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GRAPH CONNECTIVITY AFTER PATH REMOVAL

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Let G be a graph and u, v be two distinct vertices of G. A $u-v$ path P is called nonseparating if $G-V(P)$ is connected. The purpose of this paper is to study the number of nonseparating $u-v$ path for two arbitrary vertices u and v of a given graph. For a positive integer k, we will show that there is a minimum integer $\alpha(k)$ so that if G is an $\alpha(k)$-connected graph and u and v are two arbitrary vertices in G, then there exist k vertex disjoint paths $P_1[u,v]$, $P_2[u,v]$, …, $P_k[u,v]$ such that $G-V(P_i[u,v])$ is connected for every i ($i=1,2,\dots,k$). In fact, we will prove that $\alpha(k)\le 22k+2$. It is known that $\alpha(1)=3$. A result of Tutte showed that $\alpha(2)=3$. We show that $\alpha(3)=6$. In addition, we prove that if G is a 5-connected graph, then for every pair of vertices u and v there exists a path $P[u,v]$ such that $G-V(P[u,v])$ is 2-connected.

1. Introduction

The purpose of this article is to investigate graphs, which preserve some connectivity properties after the removal of the vertex set of some paths. The following result was conjectured to be true by Lovász [7] and proved by Thomassen [11].

Theorem 1. *If G is a $(k+3)$-connected graph, then G contains a cycle C such that $G-V(C)$ is k-connected.*

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However, the problem becomes more difficult if one requires the cycle to contain a specific edge. Given a pair of vertices u and v, a $u-v$ path P is a path from u to v. The following conjecture due to Lovász [7] is still open.

Conjecture 1. For each natural number k, there exists a least natural number $\beta(k)$ such that, for any two vertices u, v in any $\beta(k)$-connected graph G, there exists a $u-v$ path P such that $G-V(P)$ is k-connected.

By a theorem of Tutte [13], we have that $\beta(1)=3$. We will prove that $\beta(2)\le 5$ in this paper. In fact, results in [3] and [13] show that if u and v are two vertices in a 3-connected graph G, then there exist two internally vertex disjoint u-v paths P and Q such that both $G-V(P)$ and $G-V(Q)$ are connected.

Let $P[x,y]$ be an x-y path in G. If no confusion arises, we sometimes use only P to stand for $P[x,y]$. If u and v are two vertices on P in the order from x to y along P, let $P[u,v]$ denote the subpath from u to v and $P^-[v,u]$ denote the subpath from v to u. If $P[x,y]$ is not an edge, we will use either IP or $P(x,y)$ to denote $P[x,y]-\{x,y\}$, the internal segment of the path. If $P[x,y]=xy$, we define that $IP=P(x,y)=\emptyset$. For two paths P and Q, P *is internally disjoint from* Q if $V(IP)\cap V(Q)=\emptyset$. If P and Q have the same end vertices, then the statement P is internally disjoint from Q is equivalent to the statement Q is internally disjoint from P. In this case, we say P and Q are two internally vertex disjoint paths. If H is a connected subgraph of G and both a and b are either in H or adjacent to some vertices in H, let aHb denote an arbitrary path joining a and b such that all internal vertices are in H.

A path P is called a *nonseparating path* if $G-V(P)$ is connected. Given a positive number k, we will investigate the minimum number $\alpha(k)$ such that if G is an $\alpha(k)$-connected graph and u, v are two vertices of G, then there exist k internally vertex disjoint nonseparating $u-v$ paths P_1, P_2, …, P_k.

The value $\alpha(k)$ is related to a property called (s,t)-*linked* described below. A graph G is (s,t)-*linked* if for every two vertex disjoint sets S and T with $|S|=s$ and $|T|=t$, G contains two vertex disjoint connected subgraphs F and H such that $S\subseteq V(F)$ and $T\subseteq V(H)$. Let G be a graph and let $x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k$ be $2k$ distinct vertices of G. We say that G has an $(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k)$-*linkage* if G contains k vertex disjoint paths $P_1[x_1,y_1]$, $P_2[x_2,y_2]$, $\cdots$, $P_k[x_k,y_k]$. A graph is said to be k-*linked* if it has at least $2k$ vertices and for any choice of $2k$ distinct vertices $x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k$, G has an $(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k)$-linkage. Larman and Mani [6] and Jung [4] proved independently that there exists a (smallest) integer $f(k)$ such that every $f(k)$-connected graph is

k-linked. The proof is based on a result of Mader [8] dealing with subdivisions of large complete graphs. Bollobás and Thomason [2] proved that $f(k) \leq 22k$. A complete characterization of k-linked graphs is not known. Clearly, G is $(2,2)$-linked if and only if G is 2-linked. Jung [4] proved that all 4-connected nonplanar graphs are 2-linked. Seymour [9] and Thomassen [12] characterized graphs which do not contain an (x,u,y,v)-linkage for four specific distinct vertices u, v, x, y.

Suppose that G is an $\alpha(k+1)$-connected graph. Let $S=\{x,y\}$ and T be two disjoint subsets of $V(G)$ where $|T|=k$. Since G is $\alpha(k+1)$-connected, there are $k+1$ internally vertex disjoint nonseparating $x-y$ paths. One such path, say $P[x,y]$, does not contain any vertex of T. Let F be the subgraph induced by $V(P[x,y])$ and $H=G-V(F)$. Clearly, $S\subseteq V(F)$ and $T\subseteq V(H)$ and both F and H are connected. Thus, every $\alpha(k+1)$-connected graph (if it exists) is $(2,k)$-linked. In particular, every $\alpha(3)$-connected graph is 2-linked.

Let G be a graph and H be a subgraph or a subset of $V(G)$. We define

$$N(H) = \{x \notin V(G) - V(H) \ : \ xy \in E(G) \text{ for some } y \in V(H) \text{ or } H\}.$$

The following result from [3] will be used quite often in our proofs.

Lemma 1.1. *Let G be a 3-connected graph and let H be a connected induced subgraph of G. Let F be a component of $G-V(H)$. Then, for every two vertices x and y in H, there is a path $Q[x,y]$ in H such that each component C of $H-V(Q[x,y])$ is adjacent to F, that is, $N(C)\cap V(F)\neq\emptyset$.*

A set of internally vertex disjoint $x-y$ paths (walks) $P_1,P_2,\ldots,P_m$ are called *unified* paths (walks) if for each i $(1\leq i\leq m)$ IP_1, IP_2, $\cdots$, IP_{i-1}, IP_{i+1}, $\cdots$, IP_k are in the same component of $G-V(P_i)$. Unified paths play a fundamental role in the results we develop. Lemma 1.2 follows from Lemma 1.1.

Lemma 1.2. *Let G be a 3-connected graph and let x and y be two distinct nonadjacent vertices of G. Then, G contains k internally vertex disjoint nonseparating $x-y$ paths if, and only if, G contains a set of k unified $x-y$ paths.*

Proof. Let G be a 3-connected graph. We will only show that if G contains a set of k unified $x-y$ paths then G contains k internally vertex disjoint nonseparating x–y paths, since the reverse implication is trivial. Let $P_1[x,y]$, $P_2[x,y]$, $\cdots$, $P_k[x,y]$ be a set of k unified $x-y$ paths such that the number of nonseparating paths among these k-paths is maximum. If all $P_1[x,y]$, $P_2[x,y]$, $\cdots$, $P_k[x,y]$ are nonseparating paths, we are done. Suppose, to the contrary, $P_1[x,y]$ is not a nonseparating path. Since $P_1[x,y]$, $P_2[x,y]$, $\cdots$,

$P_k[x,y]$ form a set of unified paths, let C be the component of $G-V(P_1[x,y])$ containing $\cup_{i\neq 1}V(IP_i)$. Let $H = G-V(C)$. By Lemma 1.1, there is an $x-y$ path $Q_1[x,y]$ in H such that $G-V(Q_1[x,y])$ is connected. We claim that $Q_1[x,y]$, $P_2[x,y]$, $\cdots$, $P_k[x,y]$ is a set of unified paths, which yields a contradiction to the maximality of the number of nonseparating paths among the k unified paths. Suppose, to the contrary, there is an $i\geq 2$ such that IQ_1 and $\cup_{j\neq 1,i}IP_j$ are in different components of $G-V(P_i[x,y])$. Then, all vertices of $\cup_{j\neq 1,i}IP_j$ are in the same component and $V(IQ_1)\cap V(IP_1)=\emptyset$, since $P_1[x,y]$, $P_2[x,y]$, $\cdots$, $P_k[x,y]$ is a set of unified paths. Let D be the component of $H-V(P_1)$ containing $V(IQ_1)$. Then, by the definition of H, D is a component of $G-V(P_1)$ different from C and hence $N(D)\subseteq V(P_1)$. Moreover, again from the fact that P_1, $\ldots$, P_k are unified paths while Q_1, P_2, $\cdots$, P_k are not unified paths, it follows that $N(D)\cap V(IP_1)=\emptyset$. Consequently, $\{x,y\}$ is a cut set separating D, a contradiction to the fact that G is 3-connected. ∎

The following lemma will be used several times in this paper. We will not give a proof since it follows directly from the definitions of unified paths and walks.

Lemma 1.3. *Let G be a connected graph and let x and y be two distinct nonadjacent vertices of G. Then, G contains k unified $x-y$ paths if and only if G contains k unified $x-y$ walks.*

A graph G is said to be *unified $k-$linked* if it has at least $2k$ vertices and for any choice of distinct $s_1, s_2,\ldots,s_k, t_1, t_2,\ldots,t_k$, G contains vertex disjoint paths $P_1[s_1,t_1],P_2[s_2,t_2],\ldots,P_k[s_k,t_k]$ such that all s_j $(j\neq i)$ and all t_j $(j\neq i)$ are in the same component of $G-V(P_i[s_i,t_i])$ for each i $(1\leq i\leq k)$. Bollobás and Thomason [2] proved that every $22k$-connected graph is k-linked. In fact, following their proof, it is easy to show that the following result is true.

Theorem 2. *Let G be a graph with vertex connectivity $\kappa(G)\geq 22k$. Then G is unified k-linked.*

The following results is a consequence of Theorem 2 and Lemma 1.2.

Corollary 3. *Let G be a graph with vertex connectivity $\kappa(G)\geq 22k+2$. Then for every two vertices u and v there are k internally vertex disjoint nonseparating $u-v$ paths $P_1,\ldots,P_k$, that is, $\alpha(k)\leq 22k+2$.*

Proof. Let $u_1,\ldots,u_k$ be k neighbors of u and $v_1,\ldots,v_k$ be k neighbors of v in $G-\{u_1,u_2,\ldots,u_k\}$ and let $G^*=G-\{u,v\}$. By Theorem 2 and Lemma 1.2, G^* contains k vertex disjoint nonseparating paths $P_1[u_1,v_1],\ldots, P_k[u_k,v_k]$. Let $P_i=uP[u_i,v_i]v$ for each $i=1,2,\ldots,k$. ∎

2. The exact value of $\alpha(1), \alpha(2)$,and $\alpha(3)$

Let $\{x,y\}$ be the part of two vertices of a complete bipartite graph $K_{2,n}$ ($n \geq 3$). Clearly, there does not exist a nonseparating x-y path. Thus, $\alpha(1) \geq 3$. As stated earlier, a theorem of Tutte [13] and a result in [3] state that if G is a 3-connected graph and u and v are two nonadjacent vertices of G, then there exist two internally vertex disjoint paths $P_1[u,v]$ and $P_2[u,v]$ such that $G - V(P_1[u,v])$ and $G - V(P_2[u,v])$ are connected. From which, we conclude that $\alpha(1) = \alpha(2) = 3$. Since there do not exist three vertex disjoint nonseparating paths for two nonadjacent vertices on the unbounded face of a 5-connected plane graph, then $\alpha(3) \geq 6$. The remainder of this section is devoted to proving that $\alpha(3) \leq 6$. Let x and y be two distinct vertices in a graph G. If x and y are adjacent and G is 3-connected, following the proof of $\alpha(2) = 3$ in [3], it is not difficult to show that G contains three unified $x-y$ paths (one of them is the edge xy). If x and y are not adjacent in G, we will describe all graphs which do not contain three unified $x-y$ paths. Our approach is inspired by the following result of Jung and a stronger result obtained independently by Seymour and Thomassen.

Theorem 4. (**Jung** [4]) *Every 4-connected non-planar graph is 2-linked.*

Let G be a graph and let x and y be two nonadjacent vertices of G. The following definition is from [12]. Let G_0 be a plane graph such that the unbounded face is bounded by a 4-cycle, say $S_0 = x_1x_2y_1y_2x_1$ and such that every other face is bounded by a 3-cycle. Suppose in addition that G_0 has no separating 3-cycle (i.e. a 3-cycle which is not a facial cycle). For each 3-cycle S of G_0 we add K^S, a possibly empty complete graph vertex disjoint from G_0, and we join all vertices of K^S to all vertices of S. The resulting graph is called an (x_1,x_2,y_1,y_2)-*web* with frame S_0 and *rib* G_0. If G_0 has more than four vertices, S_0 and the rib G_0 are uniquely determined, and it follows from well-known results on planar graphs that G_0 (and hence also G) is 3-connected and that any cut set of three vertices of G_0 is of the form $\{x_1,y_1,z\}$ or $\{x_2,y_2,z\}$. A simple argument shows that G does not contain three internally vertex disjoint nonseparating $x_1 - y_1$ paths if x_1 and y_1 are not adjacent.

Theorem 5. (*Seymour [9] and Thomassen [11]*) *Let x_1,x_2,y_1,y_2 be vertices of a graph G. If G has no $(x_1,x_2,y_1,y_2)-$linkage and the addition of any edge results in a graph containing an (x_1,x_2,y_1,y_2)-linkage, then G is an (x_1,x_2,y_1,y_2)-web. Conversely, any (x_1,x_2,y_1,y_2)-web is maximal with respect to the property of not containing an (x_1,x_2,y_1,y_2)-linkage.*

In this paper, we obtain a similar characterization for graphs, which do not contain three unified x-y paths.

Theorem 6. *Let x, y be two nonadjacent vertices of a graph G of order $n \geq 4$. If G does not contain three unified $x-y$ paths and the addition of any edge to G results in a graph which has three unified $x-y$ paths, then there are two vertices u and v such that G is a (x,u,y,v)-web. Conversely, any (x,u,y,v)-web with x and y nonadjacent is maximal with respect to the property that the graph does not contain three unified $x-y$ paths.*

Proof. To prove the second part of Theorem 6, we consider an (x,u,y,v)-web G and let w and z be two nonadjacent vertices of G. By Theorem 5, $G \cup \{wz\}$ has an (x,u,y,v)-linkage. Let $P[x,y]$ and $Q[u,v]$ be two vertex disjoint paths. Then $P_1 = xuy$, $P_2 = P[x,y]$, and $P_3 = xvy$ are three unified $x-y$ paths. The proof is complete.

The first part of Theorem 6 is proved by induction on the number of vertices of G. If G has only four vertices, the statement is trivial. Suppose the result is true for graphs of order less than n $(n \geq 5)$. Let G be a graph of order n satisfying the conditions of Theorem 6 and hence, G does not contain three unified $x-y$ paths. We proceed with the following claims.

Claim 2.1. *G is 2-connected.*

Proof. Suppose z is a cutvertex of G with a,b neighbors of z belonging to distinct components of $G-z$. Since G does not contain three unified $x-y$ paths, adding the edge ab to G does not create three unified $x-y$ paths, contradicting the edge maximality of G. ∎

Claim 2.2. *G is 3-connected.*

Proof. To the contrary, let $\{a,b\}$ be a cutset of G and let G_1 and G_2 be two induced subgraphs of G such that $V(G_1) \cap V(G_2) = \{a,b\}$ and $V(G_1) \cup V(G_2) = V(G)$.

Suppose $\{a,b\} = \{x,y\}$. By the induction hypothesis, G_i is a subgraph of an (x,u_i,y,v_i)-web or G_i is a path xv_iy for each $i = 1,2$. It is then readily seen that $G \cup \{v_1v_2\}$ is a subgraph of an (x,u_1,y,u_2)-web. Thus, $G \cup \{v_1v_2\}$ does not contain three unified x-y paths, a contradiction.

Suppose that $\{a,b\} \neq \{x,y\}$. If both x and y are in G_1, the maximality property of G implies that the edge ab is present and G_1 is also edge maximal with respect to the property of not containing three unified x–y paths. Then, G_1 is an $\{x,u,y,v\}$-web. Now clearly, G is a subgraph of a web. Similarly, it is impossible to have both x, y in G_2. Therefore, without loss of generality, we assume that $x \in V(G_1) - \{a, b\}$ and $y \in V(G_2) - \{a, b\}$. Now by the maximality of G, we can see that both G_1 and G_2 are complete graphs, which implies that G is an $\{x, a, y, b\}$-web. ∎

Claim 2.3. *For every triangle A, each component of $G - V(A)$ intersects $\{x, y\}$.*

Proof. To the contrary, assume there is a triangle A such that there is a component of $G - V(A)$ that fails to intersect $\{x, y\}$. Let H be the union of all such components. Further, select A such that $|V(H)|$ is maximum. Since $xy \notin E(G)$, at most one of x and y is on A. If there are two vertices $u, v \in V(H) \cup A$ such that $uv \notin E(G)$, then $G \cup \{u, v\}$ contains three unified x–y paths, implying that G contains three unified x-y paths. Thus, $G(V(H) \cup A)$ is a clique. Moreover it is easy to see that $G - V(H)$ is maximal with respect to the property of containing no three unified $x - y$ paths. Thus, by the induction hypothesis, $G - V(H)$ is an (x, u, y, v)-web with rib, say, G_0. Let S be the unique triangle of G_0 such that every path from $V(H)$ to $\{x, y\}$ intersects S. The maximality of G implies that every vertex of H is joined to every vertex of S. Thus, $A = V(S)$ and G is an (x, u, y, v)-web. ∎

Claim 2.4. *Let A be a set of three vertices in G such that $G - A$ is disconnected. Then, each component of $G - A$ intersects $\{x, y\}$.*

Proof. Suppose, to the contrary, there is a component H of $G - A$ not intersecting $\{x, y\}$. We consider three cases according to the cardinality of $A \cap \{x, y\}$.

Case 1. Suppose that $|\{x, y\} \cap A| = 2$.

Let $A = \{x, y, z\}$. Then, $xz \in E(G)$ (and by a similar argument $yz \in E(G)$), for if $xz \notin E(G)$, $G \cup \{xz\}$ contains three unified $x - y$ paths, P_1, P_2, P_3. Clearly, one of P_1, P_2, P_3, say P_1, contains the edge xz and the other two paths, P_2, P_3 must be either both in $G - V(H)$ or $G[V(H) \cup \{x, y\}]$. Without loss of generality, say that they both are in $G - V(H)$. Then, $xHzP_1[z, y]$, P_2, and P_3 are three unified $x - y$ walks in G, a contradiction by Lemma 1.3.

Let $G_1 = G[V(H) \cup A]$ and $G_2 = G - V(H)$. Since both xz, yz are present in G and G is edge maximal with respect to the property of not containing three unified x–y paths, both G_1 and G_2 are also edge maximal with respect to the same property. By our induction hypothesis, we assume that G_i is an (x, u_i, y, v_i)-web for $i = 1, 2$. If $z \in \{u_1, v_1\}$ and $z \in \{u_2, v_2\}$, we assume $z = u_1 = u_2$. Then G is an (x, v_1, y, v_2)-web. Thus, we can assume that $z \notin \{u_1, v_1, u_2\}$. Then, xu_1y, xv_1y, and xu_2y form a set of unified paths, a contradiction completing Case 1. ∎

We now can assume that at most one of x and y is in A. The subclaim below is needed in dealing with the following two cases.

Subclaim 2.4.1. For any two vertices a, $b \in A-\{x,y\}$, $ab \in E(G)$.

Proof. Assume, to contrary, $ab \notin E(G)$. Then $G \cup \{ab\}$ contains three unified $x-y$ paths, P_1, P_2, P_3. If ab is on P_1, then $V(P_2 \cup P_3) \cap V(H) = \emptyset$. We may assume a precedes b on P_1.

Then, $P_1[x,a]aHbP_1[b,y]$, P_2, P_3 are unified $x-y$ walks in G, a contradiction by Lemma 1.3. Thus, ab is not on any of P_1, P_2, P_3. Without loss of generality, we assume that ab connects two paths P_1 and P_2 in $G \cup \{ab\} - V(P_3)$. Since $|A| = 3$ and $\{a,b\} \subset A$, we have $V(P_3) \cap V(H) = \emptyset$. So, P_1 and P_2 are connected in $G - V(P_3)$, a contradiction. ∎

Case 2. Suppose that $|A \cap \{x,y\}| = 0$.

A contradiction follows directly from Claim 2.3 and Subclaim 2.4.1. ∎

Case 3. Suppose that $|A \cap \{x,y\}| = 1$.

Without loss of generality, we assume that $A = \{x, z_1, z_2\}$. By Subclaim 2.4.1, $z_1z_2 \in E(G)$. Since $G[A] \neq K_3$, without loss of generality, assume that $xz_1 \notin E(G)$. From the maximality of G, $G^* = G \cup \{xz_1\}$ contains three unified $x-y$ paths P_1, P_2, P_3. We will show $|V(H)| = 1$ and $xz_2 \notin E(G)$.

Note that xz_1 must be on one of P_1, P_2, and P_3 since $G \cup \{xz_1\} - V(P_i) = G - V(P_i)$. Without loss of generality, we assume that $xz_1 \in E(P_1)$.

If $V(P_2 \cup P_3) \cap V(H) = \emptyset$, then $xHz_1P_1[z_1,y]$, P_2, and P_3 are three unified $x-y$ walks in G, a contradiction by Lemma 1.3. Without loss of generality, we assume $V(P_2) \cap V(H) \neq \emptyset$. Therefore $z_2 \in V(P_2)$. If $xz_2 \in E(G)$, then $xHz_1P_1[z_1,y]$, $xz_2P_2[z_2,y]$, P_3 are three unified $x-y$ walks in G, again a contradiction by Lemma 1.3.

Moreover, we assume that $G[V(H) \cup A]$ does not contain two paths $Q_1[x,z_1]$ and $Q_2[x,z_2]$ such that $V(Q_1(x,z_1]) \cap V(Q_2(x,z_2]) = \emptyset$. By Menger's theorem, $G[V(H) \cup A]$ contains a cutvertex w separating x and $\{z_1, z_2\}$. If $|V(H)| \geq 2$, then since G is 3-connected so $\{x, w\}$ is not a cutset. Then $B = \{w, z_1, z_2\}$ is a cut of G which contains neither x or y, a contradiction to Case 2. Thus, $V(H) = \{w\}$. In particular, we have that x, z_1, z_2 are all adjacent to w. Furthermore, by our assumption, xz_1 is on P_1 and xwz_2 is a segment of P_2. For convenience, we let

$$Q_1 = xwz_1P_1[z_1,y],\ Q_2 = P_2,\ Q_3 = P_3.$$

We easily see that Q_1, $xz_2Q_2[z_2,y]$, Q_3 are unified paths in $G \cup \{xz_2\}$.

Subclaim 2.4.2. Suppose that G contains a path $R[x,v]$ internally disjoint from Q_1, Q_2, and Q_3 with $v(\neq w)\in V(IQ_1)$, and let $Q_1^{'}=R[x,v]Q_1[v,y]$. Suppose further that in $G{-}V(Q_1^{'})$, IQ_2 and IQ_3 are in the same component. Then, $Q_1^{'}$, Q_2, Q_3 are unified paths.

Proof. In $G-V(Q_2)$ and $G-V(Q_3)$, $Q_1[z_1,v)$ is in the same component as $Q_1[v,y)$. Hence the desired conclusion follows from the fact that P_1, P_2, and P_3 are unified paths in $G\cup\{xz_1\}$. ∎

Subclaim 2.4.3. There does not exist a path $R[x,v]$ with $v(\neq w)\in V(IQ_1\cup IQ_2)$ and $V(R(x,v))\cap V(Q_1\cup Q_2\cup Q_3)=\emptyset$.

Proof. Suppose, to then contrary, G contains a path $R[x,v]$ such that $V(R(x,v))\cap V(P_1\cup P_2\cup P_3)=\emptyset$ and $v(\neq w)\in V(IQ_1)\cup V(IQ_2)$, say $v\in V(IQ_1)$. Let $Q_1^*=R[x,v]Q_1[v,y]$. To prove Subclaim 2.4.3 we will show that we can choose $R[x,v]$ such that Q_1^*, Q_2,and Q_3 are unified $x-y$ paths. By Subclaim 2.4.2, we only need to show that IQ_2 and IQ_3 are in the same component in $G{-}V(Q_1^*)$. To the contrary, assume that IQ_2 and IQ_3 are in different components of $G-V(Q_1^*)$. Since Q_2 and Q_3 are in the same component of $G-V(P_1)$, $V(R(x,v))$ separates Q_2 and Q_3 in $G-V(P_1)$. In particular, there are two internally vertex-disjoint paths $T_2[u_2,v_2]$ and $T_3[v_3,u_3]$ with $u_2\in V(Q_2[z_2,y))$, $u_3\in V(Q_3(x,y))$, $v_2,v_3\in V(R(x,v))$, and

$$V(T_2\cup T_3)\cap V(Q_1\cup Q_2\cup Q_3\cup R)=\emptyset.$$

If $v_2\in V(R(x,v_3))$, let $Q_2^*=R[x,v_2]T_2^-[v_2,u_2]Q_2[u_2,y]$. In $G-V(Q_2^*)$, IQ_1 and IQ_3 are in the same component. By Subclaim 2.4.2, Q_1, Q_2^*, and Q_3 are three unified $x-y$ paths, a contradiction. Thus, $v_2\in V(R[v_3,v))$.

We choose T_2 and T_3 so that v_3 is as close as possible to v on R. We may assume we have chosen the path $R(x,v)$ such that $|R(v_3,v)|$ is minimum. We will show that $Q_3\cup R[x,v_3)\cup IT_3$ and $Q_1[z_1,y)\cup Q_2[z_2,y)\cup R(v_3,v)\cup IT_2$ are in different components of $G-\{y,v_3,w\}$. Thus, $\{y,v_3,w\}$ is a cut of G, a contradiction since $v_3w\notin E(G)$.

Suppose, to the contrary, there is a path $S[u^*,v^*]$ in $G-\{y,v_3,w\}$ such that $u^*\in V(Q_3\cup R(x,v_3)\cup IT_3)$, $v^*\in V(Q_1[z_1,y)\cup Q_2[z_2,y)\cup R(v_3,v)\cup IT_2)$, and

$$V(IS)\cap V(Q_1\cup Q_2\cup Q_3\cup T_2\cup T_3\cup R)=\emptyset.$$

We will show a contradiction case by case as follows.

Case I. $u^*\in V(R[x,v_3))$. In this case, if $v^*\in V(R(v_3,v))$, path $R^*=R[x,u^*]S[u^*,v^*]R[v^*,v]$ will give a contradiction to the minimality of $|V(R(v_3,v))|$. If $v^*\in V(IT_2)$,
let $Q_2^*=R[x,u^*]S[u^*,v^*]T_2^-[v^*,u_2]Q_2[u_2,y]$. IQ_1 and IQ_3 are in the same

component of $G-V(Q_2^*)$, a contradiction to Subclaim 2.4.2. If $v^* \in V(Q_1[z_1,y))$, let $Q_1^{**}=R[x,u^*]S[u^*,v^*]Q_1[v^*,y]$. In $G-V(Q_1^{**})$, IQ_2 and IQ_3 are in the same component, a contradiction to Subclaim 2.4.2. Thus, $v^* \in V(Q_2[z_2,y))$. Let $Q_2^*=R[x,u^*]S[u^*,v^*]Q_2[v^*,y]$. In $G-V(Q_2^*)$, IQ_1 and IQ_3 are in the same component, a contradiction to Subclaim 2.4.2 again.

Case II. $u^* \in V(Q_3(x,y))$. From the minimality of $|V(R(v_3,v))|$, $v^* \notin V(R(v_3,v))$. Since IQ_2 and IQ_3 are in different components of $G-V(Q_1^*)$ where $Q_1^*=R[x,v]Q_1[v,y]$, then $v^* \notin V(IT_2)$ and $v^* \notin V(IQ_2)$. Thus, $v^* \in V(Q_1[z_1,y))$. Let $Q_2^*=R[x,v_2]T^-[v_2,u_2]Q_2[u_2,y]$. Then, S connecting IQ_1 and IQ_3 in $G-V(Q_2^*)$, a contradiction to Subclaim 2.4.2.

Case III. $u^* \in V(IT_3)$. In the same manner as Case 2, we can show that there is a contradiction. ∎

Let z_3 be the successor of x on Q_3. Then, by Subclaim 2.4.1, $\{y,w,z_3\}$ is not a cut, and hence there is a path $R_1[x,x_1]$ connecting x to a vertex $x_1 \in V(Q_3(z_3,y))$ which is internally disjoint from $Q_1 \cup Q_2 \cup Q_3$. Pick $R_1[x,x_1]$ such that $|V(Q_3[x,x_1])|$ is maximum with the above property. If $Q_3(x,x_1)$ and $IQ_1 \cup IQ_2$ are in the same components of $G-V(Q_3[x_1,y]) \cup \{x\}$, we stop. Otherwise, let $R_2[y_1,x_2]$ be a path from $y_1 \in Q_3(x,x_1)$ to $x_2 \in Q_3(x_1,y)$ which is internally vertex disjoint from Q_3 with $|V(Q_3[x,x_2])|$ maximum. Suppose we have constructed $R_i[y_{i-1},x_i]$ for $i=1,2,\cdots,m$ with $y_{i-1} \in Q_3(x,x_{i-1})$ and $x_i \in Q_3(x_{i-1},y)$ and $|Q_3[x,x_i]|$ maximum, where $y_0=x$. If $Q_3(x,x_m)$ and $IQ_1 \cup IQ_2$ are in the same component of $G-V(Q_3[x_m,y]) \cup \{x\}$, we stop. Otherwise by Case 1 there is a path $R_{m+1}[y_m,x_{m+1}]$ from $y_m \in Q_3(x,x_m)$ to $x_{m+1} \in Q_3(x_m,y]$. Because G is finite and 3$-$connected, this process must stop. So we obtain a set of vertex disjoint paths.

$$R_1[x,x_1],\ R_2[y_1,x_2],\ R_3[y_2,x_3],\ \cdots, R_k[y_{k-1},x_k],\ S[y_k,v]$$

satisfying:

1. The endvertices are in the order $x,y_1,x_1,y_2,x_2,y_3,x_3,\cdots,y_k,x_k$ along the path Q_3 from x to y;
2. Vertex $v \in V(IQ_1 \cup IQ_2)$, without loss of generality say $v \in V(IQ_1)$.
3. All vertices on these paths are not in $V(Q_1 \cup Q_2)$ except the vertices x and v.

We now attempt to revise Q_3 to Q_3^* such that there is a path $R[x,v]$ from x to $v \in IQ_1 \cup IQ_2$ internally disjoint from Q_1, Q_2, Q_3^*. By our construction, $Q_3[x,x_k] \cup R_1[x,x_1] \cup \cdots \cup R_k[y_k,x_k]$ contains two vertex disjoint (except x) paths $X_1[x,y_k]$ and $X_2[x,x_k]$. Let $Q_3^*=X_2[x_1,x_k]Q_3[x_k,y]$. It is easy to see that for Q_1, Q_2, Q_3^*, the deletion of any one path does not separate the internal parts of the other two. Hence, these paths satisfy the properties of Q_1, Q_2, and Q_3 from the proceeding paragraph and there is a path $R[x,v]$ with $v \in V(IQ_1 \cup IQ_2)$ satisfying $V(IR) \cap V(IQ_1 \cup IQ_2 \cup IQ_3) = \emptyset$, which contradicts Subclaim 2.4.3. ∎

Let S_0 be a smallest cut separating x and y. Clearly, $|S_0| \geq 3$. By Menger's theorem, there are $k = |S_0|$ internally vertex disjoint $x-y$ paths P_1, P_2, $\cdots, P_k$. Let $S_0 \cap V(P_i) = \{v_i\}$ for each $i = 1, 2, \cdots, k$.

For each P_i, let R_i be the subgraph induced by $V(P_i)$ and the components of $G - V(IP_i)$ which do not contain either x or y. Applying Lemma 1.1 on each R_i, we obtain k internally vertex disjoint $x-y$ paths Q_1, Q_2, $\ldots$, Q_k such that each component of $G - V(\cup_{i=1}^{k} Q_i)$ is adjacent to at least two of IQ_1, IQ_2, $\ldots$, IQ_k. For convenience, we still denote these paths by P_1, P_2, $\ldots$, P_k.

Define a graph $\mathcal{G}$ on $\{1, 2, \ldots, k\}$ by joining i and j if and only if there exists a path $Q[v, w]$ with $v \in V(IP_i)$ and $w \in V(IP_j)$ and $V(IQ) \cap V(\cup_{i=1}^{k} P_i) = \emptyset$. From the fact that $G - \{x, y\}$ is connected, it follows that $\mathcal{G}$ is connected. On the other hand, from the assumption that there does not exist a set of three unified paths, it follows that $\mathcal{G}$ is a forest with maximum degree at most 2. Consequently, $\mathcal{G}$ is a path. Relabeling if necessary, we assume that there does not exist a path $Q[w_i, w_j]$ with $w_i \in V(IP_i)$ and $w_j \in V(IP_j)$ and $V(IQ) \cap V(\cup_{i=1}^{k} P_i) = \emptyset$. In addition, each component C of $G - V(\cup_{i=1}^{k} P_i)$ is adjacent to exactly two consecutive paths P_j and P_{j+1} for some $j = 1, 2, \ldots, k-1$. Thus, $V(G - V(\cup_{i=1}^{k} P_i))$ can be partitioned into

$$V_{1,2},\ V_{2,3},\ \cdots, V_{k-1,k}$$

such that each component of $V_{i,i+1}$ is adjacent to vertices of $V(P_i)$ and $V(P_{i+1})$ in $G - \{x, y\}$. Note that some $V_{i,i+1}$ may be empty. Under the above conditions, we assume that $\sum |V(P_i)|$ is minimum. Then, by the minimality of $\sum |V(P_i)|$, the following holds.

Claim 2.5. *Let $v \in V(IP_i)$ such that $N(v) \cap V(IP_{i-1} \cup IP_{i+1}) \neq \emptyset$. Then there is no edge in G with one end on $P_i(x, v)$ and the other end on $P_i(v, y)$.*

We note that $G - S_0$ consists of exactly two components, since otherwise there is a component which contains neither x nor y and would thus have to be adjacent to at least three of the vertices of S_0, say v_i, v_j, v_k, $i < j < k$, a contradiction. Let H_x be the component of $G - S_0$ containing x and H_y be the one containing y.

Claim 2.6. *Either $|V(H_x)| = 1$ or $|V(H_y)| = 1$ holds.*

Proof. Suppose to the contrary, $|V(H_x)| \geq 2$ and $|V(H_y)| \geq 2$. Let G_x be the graph obtained from G by contracting H_y to a vertex $y^*$and adding the edge v_iv_{i+1} if both $v_iv_{i+1} \notin E(G)$ and there is a path connecting $P_i[v_i,y)$ and $P_{i+1}(v_{i+1},y)$ or $P_i(v_i,y)$ and $P_{i+1}[v_{i+1},y)$ which is internally disjoint from $P_i \cup P_{i+1}$. We define G_y in the same manner.

Clearly, G_x does not contain three unified $x-y^*$ paths. By our induction hypothesis, G_x is a subgraph of an $\{x,u,y^*,v\}$-web with rib G_x^0. We claim that G_x is planar.

Otherwise, by the definitions of web and rib, G_x has a cut A of three vertices from the rib G_x^0 such that the subgraph H induced by the components of G_x-A with $V(H)\cap\{x,y^*,u,v\}=\emptyset$, has at least two vertices. Furthermore, $A \not\supseteq \{x,y^*\}$. If $y^* \notin A$, then A is a cut of G, a contradiction to Claim 2.4. Thus, $y^* \in A$. Since G is 3-connected and the neighborhood of y^* in G_x is S_0, then $S_0\cap V(H)\neq\emptyset$. If $|S_0\cap V(H)|=1$, then, $(A-\{y^*\})\cup(S_0\cap V(H))$ is a cut of G with three vertices, again a contradiction. Assume that v_i, $v_j \in V(H)$ where $1\leq i<j\leq k$. Since $x\notin V(H)$, then $A=\{y^*,u_i,u_j\}$ where $u_i \in P_i(x,v_i)$ and $u_j \in P_j(x,v_j)$. If $i>1$, $v_{i-1}v_i$ is not present in G_x. There does not exist a path from $P_{i-1}[v_{i-1},y)$ to $P_i[v_i,y)$ which is internally vertex disjoint from $P_{i-1}\cup P_i$ in G. Similarly from above, if $j\geq i+2$, there does not exist a path from $P_i[v_i,y)$ to $P_{i+1}[v_{i+1},y)$ internally disjoint from $P_i \cup P_{i+1}$ and there does not exist a path from $P_{j-1}[v_{j-1},y)$ to $P_j[v_j,y)$ internally disjoint from $P_{j-1}\cup P_j$. If $j<k$, there does not exist a path from $P_j[v_j,y)$ to $P_{j+1}[v_{j+1},y)$ which is internally vertex disjoint from $P_j \cup P_{j+1}$ in G. Thus, $\{y,\ u_i,\ u_j\}$ is a cut of G, a contradiction, and G_x is therefore planar.

Since G is 3-connected, $G[IP_i\cup IP_{i+1}\cup V_{i,i+1}]$ is connected for each $i=1$, $2, \cdots, k-1$. Thus, G_x has a path R_i from $s_i \in P_i(x,v_i]$ to $s_{i+1} \in P_{i+1}(x,v_{i+1}]$ such that $V(R_i(s_i,s_{i+1})) \subseteq V_{i,i+1}$. Hence, for each plane embedding of G_x, x, y^*, v_1, v_k are cofacial and v_i, v_{i+1} are cofacial for each $i=1, 2, \ldots, k-1$.

Similarly, we can show that G_y is a planar graph and for each plane embedding of G_y, x^*, y, v_1, v_k are cofacial and v_i, v_{i+1} are cofacial for each $i=1, 2, \ldots, k-1$. Then, G is a plane graph and has a plane embedding such that the unbounded face is $P_1[x,y]P_k^-[y,x]$. By the maximality of G, we see that G is a maximal planar graph, and hence, G is a web. Thus, either H_x or H_y only contains one vertex, completing Claim 2.6. ∎

Without loss of generality, we assume that $H_y=\{y\}$. Two vertex disjoint paths $Q[a,b]$ and $R[c,d]$ with a and $c\in V(P_1(x,y))$ and b and $d\in V(P_2[x,y))$ are called a pair of *crossing paths between* P_1 and P_2 if $a\in V(P_1(x,c))$ and $d \in V(P_2[x,b))$ and all internal vertices are in $V_{1,2}$. Similarly, two vertex disjoint paths $Q[a,b]$ and $R[c,d]$ are called a pair of *crossing paths between*

P_{k-1} and P_k if $a \in V(P_{k-1}[x,c))$ and $d \in V(P_k(x,b))$ and all internal vertices are in $V_{k-1,k}$.

Claim 2.7. *There do not exist a pair of crossing paths between P_1 and P_2 and there do not exist a pair of crossing paths between P_{k-1} and P_k.*

Proof. Suppose, to the contrary, that there are a pair of crossing paths $Q[a,b]$ and $R[c,d]$ between P_1 and P_2 such that $a \in V(P_1(x,c))$ and $d \in V(P_2(x,b))$.

We first show that we can pick two crossing paths $Q[a,b]$ and $R[c,d]$ such that

$$N(P_2(x,b)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset.$$

Suppose, to the contrary, there do not exist two such crossing paths. We assume that $|P_1[x,a]| + |P_2[x,d]|$ is minimum with the crossing property described above. The following statements hold:

$$N(P_2(d,b)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset, \text{ and either}$$

$$N(P_2(x,d]) \cap (V_{2,3} \cup V(IP_3)) = \emptyset \text{ or } N(P_2[b,y)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset.$$

Otherwise, $P_1[x,a]Q[a,b]P_2[b,y]$, $P_2[x,d]R^-[d,c]P_1[c,y]$, P_3 are unified $x-y$ paths, a contradiction. Thus,

$$N(P_2(b,y)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset.$$

By Claim 2.5, there is no edge with one end on $P_1[x,a)$ and the other end on $P_1(a,y]$ and there is no edge with one end on $P_2[x,d)$ and the other end on $P_2(d,y]$. Since $\{a,d,y\}$ is not a cut of G, there is a path $L[w,w']$ with $w \in P_1[x,a) \cup P_2[x,d)$ and $w' \in P_1(a,y) \cup P_2(d,y)$.

Without loss of generality, we assume that $w \in P_2[x,d)$. If $IL \subseteq V_{2,3}$, we get a contradiction to the assumption that $N(P_2(d,y)) \cap V_{2,3} = \emptyset$. Thus, $IL \subseteq V_{1,2}$. Assume first that L does not intersect $IQ \cup IR$. By minimality of $|P_1[x,a]| + |P_2[x,d]|$, $w' \in P_2(d,y)$. Since every component of $V_{1,2}$ is adjacent to both IP_1 and IP_2, there is a path $L'[z'z]$ with $z' \in IL$ and $z \in IP_1 \cup IQ \cup IR$. By the the minimality of $|P_1[x,a]| + |P_2[x,d]|$, $z \in Q[a,b)$. Hence $L[w,z']L'$ joins $P_2[x,d)$ and $Q[a,b)$. Assume now that L intersects $IQ \cup IR$. Let z be the first vertex on L that belongs to $IQ \cup IR$. Then, $z \in IQ$ by the minimality of $|P_1[x,a]| + |P_2[x,d]|$. Thus in either case, there is a path $S[w,z]$ with $w \in P_2[x,d)$ and $z \in Q[a,b)$. We choose S such that $|P_2[x,w]|$ is minimum. By Claim 2.5, there is no edge with one endvertex in $P_2[x,w)$ and the other one in $P_2(w,y]$. Since P_1, $P_2[x,w]S[w,z]Q^-[z,b]P_2[b,y], P_3$ do not form a set of three unified paths, $N(P_2(w,d)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset$. From the minimality of both $|P_2[x,w]|$ and $|P_1[x,a]| + |P_2[x,d]|$ and the fact that every component of $V_{1,2}$ is adjacent to both IP_1 and IP_2, there is no path from $P_1(a,y) \cup P_2(w,y)$ to $P_1[x,a) \cup P_2[x,w)$ with all internal vertices in $V_{1,2}$. Recall that $N(P_2(w,y)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset$. Thus, we see that $\{y,a,w\}$ is a cut of G, a contradiction.

We now pick two crossing paths $Q[a,b]$ and $R[c,d]$ between P_1 and P_2 such that $a \in V(P_1(x,c))$, $d \in V(P_2(x,b))$, and

$$N(P_2(x,b)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset.$$

Further, $|P_1[c,y]|+|P_2[b,y]|$ is minimum with respect to the above properties. In the same manner as the argument above, we can show that $N(P_2[b,y)) \cap (V_{2,3} \cup V(IP_3)) = \emptyset$. Thus, $N(IP_2) \cap (V_{2,3} \cup V(IP_3)) = \emptyset$, which implies that G is 2-connected, a contradiction. ∎

Claim 2.8. *$P_1[x,y] = xv_1y$ and $P_2[x,y] = xv_2y$.*

Proof. Suppose, to the contrary, $|V(P_1[x,y])| \geq 4$. Let u_1 be the predecessor of v_1 along path $P_1[x,y]$. Contracting the edge v_1y to a new vertex y^*, we obtain a new graph G^*. Since G does not contain three unified $x-y$ paths, G^* does not contain three unified $x-y^*$ paths. By the induction hypothesis, G^* is a spanning subgraph of an (x,u,y^*,v)-web G^{**} with rib G_0^* say. If $G^{**} - V(G_0^*)$ has a component K^A such that $|V(K^A)| \geq 2$, let $A = N(K^A) \cap V(G_0^*)$. By the definitions of web and rib, we see that $|A| = 3$ and $|A \cap \{x,y^*\}| \leq 1$. Uncontracting y^*, we obtain a cut B of G from A. Since G does not contain three vertices separating $V(K^A)$ from $\{x,y\}$, then $|B| = 4$ and y, $v_1 \in B$. Furthermore, if $V(K^A) \cap S_0 = \emptyset$, then $B - \{y\}$ is a cut set of G, a contradiction. Thus, $V(K^A) \cap S_0 \neq \emptyset$.

If $v_i \in V(K^A)$ then there is a $u_i \in V(P_i(x,v_i))$ such that $u_i \in A$. On the other hand, if there are two distinct vertices v_i, $v_j \in V(K^A)$, then $S_0 \cup \{u_i,u_j\} - \{v_i,v_j\}$ is cut separating x and y. However, $S_0 \cup \{u_i,u_j\} - \{v_i,v_j\}$ is neither a neighborhood of x nor a neighborhood y, a contradiction to Claim 2.6. Thus, $|V(K^A) \cap S_0| = 1$. Furthermore, $V(K^A) \cap S_0 = \{v_2\}$ since $B - \{v_1\}$ is not a cut. In particular, we see that in $G^{**} - G_0^*$, K^A is the unique component which contains at least two vertices. Thus, the resulting graph G^* is planar if we contract $V(K^A)$ to a vertex. Let $B = \{v_1,u_2,w,y\}$, where $u_2 \in V(P_2(x,y))$.

Subclaim 2.8.1. If w is adjacent to some vertices in $V_{2,3} \cup V(IP_3)$, then w is adjacent to y, which implies that $w = v_3$ and G contains a cycle $v_1ywu_2v_1$.

Proof. Let H' be the subgraph of G induced by $B \cup V(K^A)$. Since K^A has at least two vertices, there is no vertex z in H' which separates $\{v_1,u_2\}$ from $\{y,w\}$, for if this were the case, then either $\{v_1,u_2,z\}$ or $\{y,w,z\}$ would separate G, a contradiction. So by Menger's theorem, H' contains paths R_1, R_2 connecting $\{v_1,u_2\}$ and $\{y,w\}$. If R_1 is a v_1-w path and R_2 is a u_2-y path, then P_1, $P_2[x,u]R_2[u_2,y]$, and P_3 are three unified walks, a contradiction by Lemma 1.3. Thus, R_1 is a v_1-y path while R_2 is a u_2-w path. We claim that the edge $e=u_2w$ is present in G. For otherwise we add this edge to G and obtain a set of three unified $x-y$ paths, L_1, L_2, L_3. If one of them, say L_1, contains u_2w, we replace the edge by R_2 to obtain a walk L_1^* from x to y. It is then readily seen that L_1^*, L_2, L_3 are unified walks in G, again a contradiction by Lemma 1.3. Thus, without loss of generality, we assume that e is a bridge connecting L_1 and L_2 in $G-V(L_3)$. In this case, we can assume that $V(L_1\cup L_2\cup L_3)\cap V(K^A)=\emptyset$. Thus, L_1, L_2, L_3 are unified paths in G, a contradiction. By considering the sets $\{v_1,y\}$ and $\{u_2,w\}$ instead of $\{v_1,u_2\}$ and $\{y,w\}$, respectively, we conclude as above that v_1u_2 and yw are also present. Hence, $G[B]=v_1ywu_2v_1$. In particular, we have shown $w=v_3$, completing Subclaim 2.8.1. ∎

Let G' denote the graph obtained by contracting K^A into a vertex z_0. It is easy to see that G' does not contain a set of three unified $x-y$ paths. By the induction hypothesis, G' is a spanning subgraph of an (x,u,y,v)-web G'' with rib say G_0'. For each vertex $z\in V(B)\cup\{z_0\}$, we note that G' contains at least four internally vertex disjoint $z-\{x,y\}$ paths. Since x and y are not on a triangle of G'', $B\cup\{z_0\}\subseteq V(G_0')$. Since two consecutive vertices of the cycle $v_1ywu_2y_1$ do not separate G, each of the four 3 -cycles of G_0' containing z_0 are facial cycles of G_0' and each complete graph of G'' attached to these 3-cycles empty. Also, by the connectivity properties of G, each complete graph of G' attached to any other 3-cycle of G_0' is empty. So it follows that $G-V(H)$ has a plane embedding such that x and y lie on the boundary of the unbounded face and $G[B]=v_1ywu_2v_1$ is a facial cycle. By the maximality of G all other facial cycles are 3-cycles.

Now $H'=G[V(K^A)\cup B]$ has no (u_2,v_1,y,w)-linkage and is therefore contained in a (u_2,v_1,y,w)-web H''. By the connectivity property of G it follows that H'' has no separating 3-cycle. Then, H'' is planar and $G[B]$ is the facial cycle of the unbounded face. It follows that G is planar and has a plane embedding such that x and y are on the boundary of the unbounded face. By maximality we see that G is an $(x,u,y,v)-$web for some u and v. Thus, we can assume that $w\in V(IP_1\cup IP_2)\cup V_{1,2}$ and $N(w)\cap(V_{2,3}\cup V(IP_3))=\emptyset$.

If $w\in V(P_1(x,v_1))$, then since $\{w,u_2,y\}$ is not a cut of G, there is a path $R[v_1^*,u_2^*]$ from $v_1^*\in V(P_2(w,v_1])$ to $u_2^*\in V(P_1[x,w)\cup P_2[x,u_2))$. If $u_2\in V(P_1[x,w))$, by the minimality of $\sum_{i=1}^k|V(P_i)|$, we see that $R[v_1^*,u_2^*]$ is not an edge. Since every component of $V_{1,2}$ is adjacent to both paths IP_1 and IP_2, we can choose $u_2^*\in V(P_2[x,u_2))$. Since $B=\{y,v_1,w,u_2\}$ is a cut of

G, $R[v_1^*,u_2^*]\cap V(K^A)=\emptyset$ holds. Contracting K^A to one vertex, we obtain a subdivision $K_{3,3}$ with vertex set $\{x,v_1^*,v_2\}\cup\{y,w,u_2^*\}$, which contradicts the fact that if we contract K^A to a vertex in G^* the resulting graph is a planar graph. Thus, $w\notin V(P_1)$.

By the minimality of $\sum_{i=1}^{k}|V(P_i)|$, if $w\in V(P_2(u_2,v_2))$, then $N(w)\cap V(P_2[x,u_2))=\emptyset$. Since $N(w)\cap(V_{2,3}\cup V(IP_3))=\emptyset$ and $B-\{w\}$ is not a cut of G, there is a path $R_1[w,u_1^*]$ with $u_1^*\in V(P_1(x,v_1))$ and such that all vertices of $R_1[w,u_1^*]$ (except u_1^*) are in $V_{1,2}$. We pick the path $R_1[w,u_1^*]$ such that $|V(P_1[x,u_1^*])|$ is minimum.

If there is a path $R_2[w,u_2^*]$ with $u_2^*\in P_2(x,u_2)$, let u_2^* such that $|P_2[x,u_2^*]|$ is minimum. Otherwise, let $u_2^*=u_2$. If $N(P_2(u_2^*,y))\cap(V_{2,3}\cup V(IP_3))\neq\emptyset$, then $G[B\cup V(K^A)]$ does not have two vertex disjoint paths $X_1[w,y]$ and $X_2[v_1,u_2]$. Otherwise, P_1, $P_2[x,u_2^*]R_2^-[u_2^*,w]X_1[w,y]$, P_3 form a set of three unified $x-y$ paths, a contradiction. In the same manner as before, we see that $G[B]$ forms a 4-cycle wv_1yu_2w. In particular, the edge yu_2 is present in G, a contradiction. Hence, $N(P_2(u_2^*,y))\cap(V_{2,3}\cup V(IP_3))=\emptyset$.

Since $\{u_1^*,u_2^*,y\}$ is not a cut of G, there is a path $S[z_1,z_2]$ such that either $z_1\in P_1[x,u_1^*)$ and $z_2\in P_2(u_2^*,u_2)$ or $z_1\in P_1(u_1^*,v_1)$ and $z_2\in P_2[x,u_2^*)$. In either case, we obtain a pair of crossing paths between P_1 and P_2, which contradicts Claim 2.7 and completing the proof of Claim 2.8. ∎

Since $P_1=xv_1y$ and $P_2=xv_ky$, G does not have an (x,v_1,y,v_2)-linkage. Otherwise, suppose that there are two vertex disjoint paths $Q[x,y]$ and $R[v_1,v_k]$. It is not difficult to see that P_1, $Q[x,y]$, and P_3 are three unified $x-y$ paths, a contradiction. Thus, G is contained in an (x,v_1,y,v_2)-web by Thomassen's theorem. Since G does not contain a three cut separating some vertices from $\{x,y\}$, then G is a planar graph. By maximality, G is a maximum planar graph so G is an (x,v_1,y,v_2)-web, a contradiction, completing the proof. ∎

By Lemma 1.1 and the fact $\alpha(2)=3$, we obtain the following result.

Corollary 7. *Let x, y be two distinct vertices of a 3-connected graph G of order $n\geq 4$. If G does not contain three internally vertex disjoint nonseparating $x-y$ paths and adding any edge to G results in a graph which has three internally vertex disjoint nonseparating $x-y$ paths, then there are another two vertices u and v such that G is an (x,u,y,v) -web. Conversely, any (x,u,y,v)-web in which x and y are not adjacent is maximal with respect to the property of not contain three internal vertex disjoint $x-y$ paths.*

3. The case $k=2$ of Lovász's conjecture

The following result will be proved in this section.

Theorem 8. *If G is a 5-connected graph, then for every pair of vertices u and v there exists a u–v path $P[u,v]$ such that $G-V(P[u,v])$ is 2-connected.*

Proof. Suppose, to the contrary, that $G-V(P[u,v])$ is not 2-connected for all u-v paths $P[u,v]$. Let $P[u,v]$ be a $u-v$ path, $H=G-V(P[u,v])$, and B be a block of H with the maximum number of vertices.

Let C_1, C_2, … , C_m be the components of $H-V(B)$. Without loss of generality, we assume that

$$|V(C_1)| \geq |V(C_2)| \geq \ldots \geq |V(C_m)|.$$

To reach a contradiction, we suppose that $P[u,v]$ satisfies the following properties.

1. The number of vertices in the block B is maximum among all possible $u-v$ paths.
2. The number of vertices in each of the components C_1, C_2, … , C_m is as large as possible with the larger order components having priority, that is, we assume that $|V(C_1)|$ is as large as possible, then, under this constraint, $|V(C_2)|$ is as large as possible, …, $|V(C_m)|$ is as large as possible if all the above constraints are satisfied.
3. Under both of the above constraints, we assume that $|V(P[u,v])|$ is as small as possible.

By property 3, we also note that $P[u,v]$ is an induced path. Since B is a block of H_1, we have $|N(C_m)\cap V(B)|\leq 1$. Let w be the neighbor of C_m in B if $N(C_m)\cap V(B)\neq\emptyset$. Since G is 5-connected, we see that $|N(C_m)\cap V(P[u,v])|\geq 4$. Let x be the first vertex of $N(C_m)$ on $P[u,v]$ along the order of $P[u,v]$ from u to v and let y be the last vertex of $N(C_m)$ on $P[u,v]$ along the order of $P[u,v]$ from u to v.

Claim 3.1. *There do not exist two independent edges such that each has one endvertex on $P(x,y)$ and the other endvertex in $V(H-\{w\}\cup V(C_m))$.*

Proof. Suppose, to the contrary, x_1y_1 and x_2y_2 are two vertex disjoint paths with x_1, $x_2\in V(P(x,y))$ and y_1, $y_2\in V((H)-\{w\}\cup V(C_m))$. Without loss of generality, we assume that x_1 and x_2 occur in that order from x to y along the subpath of $P(x,y)$. Let xC_my be a path connecting vertices x and y with all its internal vertices in C_m and $Q[u,v]=P[u,x]xC_myP[y,v]$. Let $H^*=G-V(Q[u,v])$.

If $N(V(P(x,y)) \cap (\cup_{i=1}^{m-1} V(C_i)) = \emptyset$, then y_1, $y_2 \in V(B)$. $G[V(B) \cup V(P[x_1,x_2])]$ is a 2-connected subgraph, which contradicts the maximality of $|V(B)|$. Thus, $\{y_1,y_2\} \cap (\cup_{i=1}^{m-1} V(C_i)) \neq \emptyset$. Since $H - V(C_m) \subseteq H^*$, B is a 2-connected subgraph of H^* and C_i is connected in H^* for each $i = 1, 2, \ldots, m-1$. Then, either H^* has a block larger than B or there is an i such that $H^* - V(B)$ contains a component larger than C_i and $C_1, C_2, \ldots, C_{i-1}$ are components of $H^* - V(B)$, a contradiction. ∎

By the above Claim, we see that there is a vertex z such that all edges with one endvertex in $P(x,y)$ and the other one in $V(H) - \{w\} \cup V(C_m)$ must contain z as an endvertex. Since $P[u,v]$ is an induced path, $\{x, y, w, z\}$ is a cut set which separates C_m and B, which contradicts the fact that G is 5-connected. ∎

We proved that $\beta(2) \leq 5$. The complete bipartite graph $K_{3,n}$ shows that $\beta(2) > 3$. Kawarabayashi [5] recently constructed some examples showing $\beta(2) \neq 4$. Thus, $\beta(2) = 5$.

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