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Bigraphical Sets

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ABSTRACT

Two finite nonempty sets S_1 and S_2 of positive integers are called *bigraphical* if they are the degree sets of the partite sets of a bipartite graph. It is shown that every two such sets are bigraphical. The minimum order of a corresponding bipartite graph is determined for certain sets S_1 and S_2.

1. Bigraphical sets.

The *degree set* of a graph G is the set of degrees of the vertices of G. A finite nonempty set S of positive integers is called *graphical* if there exists a graph whose degree set is S. In [1] it was shown that every finite nonempty set of positive integers is graphical and the minimum order of a graph with a given degree set was determined.

The *degree sets of a bipartite graph* G with partite sets V_1 and V_2 are the sets $S_1' = \{\deg v | v \in V_1\}$ and $S_2 = \{\deg v | v \in V_2\}$. Two finite nonempty sets S_1 and S_2 of positive integers are called *bigraphical* if they are the degree sets of a bipartite graph. It is the object of this article to show that every two finite nonempty sets of positive integers is bigraphical and to determine, for certain pairs of sets S_1 and S_2, the minimum order of a bipartite graph whose degree sets are S_1 and S_2. We begin with the first of these objectives.

Proposition 1. Every two finite nonempty sets of positive integers are bigraphical and can be realized by a connected graph.

Proof. Let $S_1 = \{a_1, a_2, \ldots, a_m\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ be sets of positive integers, where $m \le n$. For $1 \le i \le m$, let G_i denote the complete bipartite graph $K(a_i, b_i)$ with partite sets U_i and V_i, where $|U_i| = b_i$ and $|V_i| = a_i$. If $m < n$, then for $m + 1 \le i \le n$, let $G_i = K(a_1, b_i)$ with partite sets U_i and V_i, where $|U_i| = b_i$ and $|V_i| = a_1$. Now let $G = \bigcup_{i=1}^{n} G_i$. If we let $U = \bigcup_{i=1}^{n} U_i$ and $V = \bigcup_{i=1}^{n} V_i$, then G may be considered as a bipartite graph with partite sets U and V and degree sets S_1 and S_2. To obtain a connected graph, delete an edge $u_i v_i$ in G_i and $u_{i+1}v_{i+1}$ in G_{i+1} $(i = 1, 2, \ldots, n-1)$ and insert $u_i v_{i+1}$ and $u_{i+1} v_i$. ■

In view of Proposition 1, for finite nonempty sets $S_1 = \{a_1, a_2, \ldots, a_m\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ of positive integers, let $\mu(S_1;S_2) = \mu(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n)$ denote the minimum order of a bipartite graph having degree sets S_1 and S_2. We now investigate the numbers $\mu(S_1;S_2)$ for certain finite nonempty sets S_1 and S_2 of positive integers. For the remainder of this article we assume that if $S_1 = \{a_1, a_2, \ldots, a_m\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ then $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_n$. The following observation is elementary but useful.

Proposition 2. If $S_1 = \{a_1, a_2, \ldots, a_m\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ are sets of positive integers, then $\mu(S_1;S_2) \ge a_m + b_n$.

If S_1 and S_2 have the same cardinality, then $\mu(S_1;S_2)$ can be evaluated.

Theorem 3. If $S_1 = \{a_1, a_2, \ldots, a_n\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ are nonempty sets of positive integers, then $\mu(S_1;S_2) = a_n + b_n$.

Proof. By Proposition 2, $\mu(S_1;S_2) \geq a_n + b_n$. To show that $\mu(S_1;S_2) \leq a_n + b_n$, it suffices to show the existence of a bipartite graph of order $a_n + b_n$ having degree sets S_1 and S_2. Let G be a bipartite graph with partite sets U and W such that $U = U_1 \cup U_2 \cup \cdots \cup U_n$ and $W = W_1 \cup W_2 \cup \cdots \cup W_n$, where $|U_1| = a_1$ and $|U_i| = a_i - a_{i-1} (2 \leq i \leq n)$ and where $|W_1| = b_1$ and $|W_i| = b_i - b_{i-1} (2 \leq i \leq n)$. To construct the edge set of G we join each vertex of $W_i (1 \leq i \leq n)$ to each of the vertices in $U_1, U_2, \ldots, U_{n-i+1}$. Each vertex in $W_i (1 \leq i \leq n)$ has degree a_{n-i+1} while each vertex in $U_i (1 \leq i \leq n)$ has degree b_{n-i+1}, completing the proof. ∎

If $|S_1| \neq |S_2|$ we have no general formula for $\mu(S_1;S_2)$. However, Proposition 1 yields an obvious upper bound on $\mu(S_1;S_2)$ and with the aid of Theorem 3 this bound can be improved.

Corollary 4. If $S_1 = \{a_1, a_2, \ldots, a_m\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ are sets of positive integers where $m < n$ and t is a positive integer, then

(I) $\mu(S_1;S_2) \leq ta_m + \sum_{i=1}^{t} b_{im}$ if $n = tm$ or,

(II) $\mu(S_1;S_2) \leq ta_m + \sum_{i=1}^{t} b_{im} + a_r + b_n$ if $n = tm + r$, $0 < r < n$.

Proof. For $1 \leq i \leq t$ let $B_i = \{b_{(i-1)m+1}, b_{(i-1)m+2}, \ldots, b_{im}\}$ and if $n = tm + r$, $0 < r < n$, let $B_{t+1} = \{b_{tm+1}, b_{tm+2}, \ldots, b_n\}$. By Theorem 3, there exists a bipartite graph G_i of order $a_m + b_{im}$ with degree sets S_1 and $B_i (1 \leq i \leq t)$, and a bipartite graph G_{t+1} with degree sets B_{t+1} and $\{a_1, a_2, \ldots, a_r\}$.

If $n = tm$, let $G = \bigcup_{i=1}^{t} G_i$ and if $n = tm + r$ let $G = \bigcup_{i=1}^{t+1} G_i$. Then G is bipartite and has order $ta_m + \sum_{i=1}^{t} b_{im}$, if $n = tm$, and G has order $ta_m + \sum_{i=1}^{t} b_{im} + a_r + b_n$ if $n = tm + r$ and G has degree sets S_1 and S_2. As in Proposition 1 we may modify G to obtain a connected graph. ∎

In particular, note that Corollary 4 implies that $\mu(a;b_1,b_2,\ldots,b_n) \leq na + \sum_{i=1}^{n} b_i$ and if $a = 1$ it is easily seen that

$$\mu(1;b_1,b_2,\ldots,b_n) = n + \sum_{i=1}^{n} b_i .$$

In what follows we deal with the case $|S_1| = 1$ and $|S_2| = 2$.

Theorem 5. If a and b are integers such that $1 \leq a < b$ then

$$\mu(a;a,b) = \begin{cases} a + b + 1 & \text{if } 1 < a < b - 1 \text{ and } (b-a)|b, \\ a + b + 2 & \text{otherwise} \end{cases}$$

Proof. We note that $\mu(a;a,b) \leq a + b + 2$ since the bipartite graph G with partite sets V_1, containing $b + 1$ vertices of degree a, and V_2, containing a vertices of degree b and one vertex of degree a, can easily be constructed. Also, by Proposition 2, $\mu(a;a,b) \geq a + b$.

Let G be a bipartite graph with partite sets V_1 and V_2 with degree sets $S_1 = \{a\}$ and $S_2 = \{a,b\}$ respectively and let $|V(G)| = \mu(a;a,b)$. Then $|V_1| \geq b$ and $|V_2| \geq a$. Let V_2 contain x vertices of degree a.

Suppose $\mu(a;a,b) = a + b$. Then $|V_1| = b$ and $|V_2| = a$. By counting the number of edges incident with the vertices in each partite set we arrive at the equation $ab = ax + b(a-x)$. But then, $ab = ab + (a-b)x < ab$, a contradiction. Thus, $\mu(a;a,b) > a + b$.

Now, suppose $\mu(a;a,b) = a + b + 1$. Then there are only two possibilities for the cardinalities of V_1 and V_2.

Case 1. Suppose $|V_1| = b + 1$ and $|V_2| = a$. Then the equation $a(b+1) = ax + b(a-x)$ must hold. But this implies that $a(b+1) = ab + (a-b)x < ab$, again a contradiction.

Case 2. Suppose $|V_1| = b$ and $|V_2| = a + 1$. Then we see that the equation $ab = ax + b(a+1-x)$ must hold, where $1 \leq x \leq a$. Thus, $b = (b-a)x$ so that $x = b/(b-a)$ and hence $b/(b-a) \leq a$. Since x is an integer, $(b-a)$ must divide b. By hypothesis, $a < b$, moreover, $a \neq b - 1$, for otherwise $b/(b-(b-1)) = b \leq a$, implying that $a < b - 1$. Further, $a \geq 1$ implies that $b \geq 3$. Thus, since $(b-a)$ divides b, we see that $a > 1$. Clearly, if these conditions fail, then $\mu(a;a,b) = a + b + 2$.

If these conditions are satisfied, we construct a bipartite graph G with partite sets $V_1 = \{v_0, v_1, \ldots, v_{b-1}\}$ and $V_2 = \{u_0, u_1, \ldots, u_a\}$. Join each vertex $u_i (0 \leq i \leq x-1)$ to each of the vertices $v_{ia}, v_{ia+1}, \ldots, v_{(i+1)a-1}$ (subscripts expressed modulo b). Then join u_x to each of the vertices $v_{xa}, v_{xa+1}, \ldots, v_{xa+b-1}$. Continue to cyclically join each of the remaining vertices $u_{x+1}, \ldots, u_a$ to the vertices of V_1 (so that each u_j, $x+1 \leq j \leq a$, has degree b). Then each vertex of V_1 has degree a, while in V_2, each u_i, $0 \leq i \leq x - 1$, has degree a and each u_j, $x \leq j \leq a + 1$ has degree b. Thus G has degree sets S_1 and S_2 and order $a+b+1$. ∎

For a real number x, let $\{x\}$ denote the smallest integer not less than x. Further, let $\{x\}_e$ (respectively $\{x\}_o$) denote the smallest even integer (odd integer) not less than x.

Although no general formula is known for $\mu(a;b_1,b_2)$, we can determine the value of $\mu(2;b_1,b_2)$.

Theorem 6. Let b_1 and b_2 be integers with $1 \le b_1 < b_2$ and $x = b_2/b_1$.

(I) If b_1 is even and b_2 is odd, then

$$\mu(2;b_1,b_2) = b_2 + b_1/2 + 3 .$$

(II) If b_1 and b_2 are even, then

$$\mu(2;b_1,b_2) = \min\{b_2 + b_1/2 + 3, \frac{\{x\}b_1 + b_2}{2} + \{x\} + 1\} .$$

(III) If b_1 is odd and b_2 is even, then

$$\mu(2;b_1,b_2) = \min\{b_1 + b_2 + 4, \frac{\{x\}_e b_1 + b_2}{2} + \{x\}_e + 1\} .$$

(IV) If b_1 and b_2 are odd, then

$$\mu(2;b_1,b_2) = \min\{b_1 + b_2 + 4, \frac{\{x\}_o b_1 + b_2}{2} + \{x\}_o + 1\} .$$

Proof. Suppose b_1 is even and b_2 is odd. Note that $\mu(2;b_1,b_2) \le b_2 + b_1/2 + 3$ since a bipartite graph with one vertex of degree b_1 and two vertices of degree b_2 in one partite set and $b_2 + b_1/2$ vertices of degree 2 in the other partite set exists.

To see the reverse inequality holds, note that at least two vertices of degree b_2 and hence at least $b_2 + b_1/2$ vertices of degree 2 are necessary.

(II). Suppose that b_1 and b_2 are even. As above, there exists a bipartite graph of order $b_2 + b_1/2 + 3$ having degree sets $\{2\}$ and $\{b_1,b_2\}$. It is also straightforward to construct a bipartite graph of order $\frac{\{x\}b_1 + b_2}{2} + \{x\} + 1$ with one vertex of degree b_2, $\{x\}$ vertices of degree b_1 and $(\{x\}b_1 + b_2)/2$ vertices of degree 2. Thus,

$$\mu(2;b_1,b_2) \le \min\{b_2 + b_1/2 + 3, \frac{\{x\}b_1 + b_2}{2} + \{x\} + 1\} .$$

Now suppose that G is a bipartite graph with degree sets $\{2\}$ and $\{b_1,b_2\}$. If G has exactly one vertex of degree b_2, then G has at least $\{x\}$ vertices of degree b_1 and thus, at least $(\{x\}b_1 + b_2)/2$ vertices of degree 2. Thus the order of G is at least $\frac{\{x\}b_1 + b_2}{2} + \{x\} + 1$. If on the other hand, G has at least two vertices of degree b_2, then G has at least one vertex of degree b_1 and thus at least $b_2 + b_1/2$ vertices of degree 2, implying G has order at least $b_2 + b_1/2 + 3$, thus producing the desired result.

(III). Suppose b_1 is odd and b_2 is even. It is easy to construct a bipartite graph of order $b_1 + b_2 + 4$ that has two vertices of degree b_1, two vertices of degree b_2 and $b_1 + b_2$ vertices of degree 2. Furthermore, a bipartite graph of order $\frac{\{x\}_e b_1 + b_2}{2} + \{x\}_e + 1$ with one vertex of degree b_2, $\{x\}_e$ vertices of degree b_1 and $\frac{\{x\}_e b_1 + b_2}{2}$ vertices of degree 2 also exists.

Now suppose that G is a bipartite graph having degree sets $\{2\}$ and $\{b_1,b_2\}$. If G has exactly one vertex of degree b_2, then G has at least $\{x\}_e$ vertices of degree b_1, since b_1 is odd and b_2 is even. But then G has at least $\frac{\{x\}_e b_1 + b_2}{2}$ vertices of degree 2, implying G has order at least $\frac{\{x\}_e b_1 + b_2}{2} + \{x\}_e + 1$. If G contains at least two vertices of degree b_2, then G must contain at least two vertices of degree b_1 and, hence, at least $b_1 + b_2$ vertices of degree 2. Therefore, G has order at least $b_1 + b_2 + 4$. It is easily seen that if G contains more than two vertices of degree b_2, then G has order greater than $b_1 + b_2 + 4$, giving the result.

(IV). Suppose that b_1 and b_2 are odd. In a manner analogous to (III), the inequality $\mu(2;b_1,b_2) \leq \min\{b_1+b_2+4, \frac{\{x\}_o b_1+b_2}{2}+\{x\}_o+1\}$ must hold. To verify the lower bound, let G denote a bipartite graph having degree sets $\{2\}$ and $\{b_1,b_2\}$. If G has exactly one vertex of degree b_2, then G has at least $\{x\}_o$ vertices of degree b_1 since b_1 and b_2 are odd. Thus, G has at least $(\{x\}_o b_1+b_2)/2$ vertices of degree 2, and hence, the order of G is at least

$\frac{\{x\}_o b_1 + b_2}{2} + \{x\}_o + 1$. If G has exactly two vertices of degree b_2, it has at least two vertices of degree b_1 and therefore, at least $b_1 + b_2$ vertices of degree 2. Hence, the order of G is at least $b_1 + b_2 + 4$. Finally, if G has at least three vertices of degree b_2, it must have at least one vertex of degree b_1 and so, at least $(3b_2+b_1)/2$ of degree 2 and order at least $(3b_2+b_1)/2 + 4$. However, since $b_1 < b_2$,

$$b_1 + b_2 + 4 < \frac{3b_1 + b_2}{2} + 4, \quad \text{so that}$$

$$\mu(2;b_1,b_2) \geq \min\{b_1+b_2+4, \frac{\{x\}_o b_1+b_2}{2}+\{x\}_o+1\},$$

completing the proof. ∎

We note that in Theorem 6, (II) - (IV), it is possible to find pairs b_1, b_2, so that the minimum is attained by either of the two expressions.

REFERENCES

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