 18, 20\}$ *and* $\epsilon = 0$ *otherwise.*

Extremal Graph for Odd Cycles

Fact:

The Turan Graph, that is, a complete balanced bipartite graph of sufficiently large order is the extremal graph for any odd cycle.

Important Results:

Remark

It is known that the extremal number for any graph with chromatic number at least 3 is **quadratic** *in n (Erdős - Stone, 1946).*

While the saturation number for H is **linear** *in n (Kászonyi and Tuza, 1986).*

Saturation Spectrum

Definition

For a graph H, the **saturation spectrum** for H is the set of all sizes (i.e. number of edges) of an H-saturated graph G of order n.

Remark

Thus, the upper and lower bounds for the saturation spectrum of C_5 are known exactly. But for larger odd cycles, only approximately at the low end.

First Odd Cycle Spectrum

Theorem (Barefoot, Casey, Fisher, Fraughnaugh, and Harary, 1995)

There exists a $C_3 = K_3$ saturated graph of order n and size m if and only if

$$2n - 5 \leq m \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$$

or

$$m = k(n-k).$$

Other Spectrum results

K_4 with K. Amin, 2012

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Trees, with P. Horn, M. Jacobson, B. Thomas, 2019

Remark

A useful graph operation is that of identifying vertex v_1 of G with vertex v_2 of H. We denote the resulting graph as $G \bullet H$.

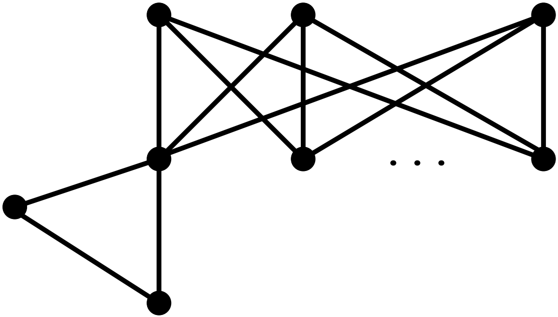

Figure: Example $K_3 \bullet K_{n.n}$.

Why this class is important

Definition

A vertex v in a graph G is k-**suitable** if for every vertex $u \neq v$ there is a $u - v$ path of length k as well as one of length at most $2k - 1$.

Lemma with Kang and Kundgen

Let G_1 and G_2 be C_{2k+1}-saturated graphs with k-suitable vertices v_1 and v_2 respectively. Then

$$G_1 \bullet G_2 \text{ is } C_{2k+1}\text{-saturated.}$$

with M. Kang and A. Kundgen

Theorem

If $n \geq 9$, then there is a C_5-saturated graph on n vertices and m edges if and only if

$$sat(n, C_5) \leq m \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 6$$

or $G = K_{p,q}$ or $G = K_3 \bullet K_{p,q}$, for some $p, q \geq 3$.

For larger odd cycles

Thoerem

If $k \geq 3$ and $n \geq 6k - 3$, then there exists a C_{2k+1}-saturated graph on n vertices and m edges if

$$\frac{k+1}{2}n - k \leq m \leq \left\lfloor \frac{(n-4k+5)^2}{4} \right\rfloor + \binom{2k+1}{2} - 6.$$

Important Classes: Here $|V_i| = n_i$.

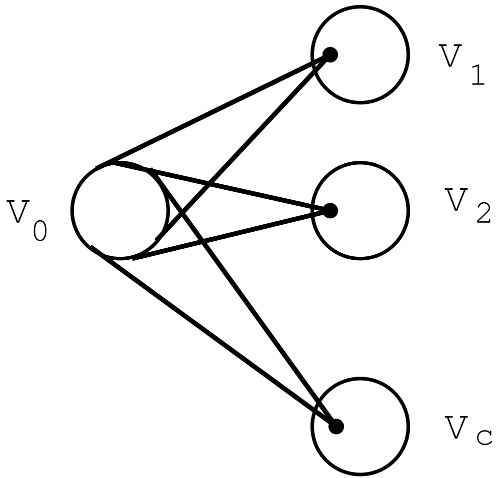

Figure: The graphs $H(n_0, n_1, \ldots, n_c)$.

Important Classes - Blowup of Cycle

Definition

$C_r(n_1, n_2, \ldots, n_r)$ is a graph whose vertices are partitioned into r parts V_i with $|V_i| = n_i \geq 1$ and in which two vertices are adjacent if and only if they are in consecutive parts V_i, V_{i+1}, where the subscripts are taken modulo r

While $C_r(n_1, \ldots, n_r) +$ ij is just $C_r(n_1, \ldots, n_r)$ along with all edges between V_i and V_j.

Examples

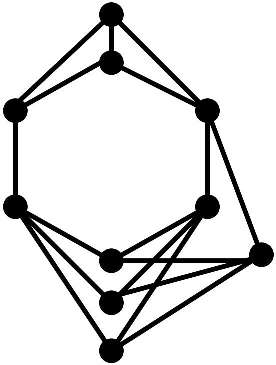

Figure: The graph $C_6(1, 1, 2^*, 1, 2, 3)$.

Examples

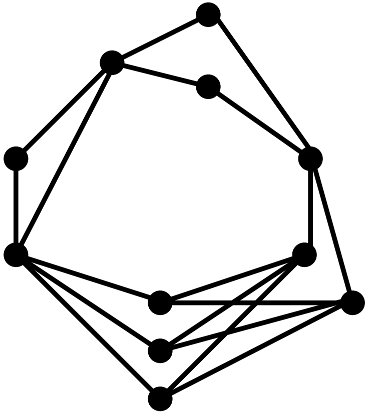

Figure: The graph $C_7(1, 1, 1, 2, 2, 3) + 13$.

And another example is of course $K_3 \bullet K_{p,q}$.

Proposition

Let m, n, k be integers with $k \geq 2$ and $\frac{(k+1)}{2}n - k \leq m$. Then there is a C_{2k+1}-saturated graph $H(k, n_1, n_2, \ldots, n_c)$ on n vertices and m edges with a k-suitable vertex if

1. $k = 2$, $n \geq 7$ and $m \leq 2n - 3$, or
2. $k \geq 3$, $n \geq 3k + 2$ and $m \leq k(n-k) - \binom{k-1}{2}$.

Proposition

Let p, q, r, k be integers with $p, q, r \geq 2$ and $k \geq 3$.
1. $C_6(1, 1, 2^*, 1, p, q)$ is a C_5-saturated graph on $p + q + 5$ vertices and $(p + 1)(q + 1) + 5$ edges.
2. $C_7(1, 1, 2, 2, r, p, q)$ is a C_5-saturated graph on $p + q + r + 6$ vertices and $(p + 2)(r + q) - q + 7$ edges.
3. $C_7(1, 1, 1, 2, r, p, q)$ +13 is a C_5-saturated graph on $p + q + r + 5$ vertices and $(p + 2)(r + q) - q + 5$ edges.
4. $C_{2k+1}(2, r, p, q, 1, (2k - 2)^*, 1, \ldots, 1)$ is a C_{2k+1}-saturated graph on $p + q + r + 4k - 3$ vertices and $(p + 2)(q + r) - q - 4 + \binom{2k+1}{2}$ edges when $r \geq p \geq k - 2$ except if $p = k - 2 = 2$.

Sketch of proof for sat spectrum of C_5

1. Establish saturation number and extremal number.

2. Fill the interval by overlaying intervals that are covered by the various families of examples.

3. Show that there are no nonbipartite graphs in the upper interval by a direct count of the edges possible.