Applications of Mathematics to Games and Puzzles

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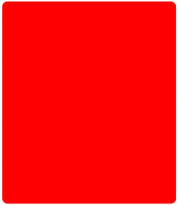
Game 1 - Three Card Game

The Tools: A man has three cards, one **red** on both sides, one **blue** on both sides, and one **blue** on one side and **red** on the other side.



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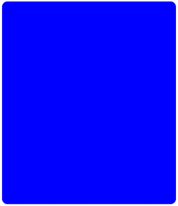

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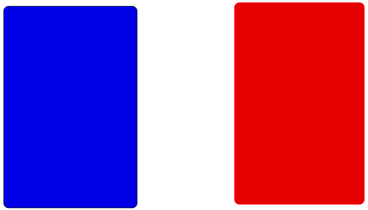

Side 1 Side 2

Play

The Game: He places the three cards into a hat and asks you to blindly (so randomly) select one card.

You are to then show him one side of the card and he offers to bet even money he can tell you the color of the other side of the card.

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Is this a fair bet?

Common Thinking

Suppose the side you show is **red**.

Then, most people say this is a fair bet since:

The card is either **red-red**

or the card is **red**-**blue**

and the man has a 50% chance of guessing correctly.

The Scam - I mean the real solution

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- The key is - 2 cards have the same color on both sides.
- Everything is decided at the time the card is selected.
- The man will always guess the same color as that shown and be correct 2/3 of the time!

Guarini's Problem - Knights on a 3 × 3 chessboard

K_1		K_2
K_1		K_2

Problem: Have the knights switch row positions using only legal moves.

Recall that knights only move in an L shape.

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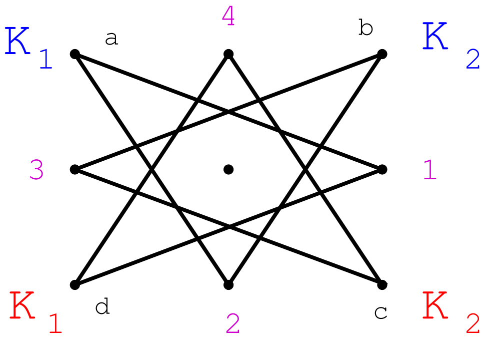
Question: How can we study this question?

We Can Use A Graph Model:



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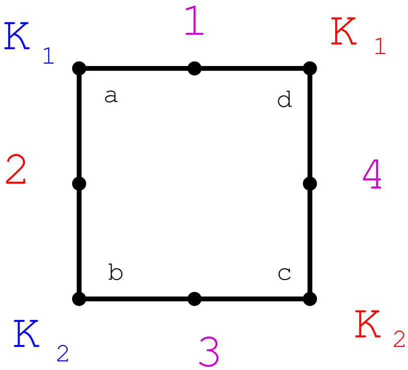

The Answer!

Thus, it takes a minimum of 16 moves to complete the transformation.

The coin game - for 2 players

Suppose 10 (or any even number) randomly selected coins are placed in a row. The sum of the values of these coins is T **which is odd**.

GAME: Player 1 selects one coin from either end of the row. Player 2 selects one coin from either end of the row. Play continues, alternating turns, until all coins have been selected. The winner is the player with the largest total on their selected coins.

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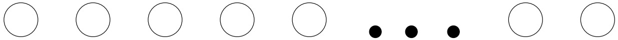
And better yet, find the winning strategy!

The Strategy!



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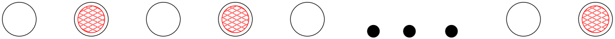

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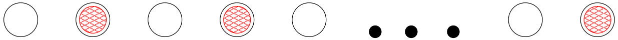

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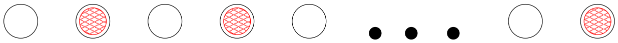

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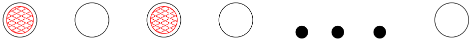

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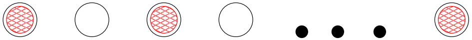

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3. Player 1 selects the end coin that is white.
4. Player 2 now has only red coins on either end to select from.
5. Player 1 now selects the white coin uncovered by the choice Player 2 just made.
6. Player 2 again has only reds to choose from. This continues until the game ends with Player 1 having all the white coins and the win.

The Game: Three friends, **Matt** (the mathematician) **Phil** (the physicist) and **Ed** (the engineer) decide to play paintball.

Matt - limited experience: 50% shooter.
Phil - plays more: 75% shooter.
Ed - plays constantly: 100% shooter.

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- **Matt** will get the first shot (50%).
- **Phil** will get the second shot (75%).
- **Ed** will get the third shot (100%).
- Should it be needed, another round then follows among any survivors.

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- If he shoots (and hits) Phil, then Ed immediately will shoot him!
- Surrender and go to a bar and wait for the winner to join him there later!
- Matt's best move is to **purposely miss!**

Why is this a good move? Find Prob. Ed wins

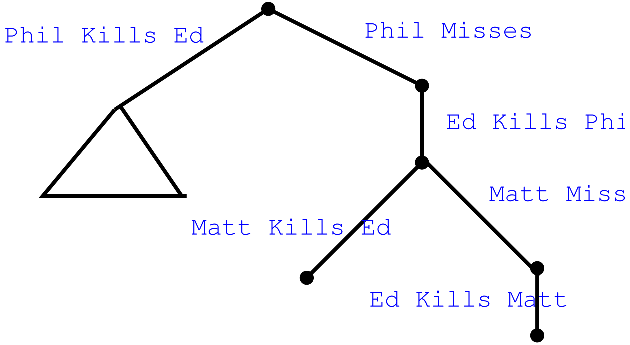

Figure: The strategy tree diagram.

$P(\text{ Ed wins }) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = .125$

What is the probability Matt wins?

A similar but more complex attack shows that Phil wins with probability about .321.

Thus,

P(Matt wins) $\approx 1 - .321 - .125 \approx .554$.

The 5 card trick - due to Fitch Chaney, 1950

Suppose you and a partner want to perform the following 5 card trick.

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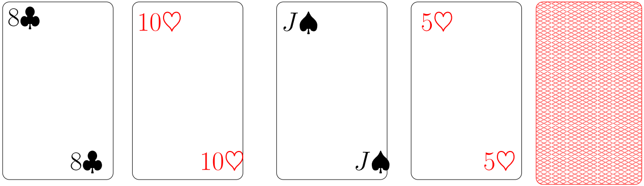

The Trick

Now your partner now enters the scene and identifies the face down card exactly!

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3. We need to encode this information somehow!

How the suit information is encrypted

Given any 5 cards from a standard deck, the

The Pigeon Hole Principle:
There must be **at least 2 cards** of the same suit!

Thus, we can use one card from that suit placed in a predetermined position to signal the suit. Say for now - the left most **position**.

But maybe there is more to this first position.

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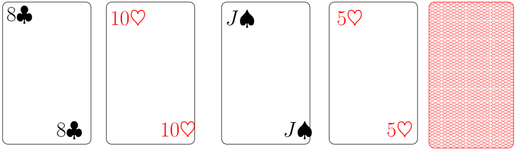
We use a counting system, starting with the rank of the first card.

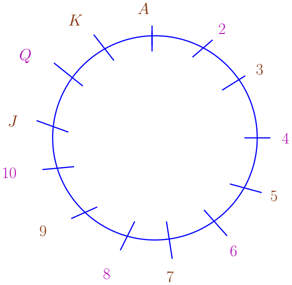
lexicographic ordering is the key!

3 "unused cards" - Use the relative ranks to have a **low (L)**, **medium (M)** and **high (H)** card among the 3 remaining cards. The lexicographic orderings of these three produce:

L M H (call this 1)
L H M (call this 2)
M L H (call this 3)
M H L (call this 4)
H L M (call this 5)
H M L (call this 6)

The solution





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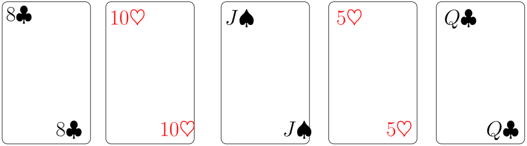

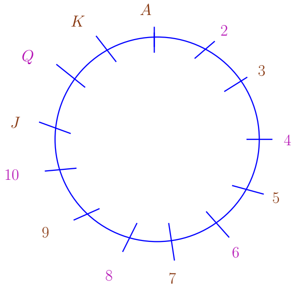

The Game of Penny Ante

This game was invented by Walter Penny in 1969.

The Game: Ask your opponent to select any pattern of **heads (H)** and **tails (T)** of length three they wish. Then you select a different pattern. Flip pennies repeatedly until one of the two patterns occurs on three consecutive tosses. The person with that pattern wins an even money bet.

Is this a fair game?

Many people think the game is unfair because the second person only gets to choose their pattern after the first person has selected. This means they do not have all the choices the first person had.

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Those are the conditions - but that is not the advantage!

Suppose, for ease of argument, your opponent selected **HHH** .

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Then you will select **THH** .

Case Analysis

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H H H

____ ____ ____ .

First 3 all heads

probability = 1/8

Case Analysis

. . . ____ H ____ H ____ H ____ .

Otherwise, it happens later.

Case Analysis

. . . $\underline{\text{T}}$ $\underline{\text{H}}$ $\underline{\text{H}}$ $\underline{\text{H}}$.

Then a tail must happen.

An analysis of all the other possibilities (more involved tree structures and infinite geometric series) shows that the following rule always wins with probability at least $2/3$.

Rule: Let the opponent select a pattern of three. Now move their first two choices to your last two choices and fill the first position with what ever choice does not form a palindrome!

2 decks problem

Suppose two standard decks of cards are individually shuffled and placed on a table. Now, the top card of each deck is turned over to determine if there is an **EXACT MATCH.** This process is continued throughout the remaining cards.

Given an even money bet - would you bet there was an exact match before the cards run out?

The quick answer

One view of this problem is that when the card from deck 1 is turned over, there is a $\frac{1}{52}$ chance the card in deck 2 will match it. Thus, the expected number of matches when going through the entire deck would be:

$$np \;=\; (52)\,\frac{1}{52} \;=\; 1.$$

Thus, the **actual length of the deck** does not seem to matter.

Can we squeeze more out of this?

That rough answer seems to say it would seem a wise bet to say there would be a match.

BUT HOW WISE???.

What is really going on here?

Another way to look at this problem, and one that brings far more to the table, is:

Think of the 2nd deck as a permutation of the first deck.

Now, the question we are asking is:

How many permutations of the deck have at least one position unchanged (a fixed point)?

This unchanged position gives us an exact match!

A permutation with no fixed points is called a **derangement.**

Counting derangements - use Inclusion - Exclusion Principle (IEP)

Suppose we want to count how many permutations (of our 52 cards) have at least $i \geq 1$ matches with deck 1. Then if we count the derangements we can then find the number that are not derangements. Let D be the number of derangements of deck 1.

This is ideal for the Inclusion - Exclusion Principle.

The IEP

Given a set of properties $P_1, P_2, \ldots, P_m$ and a set of outcomes $A_1, A_2, \ldots, A_m$ of some experiment, where the sets A_i are the outcomes with property P_i, we can count the number of outcomes that have none of these properties as follows:

The IEP

Theorem The number of outcomes S that have none of the properties P_i is

$$|\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_m| =$$

$$|S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \ldots + (-1)^m |A_1 \cap \ldots \cap A_m|.$$

Let S be the set of all permutations and let P_i be the property of matching deck 1 in at least position i.

Then $A_i \cap A_j$ is the set of permutations that match deck 1 in at least both position i and j, etc.

There are $(52-1)! = 51!$ ways to fill any permutation that matches deck 1 in at least position i and there are $(52-2)! = 50!$ ways to fill any permutation that matches deck 1 in at least 2 positions, etc.

Then, applying the IEP we obtain:

$$D = 52! - \sum_{i=1}^{i=52} 51! + \sum_{i,j} (50!) \dots (-1)^{52} 1!$$

.

$$= \frac{52!}{0!} + \binom{52}{1} 51! - \binom{52}{2} 50! \ \dots$$

Doing a bit of arithmetic we get:

$$= \frac{52!}{0!} - \frac{52!}{1!} + \frac{52!}{2!} - \frac{52!}{3!} \dots + \frac{52!}{52!}$$

$$= 52! [\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots + \frac{1}{52!}]$$

But we know from Calculus that:

$$\approx 52! \times \frac{1}{e}$$

But

$$\frac{1}{e} \approx .3679.$$

Hence, $P(derangement) \approx .3679$ and thus $P(match) \approx .63$.
Thus, it is a very good even money bet!

The Game of NIM

Game: Given $t \geq 2$ piles of chips, two players proceed as follows:

Player 1 removes some number of chips from exactly one pile.

Player 2 then removes some number of chips from exactly one pile.

Players alternate turns until no chips remain. The first player that cannot remove some chips loses.

Is there a strategy for optimal play?

To find the strategy, it really helps to start at the end of the game.

When no chips remain we call this the **ZERO POSITION.**

Your goal as a player is to put your oponent in the zero position.
How can we achieve this?

Good Example to Learn From

Suppose we have only two piles of chips, and they are of equal height.

Is there a strategy now?

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Is there a strategy now?

Yes, player two should just mimick the move of player 1.

If we represented the height of each of the two piles of chips as a base two number, then they would be identical representations, and the 1's can be thought of, base two, as summing to zero.

Hence, this is a zero position!

The Strategy: Base two representations

The key is to think of the height of each pile of chips as a number and to look at the base 2 representation of this number.

Say we have three piles of 8, 5 and 6 chips.

Base 2 view

8:	1	0	0	0
5:	0	1	0	1
6:	0	1	1	0

Base 2 view

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5:	0 1 0 1
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Base 2 view

8: 1 0 0 0
5: 0 1 0 1
6: 0 1 1 0

A: 1 0 1 1
(not a zero position)

What to do?

Player 1 would like to put the other player into a zero position.

We think of a zero position as 0 0 0 0

So to get our parity counts above to a zero position we must remove chips from one pile so that all the colums have an even number of ones:

This can always be done!

One such move

Looking at our example:

8: 1 0 0 0
5: 0 1 0 1
6: 0 1 1 0

A: 1 0 1 1
(not a zero position)

Removing 5 chips from the pile of 8 will produce what we need!

3: 0 0 1 1
5: 0 1 0 1
6: 0 1 1 0

A: 0 0 0 0
(zero position)

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Player 1 will then repeat the strategy of moving to a zero position! Eventually player 1 will reduce to the ultimate zero position - no chips, and win the game