Chorded Cycles

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Many types of cycle problems have been studied.

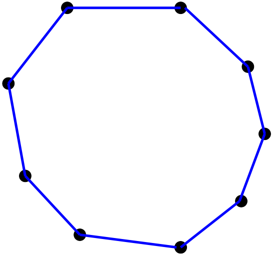
- Hamiltonian (spanning) cycles
- **Existence of long cycles**
- Pancyclic graphs
- **Vertex disjoint cycles**
- Cycles containing certain elements

One property that was greatly ignored

Question Posa, 1960

What conditions imply a graph contains a cycle with a chord?

Here a **chord** is an edge between two vertices on the cycle that is not an edge of the cycle.



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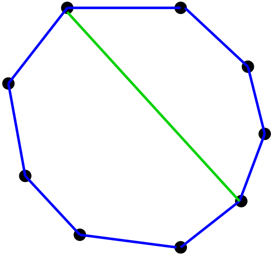

First answer by J. Czipzer, 1963

Theorem

If G has minimum degree $\delta(G) \geq 3$, then G contains a chorded cycle.

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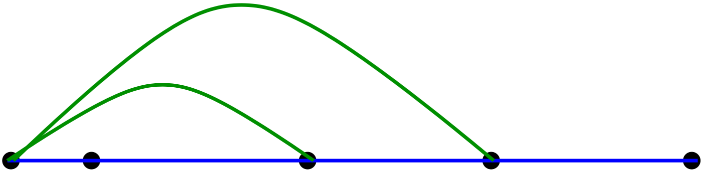
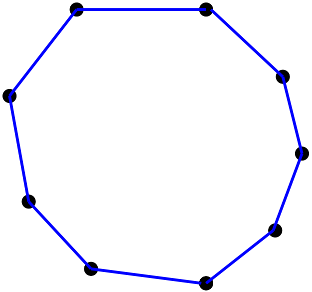


Sharpness Example

Minimum degree 2 is not enough! Simply take any cycle.



Useful Definition

For $k \geq 2$ let

$$\sigma_k(G) = \min\{deg\ x_1 + \ldots + \text{deg}\ \ x_k\}$$

where this minimum is taken over all sets of k independent vertices.

Everyone jumped on chorded cycle problems (45 years later)

Theorem (Finkel, 2008)

If G is a graph on $n \geq 4k$ vertices with minimum degree $\delta(G) \geq 3k$, then G contains at least k vertex disjoint chorded cycles.

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Theorem (Corradi and Hajnal, 1963)

Let G be a graph of order $n \geq 3k$ with minimum degree $\delta(G) \geq 2k$, then G contains k vertex disjoint cycles.

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$\delta(G) \geq 3$ **implies chorded cycle.**

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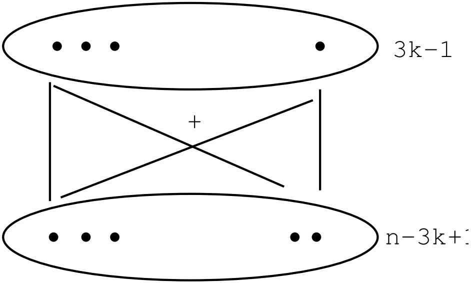
$\delta(G) \geq 2$ **implies cycle.**

Sharpness

Clearly, $n \geq 4k$ is needed as each of the k cycles need at least 4 vertices to have a chord.

For $n \geq 6k$, the graph $K_{3k-1,n-3k+1}$ has $\delta = 3k - 1$ and no collection of k vertex disjoint chorded cycles exists, as chorded cycles here require 3 vertices from each partite set.



K $_{3k-1,\ n-3k+1}$

Bialostocki, Finkel and Gyarfas, 2008

Let $f(n)$ ($g(n)$) be the minimum number of edges in a graph of order n that ensures the graph contains two vertex disjoint cycles, at least one (both) of them chorded.

Theorem

For $n \geq 10$, $f(n) = 4n - 5$ and for $n \geq 12$, $g(n) = 5n - 24$.

It remains open to determine the minimum number of edges in a graph on n vertices ensuring $k \geq 3$ vertex disjoint chorded cycles.

Chiba, Fujita and Gao, 2010

Answering a conjecture of Bialostocki, Finkel and Gyarfas

Theorem

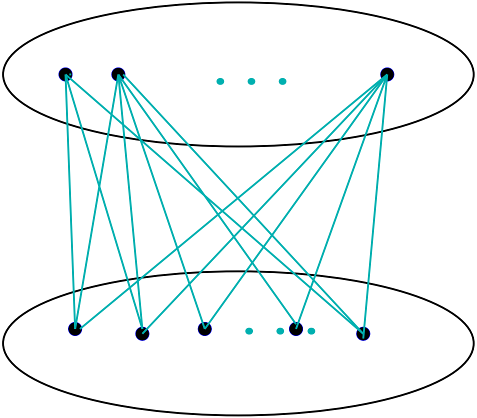
Let r and s be integers with $r + s \geq 1$. Let G be a graph of order at least $3r + 4s$. If

$$\sigma_2(G) \geq 4r + 6s - 1,$$

then G contains a collection of $r + s$ vertex disjoint cycles, such that s of them are chorded.

Sharpness example

$K_{2r+3s-1, n-2r-3s+1}$



Chiba, Fujita, Gao, Li, 2010 and Gao and Guojun - 2012

Theorem

Let k be a positive integer and G a graph of order $n \geq 4k$ with $\sigma_2(G) \geq 6k - 1$. Then G contains k vertex disjoint chorded cycles.

Further Extensions

Theorem with K. Hirohata, A. Keller Rorabaugh, 2020

Let $k \geq 1$ be an integer. If G is a graph of order at least $8k + 5$ and

$$\sigma_3(G) \geq 9k - 2,$$

then G contains k vertex-disjoint chorded cycles.

The Pattern

$\delta \geq 3k - 0$ implies k disjoint chorded cycles.

$\sigma_2 \geq 6k - 1$

$\sigma_3 \geq 9k - 2$

$\sigma_4 \geq 12k - 3$

...

Extension to arbitrary sums

Theorem with B. Elliott and K. Hirohata, 2020

For $k \geq 2$ and $t \geq 1$, if G is a graph of order $n \geq 8k^2 - 9k + 12t + 1$ with

$$\sigma_t(G) \geq 3kt - t + 1,$$

then G contains k vertex-disjoint chorded cycles. Further, this degree condition is sharp.

S. Qiao and S. Zhang, 2010

Theorem

If G is a graph with at least $4k$ vertices and minimum degree at least $\lceil \frac{7k}{2} \rceil$, then G contains k vertex disjoint cycles, each with at least 2 chords.

Call such cycles *doubly chorded cycles.*

with Horn and Hirohata, 2015

Theorem

If G is a graph of order $n \geq 6k$ with $\sigma_2(G) \geq 6k - 1$, then G contains k vertex disjoint **doubly chorded cycles**.

Extending Bondy's meta-conjecture

In 1971, Bondy noted a relation between the conditions of a number of edge density results on hamiltonian graphs and those of pancyclic graphs.

Meta-Conjecture

Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic. There may be some simple family of exceptional graphs.

Supporting work

Theorem (Ore, 1962)

If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is hamiltonian.

Theorem (Bondy, 1971)

If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is pancyclic or $G = K_{n/2,n/2}$.

Extending Bond's Meta-conjecture

Meta-Conjecture (Cream, RG, Hirohata, 2017)

Almost any condition that implies a graph is hamiltonian, also implies it is chorded pancyclic (contains chorded cycles of lengths 4 to $|V(G)|$). There may be some simple families of exceptional graphs or some small order exceptional graphs.

Cream, RG, Hirohata, 2017

Theorem

If G is a graph of order $n \geq 4$ with $\sigma_2(G) \geq n$, then G is chorded pancyclic or $G = K_{n/2,n/2}$, or $G = G_6 = K_3 \square K_2$.

Cream, RG, Hirohata, 2017

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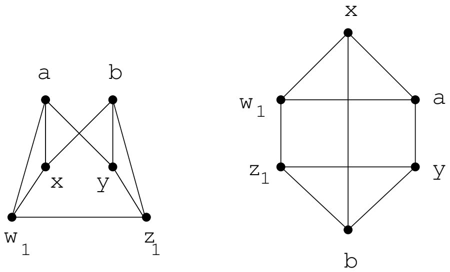


The graph G_6 two ways.

Sharpness and limitations

The graph $2K_{\frac{n-1}{2}} + K_1$ shows the degree conditon is sharp, as this graph is not hamiltonian.

Note that the graph obtained by adding one edge to either partite set of $K_{n/2,n/2}$ is chorded pancyclic, but not doubly chorded pancyclic as no 4-cycle contains 2 chords.

Further supporting results

Theorem (Bondy, 1971)

Every hamiltonian graph G of order n with at least $\frac{n^2}{4}$ edges is pancyclic unless n is even and $G = K_{n/2,n/2}$. In particular, if $E(G)| > \frac{n^2}{4}$, then G is pancyclic.

More evidence

Theorem Chen, RG, Gu, Saito, 2018

Let G be a graph of order n with $|E(G)| \geq \frac{n^2}{4}$, and let k be a positive integer. If G contains a k-cycle, then it contains a chorded k-cycle, unless $k = 4$ and G is either $K_{n/2,n/2}$ or $G = K_3 \square K_2$.

Extending Bondy's Theorem, we get the following.

Corollary

A hamiltonian graph of order $n \geq 4$ with $|E(G)| \geq \frac{n^2}{4}$ is chorded pancyclic, unless $G = K_{n/2,n/2}$ or $G = K_3 \square K_2$.

Bipartite setting - Gao, Lin, Wang, 2019

Theorem

Let k be a positive integer. Let $G = (V_1, V_2, E)$ be a bipartute graph with $|V_1| = |V_2| \geq 3k$. If $\delta(G) \geq 2k + 1$, then G contains k vertex-disjoint doubly chorded cycles.

More on degree sum

Theorem with Brandt, Chen, Faudree, and Lesniak, 1997

Let k be a positive integer and let G be a graph of order $n \geq 4k$. If

$$\sigma_2(G) \geq n,$$

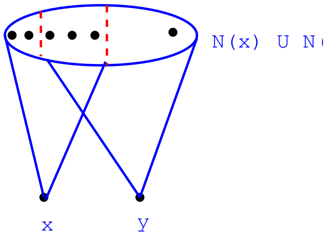
then G has a 2-factor with exactly k vertex disjoint cycles.

Theorem (Chiba, Jiang, Yan, 2019)

For positive integers k and c, there exists an integer $f(k,c)$ such that if G is a graph of order $n \geq f(k,c)$ and $\sigma_2(G) \geq n$, then G can be partitioned into k vertex-disjoint cycles, each of which has at least c chords.

Another density conditon - Neighborhood Unions

For nonadjacent vertices x, y let $N(x, y) = N(x) \cup N(y)$.



Neighborhood Unions

Theorem RG, Hirohata, Horn, 2010

If G is a graph on $n \geq 4k$ vertices such that for any pair of non-adjacent vertices x, y,

$$|N(x,y)| \geq 4k+1,$$

then H contains at least k vertex disjoint chorded cycles.

Conjecture

Conjecture

Let G be a graph of sufficiently large order n. Then if

$$|N(u,v)| \geq 4k+1$$

G contains k vertex disjoint doubly chorded cycles.

Question

When can we have many chorded cycles of the same length?

with Chen, Hirohata, Ota and Song, 2015

Theorem

Let k be a natural number. Then there exists a positive integer n_k such that if G is a graph with $\delta(G) \geq 3k + 8$ and order at least n_k, then G contains k vertex disjoint chorded cycles of the same length.

Theorem

Let G be a multigraph of order n and minimum degree at least 5. Then G contains a chorded cycle of length at most $c_0\ log_2\ n$, where $30 \leq c_0 \leq 300$ is a constant.

Specifing certain edges to be chords

Question

When can we make an independent set of k edges the chords of k vertex disjoint cycles?

with M. Cream, R. Faudree, K. Hirohata, 2016

Question

Can we make an independent set of k edges chords of k vertex disjoint cycles?

Theorem

Let $k \geq 1$ be an integer and G be a graph of order $n \geq 14k$. If $\sigma_2(G) \geq n + 3k - 2$, then for any k independent edges $e_1, e_2, \ldots, e_k$ of G, the graph G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that e_i is a chord of C_i for all $1 \leq i \leq k$. Furthermore, $4 \leq |V(C_i)| \leq 5$ for each i.

Question- Placing vertices (or edges) on chorded cycles

Question

When can we distribute k vertices on k disjoint chorded cycles?

Placing vertices on chorded cycles

Theorem with Cream, Faudree and Hirohata, 2016

Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 16k - 12$. If $\delta(G) \geq n/2$ then for any set of k vertices $\{v_1, v_2, \ldots, v_k\}$ there exists a collection of k vertex disjoint chorded cycles $\{C_1, \ldots, C_k\}$ such that $v_i \in V(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Placing Edges on Chorded Cycles

Theorem [with Cream, Faudree and Hirohata, 2016]

Let G be a graph of order $n \geq 18k - 2$ and let $e_1, e_2, \ldots, e_k$ be a set of k independent edges in G. If

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k chorded cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

With Cream, Faudree and Hirohata, 2016.

As a Corollary to the proof we obtain the fact the edges

$$e_1, e_2, \ldots, e_k$$

can be a mix of either chords or edges of the cycles (again one edge per cycle).

Further, we can show that the cycle system can also be extended to span $V(G)$.

Doubly Chorded Cycles

Theorem with Cream, Faudree and Hirohata, 2016

Let G be a graph of order $n \geq 22k - 2$ and let $e_1, \ldots, e_k$ be k independent edges in G. Then if

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k vertex disjoint doubly chorded cycles $C_i, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Corollary

The above system can be extended to span $V(G)$.

Containing Linear Forests

Fact

Given independent path $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ with each $r_i \geq 2$ let $r = \sum r_i$. Then the number of interior vertices in this path system is $r - 2k$.

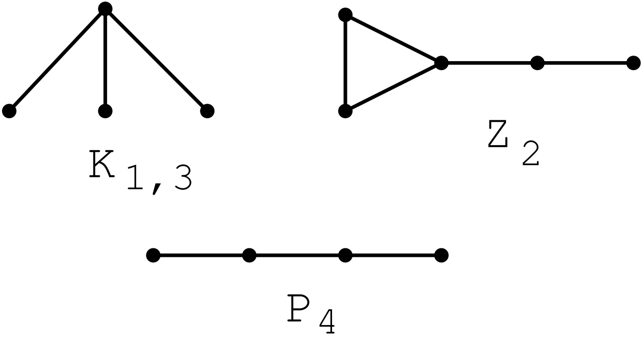
Theorem Cream, RG, Faudree, Hirohata, 2016

For an integer $k \geq 1$, let G be a graph of order $n \geq 16k + r - 3$ with

$$\delta(G) \geq \frac{n}{2} + r - k - 1,$$

and let $P_1, \ldots, P_k$ be any k independent paths in G. Then there exist k disjoint doubly chorded cycles $D_1, \ldots, D_k$ such that D_i contains P_i as a path along the cycle and $r_i + 2 \leq |V(D_i)| \leq r_i + 4$ for all $1 \leq i \leq k$.

Forbidden Subgraphs = Forbidden Induced Subgraphs



Forbidden Subraphs and Pancyclic Graphs

Theorem with Faudree, 1997

Let R and S be connected graphs $(R, S \neq P_3)$ and let G $(G \neq C_n)$ be a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is pancyclic if, and only if, $R = K_{1,3}$ and S is one of P_4, P_5, P_6, Z_1 or Z_2.

Forbidden Subgraphs

Theorem with Cream and Larson, 2017

Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, Z_2\}$−free, then $G = C_n$ or G is chorded pancyclic.

Theorem with Cream and Larson, 2017

Let G be a 2-connected graph of order $n \geq 5$. If G is $\{K_{1,3}, P_4\}-$free, then G is chorded pancyclic.

Theorem with Cream and Larson, 2017

Let G be a 2-connected graph of order $n \geq 8$. If G is $\{K_{1,3}, P_5\}-$free, then G is chorded pancyclic.

Theorem with Cream and Larson, 2017

Let G be a 2-connected graph of order $n \geq 13$. If G is $\{K_{1,3}, P_6\}-$free, then G is chorded pancyclic.

Extending vertex and edge pancyclic graphs

Theorem (Hendry, 1990)

If G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n+1}{2}$, then G is vertex pancyclic (ie. each vertex appears on a cycle of each length from 3 to n).

Theorem Cream, RG, Hirohata, 2019

If G is a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+1}{2}$, then G is chorded vertex pancyclic (ie. each vertex appears on a chorded cycle of each length from 3 to n).

Theorem Cream, RG, Hirohata, 2019

Let G be a graph of order $n \geq 5$ with $\delta(G) \geq \frac{n}{2}$. Then for $k \geq 5$ and any $x \in V(G)$, there is a doubly chorded k-cycle in G containing x. (ie. chorded 5-pancyclic).

Theorem (Randerath, et al., 2002)

If G is a graph of order $n \geq 4$ with $\sigma_2(G) \geq n+1$, then G vertex 4-pancyclic.

Theorem Cream, RG, Hirohata, 2019

If G is a graph of order $n \geq 4$ with $\sigma_2 \geq n+1$, then G is chorded vertex 5-pancyclic.

Theorem (Randerath et al. 2002)

Let G be a graph of order $n \geq 3$ with $\sigma_2(G) \geq \lceil \frac{4n}{3} \rceil - 1$, then G is vertex pancyclic.

Theorem Cream, RG, Hirohata, 2019

Let G be a graph of order $n \geq 8$ with $\sigma_2(G) \geq \lceil \frac{4n}{3} \rceil - 1$, then G is chorded vertex pancyclic.

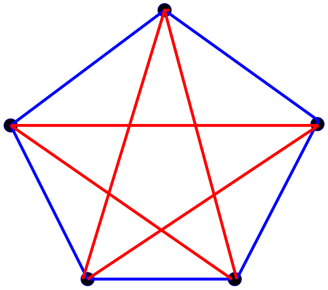
An interesting question

Question

How many chords should we expect or hope to find?

Cycles with many chords?

A special case: Cliques



K_5: 4-regular but with 5 chords

In general:

Given a K_{k+1}: It is k-regular with

$$f(k) = \frac{(k-2)(k+1)}{2}$$

chords. We think of $f(k)$ chorded cycles as "loose" K_{k+1} cliques.

Note: There are no single chorded cliques.

Theorem (Ali, Staton - 1999)

If $\delta(G) = k$, then G contains a

$$\left\lceil \frac{k(k-2)}{2} \right\rceil - \textit{ chorded cycle.}$$

Note: There are no single chorded cliques.

Theorem (Ali, Staton - 1999)

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Corollary

If $\delta(G) \geq 3$, then G contains a doubly chorded cycle - that is, a loose K_4.

RG, P. Horn, C. Magnant: Our first step -

Theorem

If $\delta(G) = k$, then G contains an

$$f(k) = \frac{(k+1)(k-2)}{2} - \text{ chorded cycle.}$$

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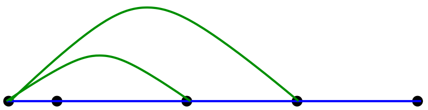

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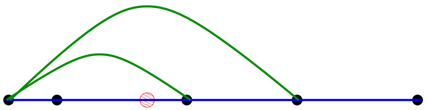

The story so far:

Lower End: $\delta(G) \geq k$ implies an $f(k) = \frac{(k-1)(k+2)}{2}-$chorded cycle.

Upper End:

Theorem (Hajnal and Szemerédi, 1970)

If $\delta(G) \geq kt$, $|G| = (k+1)t$, then G can be covered by t vertex disjoint K_{k+1}'s.

RG, P. Horn and C. Magnant, 2014

Conjecture

If $\delta(G) \geq kt$, and $|G| \geq (k+1)t$ then G contains t

$$f(k) = \frac{(k+1)(k-2)}{2} - \text{disjoint chorded cycles.}$$

RG, P. Horn and C. Magnant, 2014

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Tight End: Hajnal-Szemerédi Theorem.

RG, P. Horn and C. Magnant, 2014

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Tight End: Hajnal-Szemerédi Theorem.
If $t = 1$, this is our first Theorem.

RG, P. Horn and C. Magnant, 2014

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Tight End: Hajnal-Szemerédi Theorem.
If $t = 1$, this is our first Theorem.
We show it is true for some classes of graphs, and for graphs with some extra "room".

RG, P. Horn, C. Magnant 2014

Theorem

There exist k_0, t_0 such that if $\delta(G) \geq kt$, where $k \geq k_0$, $t \geq t_0$ and $n \geq n_0(k,t)$, then G contains t disjoint cycles with at least

$$f(k) = \frac{(k+1)(k-2)}{2}$$

chords.

RG, P. Horn, C. Magnant 2014

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- Bounds for k_0 and t_0 show a tradeoff.

RG, P. Horn, C. Magnant 2014

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chords.

- Bounds for k_0 and t_0 show a tradeoff.
- Bounds for n_0 quite large.

Possibly More Useful:

Theorem

Let $d = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil$.

- If G has average degree at least $2d$, then G contains a $f(k) = \frac{(k+1)(k-2)}{2}$-chorded cycle.
- There exist graphs with average degree $2d - o(1)$ with no $f(k)$-chorded cycle.

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- Sharpness: Bipartite graph $K_{d,n}$.

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- Sharpness: Bipartite graph $K_{d,n}$.
- $k = 2, 3, 4$: Trivial induction removing vertex of lowest degree if $< \delta$.

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- If G has average degree at least $2d$, then G contains a $f(k) = \frac{(k+1)(k-2)}{2}$-chorded cycle.
- There exist graphs with average degree $2d - o(1)$ with no $f(k)$-chorded cycle.

- Sharpness: Bipartite graph $K_{d,n}$.
- $k = 2, 3, 4$: Trivial induction removing vertex of lowest degree if $< \delta$.
- $k \geq 5$ much tougher induction.