

# Degree Conditions for $k$ -Ordered Hamiltonian Graphs

---

**Ralph J. Faudree,<sup>1</sup> Ronald J. Gould,<sup>2</sup>  
Alexandr V. Kostochka,<sup>3,4</sup> Linda Lesniak,<sup>5\*</sup>  
Ingo Schiermeyer,<sup>6</sup> and Akira Saito<sup>7</sup>**

<sup>1</sup>DEPARTMENT OF MATHEMATICAL SCIENCES  
UNIVERSITY OF MEMPHIS, MEMPHIS, TENNESSEE 38152

<sup>2</sup>DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE  
EMORY UNIVERSITY, ATLANTA, GEORGIA 30322

<sup>3</sup>DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS  
AT URBANA-CHAMPAIGN, URBANA, IL 61801

<sup>4</sup>INSTITUTE OF MATHEMATICS  
630090 NOVOSIBIRSK, RUSSIA

<sup>5</sup>DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE  
DREW UNIVERSITY, MADISON, NEW JERSEY 07940

<sup>6</sup>DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE  
FREIBERG UNIVERSITY OF MINING AND TECHNOLOGY  
D-09596 FREIBERG, GERMANY

<sup>7</sup>DEPARTMENT OF APPLIED MATHEMATICS  
NIHON UNIVERSITY, TOKYO 156, JAPAN

Received August 11, 2000; Revised August 21, 2002

DOI 10.1002/jgt.10084

---

Contract grant sponsor: Russian Foundation for Fundamental Research; Contract grant number: 99-01-00581; Contract grant sponsor: Dutch-Russian Cooperative Grant.

\*Correspondence to: Linda Lesniak, Department of Mathematics and Computer Science, Drew University, Madison, NJ 07940. E-mail: llesniak@drew.edu

© 2003 Wiley Periodicals, Inc.

**Abstract:** For a positive integer  $k$ , a graph  $G$  is *k-ordered hamiltonian* if for every ordered sequence of  $k$  vertices there is a hamiltonian cycle that encounters the vertices of the sequence in the given order. It is shown that if  $G$  is a graph of order  $n$  with  $3 \leq k \leq n/2$ , and  $\deg(u) + \deg(v) \geq n + (3k - 9)/2$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is *k-ordered hamiltonian*. Minimum degree conditions are also given for *k-ordered hamiltonicity*. © 2003 Wiley Periodicals, Inc. *J Graph Theory* 42: 199–210, 2003

Keywords: *hamiltonian cycle*

## 1. INTRODUCTION

One of the most widely studied classes of graphs are hamiltonian graphs, that is, graphs that possess spanning cycles. In this article, we consider a special family of hamiltonian graphs known as *k-ordered hamiltonian graphs*. A graph is *k-ordered hamiltonian* if for every ordered sequence of  $k$  vertices there is a hamiltonian cycle that encounters the vertices of the sequence in the given order. This concept was introduced by Chartrand. Clearly, every hamiltonian graph is 3-ordered hamiltonian. Ng and Schultz [5] showed the following.

**Theorem 1.1.** *Let  $G$  be a graph of order  $n$  and let  $k$  be an integer with  $3 \leq k \leq n$ . If  $\deg(u) + \deg(v) \geq n + 2k - 6$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is *k-ordered hamiltonian*.*

**Corollary 1.1.** *Let  $G$  be a graph of order  $n$  and let  $k$  be an integer with  $3 \leq k \leq n$ . If  $\deg(u) \geq n/2 + k - 3$  for every vertex  $u$  of  $G$ , then  $G$  is *k-ordered hamiltonian*.*

Corollary 1.1 is an analog of the well-known theorem of Dirac [1] that says that every graph of order  $n \geq 3$  with minimum degree at least  $n/2$  is hamiltonian, and Theorem 1.1 is an analog of Ore's theorem [6] that says that every graph of order  $n \geq 3$  such that  $\deg(u) + \deg(v) \geq n$  for every pair  $u, v$  of nonadjacent vertices is hamiltonian. We note that the restriction  $n$  in the statement of Ore's theorem is simply twice the restriction  $n/2$  in the statement of Dirac's theorem. The same holds for Corollary 1.1 and Theorem 1.1.

Both bounds for *k-hamiltonicity* were improved for small  $k$  with respect to  $n$ . The first result was improved by Faudree et al. [2].

**Theorem 1.2.** *Let  $k \geq 3$  be an integer and let  $G$  be a graph of order  $n \geq 53k^2$ . If  $\deg(u) + \deg(v) \geq n + (3k - 9)/2$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is *k-ordered hamiltonian*.*

The second result was improved by Kierstead et al. [3] as follows.

**Theorem 1.3.** *Let  $k \geq 2$  be an integer and let  $G$  be a graph of order  $n \geq 11k - 3$ . If  $\deg(u) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$  for every vertex  $u$  of  $G$ , then  $G$  is *k-ordered hamiltonian*.*

We note that both of these bounds are sharp for the respective values of  $k$ . Thus, a bit unexpectedly, for small  $k$ , the Dirac type bound does not follow from the Ore type bound.

Our main result says that the bound of Theorem 1.2 holds for every  $k$ .

**Theorem 1.4.** *Let  $k$  be an integer with  $3 \leq k \leq n/2$  and let  $G$  be a graph of order  $n$ . If  $\deg(u) + \deg(v) \geq n + (3k - 9)/2$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is  $k$ -ordered hamiltonian.*

The bound in this theorem is sharp. Moreover, for large  $k$  it implies the bound of the Dirac type. Thus,

- (a) for large  $k$ , the Ore type bound yields the Dirac type bound;
- (b) for small  $k$ , the Ore type bound is more than twice the Dirac type bound;
- and
- (c) for moderate  $k$ , the situation is not clear.

## 2. DEGREE SUM CONDITIONS

The following concept will be useful in establishing the primary result of this section. For a positive integer  $k$ , a graph  $G$  is  $k$ -ordered if for every sequence of  $k$  vertices there is a cycle that encounters the vertices of the sequence in the given order. The following lemma gives a condition under which a  $k$ -ordered graph is  $k$ -ordered hamiltonian. The proof uses the following notations. If  $C$  is a cycle in a graph with an understood orientation, and  $S$  is a set of vertices of  $C$ , then  $S^+$  and  $S^-$  denote the successors and predecessors of the vertices in  $S$  on  $C$ , respectively. If  $S = \{x\}$ , we simply write  $x^+$  and  $x^-$ . Also, if  $x$  and  $y$  are vertices of  $C$ , then  $x\vec{C}y$  denotes the path from  $x$  to  $y$  along  $C$  in the designated direction or, when appropriate, the vertex set of this path. The notation  $x\overleftarrow{C}y$  denotes the path from  $x$  to  $y$  in the opposite direction. Similar notation is used in the case of paths.

**Lemma 2.1.** *If  $G$  is a  $k$ -ordered,  $(k + 1)$ -connected graph of order  $n \geq 3$  such that  $\deg(u) + \deg(v) \geq n$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is  $k$ -ordered hamiltonian.*

**Proof.** Let  $x_1, x_2, \dots, x_k$  be an ordered sequence of  $k$  vertices of  $G$ . Since  $G$  is  $k$ -ordered, there is a cycle  $C$  that encounters these vertices in this order. Choose such a cycle  $C$  such that  $|V(C)|$  is as large as possible. Assume  $V(C) \neq V(G)$  and let  $H$  be a component of  $G - V(C)$ . Since  $G$  is  $(k + 1)$ -connected,  $|N_C(H)| \geq k + 1$  and hence  $|N_C(H) \cap x_i\vec{C}x_{i+1}| \geq 2$  for some  $i, 1 \leq i \leq k$ , where we consider  $x_{k+1} = x_1$ . We may assume  $|N_C(H) \cap x_k\vec{C}x_1| \geq 2$ . Choose a pair of distinct vertices  $y_1, y_2$  in  $N_C(H) \cap x_k\vec{C}x_1$  so that  $x_k, y_1, y_2$ , and  $x_1$  appear in this order along  $C$ , and so that  $y_1\vec{C}y_2$  is as short as possible. Possibly  $x_k = y_1$  or  $y_2 = x_1$ . Let  $z_i \in N_H(y_i)$  ( $i = 1, 2$ ). Note that possibly  $z_1 = z_2$ . Since  $H$  is connected, there exists a path  $P$  from  $z_1$  to  $z_2$  in  $H$ . Then  $C' = y_2\vec{C}y_1z_1\vec{P}z_2y_2$  is a cycle which

encounters  $x_1, x_2, \dots, x_k$  in this order. If  $y_2 = y_1^+$ , then  $|V(C')| > |V(C)|$ , which contradicts the maximality of  $|V(C)|$ . Therefore,  $y_2 \neq y_1^+$ . Note that possibly  $y_1^+ = y_2^-$ . By the choice of  $y_1$  and  $y_2$ ,  $N_C(H) \cap y_1^+ \vec{C}y_2^- = \phi$ . In particular,  $y_1^+ z_1 \notin E(G)$ , and hence  $\deg_G(y_1^+) + \deg_G(z_1) \geq n$ .

Let  $A_1 = N_G(y_1^+) - V(C)$ ,  $A_2 = N_G(y_1^+) \cap y_2 \vec{C}y_1$ ,  $A_3 = N_G(y_1^+) \cap y_1^+ \vec{C}y_2^-$ ,  $B_1 = N_G(z_1) - V(C)$  and  $B_2 = N_G(z_1) \cap y_2 \vec{C}y_1$ . Then  $N_G(y_1^+)$  is the disjoint union of  $A_1, A_2$ , and  $A_3$ , and since  $N_G(z_1) \cap y_1^+ \vec{C}y_2^- = \phi$ ,  $N_G(z_1)$  is the disjoint union of  $B_1$  and  $B_2$ . If  $A_1 \cap B_1 \neq \phi$ , say  $v \in A_1 \cap B_1$ , then  $v \in V(H)$  since  $v \in N_G(z_1)$ . However, this implies  $y_1^+ \in N_C(H)$ , which contradicts the fact that  $N_C(H) \cap y_1^+ \vec{C}y_2^- = \phi$ . Thus, we have  $A_1 \cap B_1 = \phi$ . Since  $z_1 \notin A_1 \cup B_1$ , we have  $n - |V(C)| - 1 \geq |A_1 \cup B_1| = |A_1| + |B_1|$ . Furthermore, since  $y_1^+ \notin A_3$ , we have  $|A_3| \leq |y_1^+ \vec{C}y_2^-| - 1$ . Therefore, we have

$$\begin{aligned} n &\leq \deg_G(y_1^+) + \deg_G(z_1) = |A_1| + |A_2| + |A_3| + |B_1| + |B_2| \\ &\leq n - |V(C)| - 1 + |y_1^+ \vec{C}y_2^-| - 1 + |A_2| + |B_2|, \end{aligned}$$

or  $|A_2| + |B_2| \geq |y_2 \vec{C}y_1| + 2$ .

Assume  $|B_2| \geq \frac{1}{2}|y_2 \vec{C}y_1| + 1$ . Then  $\{v, v^+\} \subset N_G(z_1)$  for some  $v \in y_2 \vec{C}y_1^-$ , and  $v^+ \vec{C}vz_1v^+$  is a cycle which contains  $x_1, x_2, \dots, x_k$  in this order. However, this again contradicts the maximality of  $|V(C)|$ . Therefore,  $|B_2| \leq \frac{1}{2}(|y_2 \vec{C}y_1| + 1)$ . This implies

$$|A_2| = |N_G(y_1^+) \cap y_2 \vec{C}y_1| \geq \frac{1}{2}|y_2 \vec{C}y_1| + \frac{3}{2}.$$

Applying the same arguments to  $z_2$  and  $y_2^-$  instead of  $z_1$  and  $y_1^+$ , we have  $|N_G(y_2^-) \cap y_2 \vec{C}y_1| \geq \frac{1}{2}|y_2 \vec{C}y_1| + \frac{3}{2}$ . Let  $X = (N_G(y_1^+) \cap y_2^+ \vec{C}y_1)^-$  and  $Y = N_G(y_2^-) \cap y_2 \vec{C}y_1$ . Since

$$|X| = |X^+| = |N_G(y_1^+) \cap y_2^+ \vec{C}y_1| \geq \frac{1}{2}|y_2 \vec{C}y_1| + \frac{1}{2}$$

and  $X \cup Y \subset y_2 \vec{C}y_1$ ,

$$\begin{aligned} |X \cap Y| &= |X| + |Y| - |X \cup Y| \\ &\geq \frac{1}{2}|y_2 \vec{C}y_1| + \frac{1}{2} + \frac{1}{2}|y_2 \vec{C}y_1| + \frac{3}{2} - |y_2 \vec{C}y_1| = 2 \end{aligned}$$

and hence  $X \cap Y \neq \phi$ . Let  $v \in X \cap Y$ . Then  $v^+ \in N_G(y_1^+) \cap y_2^+ \vec{C}y_1$ . Let

$$C'' = y_1 z_1 \vec{P}z_2 y_2 \vec{C}v y_2^- \overleftarrow{C}y_1^+ v^+ \vec{C}y_1.$$

Then  $C''$  contains  $x_1, x_2, \dots, x_k$  in this order and  $|V(C'')| > |V(C)|$ . (Note that the arguments are valid even if  $y_2^- = y_1^+$ .) This contradicts the maximality of  $|V(C)|$ , and the lemma follows. ■

Lemma 2.1 can now be used to improve the result of Theorem 1.2. For convenience, we restate Theorem 1.5 here as Theorem 2.1.

**Theorem 2.1.** *Let  $k$  be an integer with  $3 \leq k \leq n/2$  and let  $G$  be a graph of order  $n$ . If  $\deg(u) + \deg(v) \geq n + (3k - 9)/2$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is  $k$ -ordered hamiltonian.*

**Proof.** By the assumptions of the theorem,  $G$  is hamiltonian, and hence,  $k$ -ordered hamiltonian for  $k \leq 3$ . Thus we may assume that  $k \geq 4$ . Furthermore, the assumed degree conditions imply that  $G$  is  $(k + 1)$ -connected unless  $k = 4$  or  $5$ . In these cases it is straightforward to check that  $G$  is  $k$ -ordered hamiltonian. Thus we may assume that  $G$  is  $(k + 1)$ -connected and, by the previous lemma, it suffices to show that  $G$  is  $k$ -ordered.

Let  $K = \{x_1, x_2, \dots, x_k\}$  be an ordered sequence of  $k$  vertices of  $G$ . We show that  $G$  contains a cycle that encounters these vertices in the given order. Let  $W$  be the set of indices  $i$  such that  $x_i x_{i+1} \in E(G)$  and  $w = |W|$ . A 1-improvement of size  $s$  of  $G$  is a set of vertices  $S = \{y_1, y_2, \dots, y_s\} \subset V(G) - K$  and a set of indices  $\{i_1, i_2, \dots, i_s\}$  such that for every  $j = 1, 2, \dots, s$ , we have that  $x_{i_j} x_{i_j+1} \notin E(G)$  and  $y_j$  is adjacent to both  $x_{i_j}$  and  $x_{i_j+1}$ . The indices  $i_1, i_2, \dots, i_s$  will be called  $S$ -indices. For  $i = 1, 2, \dots, k$ , let  $\delta(i) = 1$  if neither of the edges  $x_{i-1} x_i$  and  $x_i x_{i+1}$  is in  $E(G)$ , and otherwise,  $\delta(i) = 0$ .

**Claim 2.1.** *There is a 1-improvement of size  $s \geq 3k - n - 2w$ .*

Consider the auxiliary bipartite graph  $H$  with partite sets  $P$  and  $Q$ , where  $Q = V(G) - K$ , and

$$P = \{\{x_i, x_{i+1}\} : i \in \{1, 2, \dots, k\} \setminus W, \text{ where } k + 1 \equiv 1\},$$

and a vertex  $\{x_i, x_{i+1}\} \in P$  is adjacent to a vertex  $q \in Q$  if and only if  $q$  is a common neighbor of  $x_i$  and  $x_{i+1}$  in  $G$ . By the construction of  $H$ ,  $s$  is the size of a maximum matching in  $H$ . By Ore's theorem or, more generally, by Berge's theorem on maximum matchings [4], there exists  $T \subset P$  such that

$$|N_H(T)| = |T| - (k - s - w).$$

If  $s = k - w$ , we immediately get the desired conclusion for Theorem 2.1. Thus we may assume  $s < k - w$  which, in particular, implies  $T \neq \phi$ . Let  $\{x_i, x_{i+1}\} \in T$  such that  $\delta(i) + \delta(i + 1)$  is maximum. Let  $L = Q - N_H(T)$ . By definition,  $L = (L - N_G(x_i)) \cup (L - N_G(x_{i+1}))$ . We may assume that

$$|L - N_G(x_i)| \geq |L - N_G(x_{i+1})| \tag{1}$$

and that  $y \in L - N_G(x_i)$ .

By (1),

$$\deg_G(x_i) \leq n - 2 - \delta(i) - |L|/2$$

and  $\deg_G(y) \leq n - 1 - (|T| + 1 - \delta(i))/2$ . Since  $x_i y \notin E(G)$ , we conclude that

$$\begin{aligned} n + \frac{3k - 9}{2} &\leq \left( n - 2 - \delta(i) - \frac{|L|}{2} \right) + \left( n - 1 - \frac{|T| + 1 + \delta(i)}{2} \right) \\ &= 2n - \frac{|T| + 7 + \delta(i) + n - k - |N_H(T)|}{2} \\ &= 2n - \frac{|T| + 7 + \delta(i) + n - k - |T| + k - s - w}{2}. \end{aligned}$$

Hence,

$$n + \frac{3k - 9}{2} - 2n + \frac{7 + n - s - w + \delta(i)}{2} \leq 0. \tag{2}$$

**Case 1.** Suppose  $w + \delta(i) \geq 2$ . Then we are finished by (2).

**Case 2.** Suppose  $w = 0$ . Then  $\delta(j) = 1$  for every  $j$ . In particular,  $\delta(i) = 1$ . Thus, if the inequality in (2) is strict, then we are done. In order to have equality in (2), we must have

- (a)  $|L - N_G(x_i)| = |L|/2$ , so  $|L|$  is even and, in view of (1),  $L - N_G(x_i) = L \cap N_G(x_{i+1})$ ;
- (b)  $3k - 9$  is even (and hence  $k$  is odd);
- (c)  $\deg_G(y) = n - 1 - |T|/2$ , so that  $|T|$  is even;
- (d) every nonedge at  $y$  must “spoil” exactly two elements of  $T$  to have  $\deg_G y = n - 1 - |T|/2$ ; in particular, the pair  $(x_{i-1}, x_i)$  must be in  $T$ .

Because of (a), the roles of  $x_i$  and  $x_{i+1}$  are interchangeable, and hence  $(x_{i+1}, x_{i+2})$  must be in  $T$ . Applying the above reasoning to the pair  $(x_{i+1}, x_{i+2})$ , we get that  $(x_{i+2}, x_{i+3}) \in T$ , and so on. It follows that  $T = P$  and, since  $W = \emptyset$ ,  $|T| = k$ . Then by (c),  $k$  is even, a contradiction to (b).

**Case 3.** Suppose  $w = 1$  and  $\delta(i) = 0$ . This is possible only if  $i - 1$  is the unique index in  $W$ . Because of the choice of  $i$ , we get  $|T| \leq 2$ . Thus,  $|N_H(T)| \leq 2 - (k - s - w)$ . On the other hand, since  $x_i x_{i+1} \notin E(G)$ ,

$$\begin{aligned} |N_H(T)| &\geq |N_G(x_i) \cap N_G(x_{i+1}) - K| \\ &\geq n + \frac{3k - 9}{2} - (n - 2) - (k - 2) + \delta(i) + \delta(i + 1) = \frac{k + 1}{2}. \end{aligned}$$

Since  $k - s - w \geq 0$ , we get  $(k + 1)/2 \leq 2$ . This is possible only for  $k \leq 3$ , which is impossible since  $k \geq 4$ . This proves Claim 2.1.

Let  $(S, \{i_1, i_2, \dots, i_s\})$  be a 1-improvement of  $G$ . Construct the auxiliary bipartite graph  $F$  as follows. One partite set of  $F$  is  $M = V(G) - K - S$ . The other partite set of  $F$  is  $R$ . In order to define  $R$ , let  $I = \{1, 2, \dots, k\} \setminus \{i_1, i_2, \dots, i_s\} \setminus W$ . Then

$$R = \cup_I \{(i, x_i), (i, x_{i+1})\}.$$

We join  $v \in M$  with  $(i, x_j) \in R$  if and only if  $vx_j \in E(G)$ . Note that  $|R| = 2(k - s - w)$ . Call a pair  $(i, x_j) \in R$  *senior* if  $x_j$  is adjacent in  $G$  to at least  $k - w$  vertices in  $V(G) - K$ , and *junior*, otherwise.

**Claim 2.2.** *If  $S$  is a 1-improvement of maximum size, then the auxiliary graph  $F$  contains a matching covering all senior elements of  $R$ .*

If there is no such matching, then there exists  $T \subset R$  consisting of senior vertices with  $|N_F(T)| \leq |T| - 1$ .

**Case 1.** Suppose there is an  $i$  such that  $(i, x_i) \in T$  and  $(i, x_{i+1}) \in T$ , say  $i = 1$ . Due to the maximality of  $S$ ,  $(N_G(x_1) \cap N_G(x_2)) - K \subseteq S$ . Hence,  $|N_F(T)| \geq (k - w) + (k - w) - 2s = |R|$ , a contradiction to the choice of  $T$ .

**Case 2.** Suppose  $|T| \leq k - s - w$ . Let  $(i, x_j) \in T$ . Since  $(i, x_j)$  is senior,  $x_j$  is adjacent in  $G$  to at least  $k - w - s$  vertices in  $V(G) - K - S$ . Therefore,  $|N_F(T)| \geq k - s - w \geq |T|$ , again a contradiction to the choice of  $T$ .

**Claim 2.3.** *If  $S$  is a 1-improvement of maximum size, then the auxiliary graph  $F$  contains a matching covering all elements of  $R$ .*

By Claim 2.1,  $|R| = 2(k - s - w) \leq 2k - s - 2w - (3k - n - 2w) = n - k - s = |M|$ , and hence by Claim 2.2, we can assign to every  $(i, x_i) \in R$  a vertex  $z_{i,1}$  and to every  $(i - 1, x_i) \in R$  a vertex  $z_{i-1,2}$  so that  $x_i z_{i,1} \in E(G)$  provided that  $(i, x_i)$  is senior and  $x_i z_{i-1,2} \in E(G)$  provided that  $(i - 1, x_i)$  is senior. Among all such assignments choose one with the maximum number of edges of the form  $x_i z_{i,1}$  and  $x_i z_{i-1,2}$ . We will show that all edges of this kind exist in  $G$  which will prove the claim.

Assume that, say,  $x_1$  is not adjacent to  $z_{1,1}$ . Recall then that  $(1, x_1)$  is junior. Let  $Z_i = \{z_{i,1}, z_{i,2}\}$  and  $Z = \cup_{i \in I} Z_i$ . Let  $Z^*$  be the set of vertices in  $Z$  adjacent to  $x_1$  and let  $S^*$  be the set of vertices in  $S$  adjacent to  $x_1$ . If some  $z_{i,1} \in Z^*$  and  $z_{1,1} x_i \in E(G)$ , then we can switch  $z_{i,1}$  with  $z_{1,1}$  and have  $z_{1,1} x_1 \in E(G)$ , maintaining the desired properties. Thus, in this case,  $z_{1,1} x_i \notin E(G)$ . Similarly,  $z_{1,1} x_{i+1} \notin E(G)$  if  $z_{i,2} x_1 \in E(G)$ . By the maximality of  $S$ , if  $y_j \in S^*$ , then  $z_{1,1}$  is not adjacent to both  $x_{i_j}$  and  $x_{i_j+1}$ , but possibly one of them. Therefore, since  $z_{1,1} \notin Z^*$ , it follows that  $\deg(z_{1,1}) \leq n - 1 - (|S^*| + |Z^*| + (1 - \delta(1)))/2$ , where in  $|S^*| + |Z^*|$ , one missing edge (other than  $z_{1,1} x_1$ ) is counted at most twice, and

the missing edge  $z_{1,1}x_1$  is counted at most once if  $\delta(1) = 1$  and is not counted if  $\delta(1) = 0$ . Hence

$$n + \frac{3k - 9}{2} \leq \deg(x_1) + \deg(z_{1,1}) \leq (k - 2 - \delta(1) + |S^*| + |Z^*|) + \left( n - 1 - \frac{|S^*| + |Z^*| + 2 - \delta(1)}{2} \right) = n + k - 4 + \frac{|S^*| + |Z^*| - \delta(1)}{2}.$$

It follows that  $k \leq |S^*| + |Z^*| + 1 - \delta(1)$ . Since  $x_1$  is junior,  $|S^*| + |Z^*| \leq k - 1 - w$  and thus  $k \leq k - w - \delta(1)$ . But if  $w = 0$ , then  $\delta(1) = 1$ . This contradiction proves Claim 2.3.

We now complete the proof of the theorem. Claim 2.3 says that we can assume that for every  $i = 1, 2, \dots, k$  there is either  $y_i \in S$  adjacent to both  $x_i$  and  $x_{i+1}$  in the case  $x_i x_{i+1} \notin E(G)$  or there are two vertices  $z_{i,1} \in N_G(x_i)$  and  $z_{i,2} \in N_G(x_{i+1})$ . Moreover, all the  $y_i$  and  $z_{i,j}$  are distinct. Let  $Z_i = \{z_{i,1}, z_{i,2}\}$  and  $Z = \cup_{i=1}^k Z_i$ . Among all such assignments, choose one with the maximum number of edges of the kind  $z_{i,1}z_{i,2}$ . We will show that all edges of this kind exist, which will imply the theorem.

Assume that in our assignment there is an  $i$  such that  $z_{i,1}z_{i,2} \notin E(G)$ , say  $i = 1$ . Let  $P = V(G) - K - S - Z$ . By the maximality of  $S$ , no vertex in  $P$  is adjacent to both  $x_1$  and  $x_2$ . Let  $P_0$  be the set of vertices in  $P$  that are adjacent to neither  $x_1$  nor  $x_2$ , and for  $i = 1, 2$ , let  $P_i$  be the set of vertices in  $P$  adjacent to  $x_i$ . For  $i = 0, 1, 2$ , let  $p_i = |P_i|$ . For  $j = 0, 1, 2$ , let  $S_j$  denote the set of vertices in  $S$  adjacent to exactly  $j$  of the vertices  $x_1$  and  $x_2$  and let  $s_j = |S_j|$ .

Since  $x_1x_2 \notin E(G)$ ,  $\deg(x_1) + \deg(x_2) \geq n + (3k - 9)/2$ . Recall that by the maximality of  $S$ , no vertex in  $Z$  is adjacent to both  $x_1$  and  $x_2$ . It follows that

$$n + \frac{3k - 9}{2} \leq \deg(x_1) + \deg(x_2) \leq (n - 2) + (k - 2) - \delta(1) - \delta(2) + s_2 - s_0 - p_0,$$

and hence

$$s_2 \geq -0.5 + \delta(1) + \delta(2) + s_0 + p_0 + k/2.$$

Since  $z_{1,1}$  and  $z_{1,2}$  are nonadjacent,  $\deg(z_{1,1}) + \deg(z_{1,2}) \geq n + (3k - 9)/2$ . On the other hand,  $z_{1,j}$  is adjacent to no vertex in  $P_{3-j}$  ( $j = 1, 2$ ). Furthermore, for every  $Z$ -index or  $S_2$ -index  $j$ , each of  $z_{1,1}$  and  $z_{1,2}$  is not adjacent to at least one of  $x_j, x_{j+1}$ . It follows that

$$n + \frac{3k - 9}{2} \leq \deg(z_{1,1}) + \deg(z_{1,2}) \leq 2(n - 2) - p_2 - p_1 - s_2 - \frac{|Z| + (1 - \delta(1)) + (1 - \delta(2))}{2},$$



and hence

$$3k/2 + 0.5 + p_1 + p_2 + s_2 + \frac{|Z| - \delta(1) - \delta(2)}{2} \leq n.$$

Recall that

$$s_2 \geq -0.5 + \delta(1) + \delta(2) + s_0 + p_0 + k/2.$$

Hence

$$2k + s_0 + p_0 + p_1 + p_2 + \frac{|Z|}{2} + \frac{\delta(1) + \delta(2)}{2} \leq n.$$

But

$$p_0 + p_1 + p_2 + \frac{|Z|}{2} = |V(G) - K - S| - \frac{|Z|}{2} = n - k - (k - w) = n - 2k + w.$$

These last two statements imply that  $2k + n - 2k + w + \frac{\delta(1) + \delta(2)}{2} \leq n$ , which in turn gives  $w + \frac{\delta(1) + \delta(2)}{2} \leq 0$ . But by the definitions of  $w$  and  $\delta(1)$ , we have  $w + \frac{\delta(1) + \delta(2)}{2} \geq 1$ . This contradiction completes the proof of the theorem. ■

**Corollary 2.1.** *Let  $k$  be an integer with  $3 \leq k \leq n/2$  and let  $G$  be a graph of order  $n$ . If  $\deg(v) \geq n/2 + \frac{3k-9}{4}$  for every vertex  $v$  of  $G$ , then  $G$  is  $k$ -ordered hamiltonian.*

That Theorem 2.1 is sharp is indicated by the following example which was mentioned in Ng et al. [5]. The graph  $G$  with  $n$  vertices is composed of three parts: copies of  $K_{k-1}$ ,  $K_k - C_k$ , and  $K_{n-2k+1}$ , where the vertices of the “missing” cycle  $C_k$  are indexed in the natural order. Further,  $G$  contains all the edges between the copies of  $K_{k-1}$  and  $K_{n-2k+1}$ , and all of the edges between the copies of  $K_{k-1}$  and  $K_k - C_k$ . Between the copies of  $K_{n-2k+1}$  and  $K_k - C_k$ ,  $G$  contains all edges incident to the even indexed vertices of  $C_k$ . This graph is not  $k$ -ordered because there is no cycle containing the vertices of  $C_k$  in order. However, when  $k$  is even, if  $u \in V(K_{n-2k+1})$  and  $v \in V(K_k - C_k)$ , where  $v$  is an odd-indexed vertex,  $\deg(u) + \deg(v) = n + \frac{3k-10}{2}$ .

### 3. MINIMUM DEGREE CONDITIONS

In this section, we consider minimum degree conditions that guarantee that a graph is  $k$ -ordered hamiltonian. The main interest in such results follows from the following observation. Theorem 2.1 gives a result based on the degree sums of nonadjacent vertices of a graph of order  $n$ . Here we have the condition for

$3 \leq k \leq n/2$ , that if  $\deg(u) + \deg(v) \geq n + (3k - 9)/2$  for all nonadjacent vertices  $u$  and  $v$  of a graph  $G$ , then  $G$  is  $k$ -ordered hamiltonian. However, Theorem 1.3 gives a minimum degree result that says that for  $n \geq 11k - 3$ , if  $\deg(u) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$  for every vertex  $u$  of a graph  $G$ , then  $G$  is  $k$ -ordered hamiltonian. Both of these results are sharp. However, the bound given in Theorem 2.1 is not twice the bound given in Theorem 1.3, unlike most results of this nature. So an obvious question is, for all  $n$  and  $k$ , what minimum degree condition implies that a graph  $G$  is  $k$ -ordered hamiltonian?

In the previous sections, we have seen two such results, the aforementioned Theorem 1.3 for values of  $n, k$  satisfying  $n \geq 11k - 3$ , and Corollary 2.1 for values of  $n, k$  satisfying  $k \leq n/2$ . It is also obvious that for  $n/2 < k \leq 2n/3$ , minimum degree at least  $n - 2$  guarantees a  $k$ -ordered hamiltonian graph, and that for  $k > 2n/3$  (and in fact for any  $k$ ), minimum degree  $n - 1$  gives a  $k$ -ordered hamiltonian graph. The main object of this section is to provide examples that discuss the sharpness of these known results.

For  $2 \leq k \leq n/3$ , consider the graph  $G$  that consists of three parts: two copies of  $K_{(n-k+2)/2}$  and a copy of  $K_{k-2}$ , where  $n$  and  $k$  are of the same parity. The vertices in the copy of  $K_{k-2}$  are adjacent to all other vertices of the graph. Then  $G$  is not  $k$ -ordered hamiltonian since  $G$  is not  $(k - 1)$ -connected, a necessary condition for a graph to be  $k$ -ordered hamiltonian, and  $\delta(G)$  is  $\frac{n}{2} + \frac{k}{2} - 2$ .

For  $n/3 < k < 2(n + 2)/5$ , consider the graph  $G$  that consists of three parts: copies of  $K_{3k-n-1}$ ,  $K_{2n-4k+1}$  and  $K_k - C_k$ , where the vertices of the “missing” cycle are indexed in the natural order.  $G$  contains all the edges between the copy of  $K_{3k-n-1}$  and the rest of the graph. The vertices of  $K_{2n-4k+1}$  are divided into two sets  $A$  and  $B$ , where  $|A| = n - 2k + 1$  and  $|B| = n - 2k$ . All vertices of  $A$  are adjacent to the even indexed vertices of  $C_k$  and all vertices of  $B$  are adjacent to the odd indexed vertices of  $C_k$ . Then  $G$  is not  $k$ -ordered hamiltonian since there is no hamiltonian cycle containing the vertices of  $C_k$  in order, and  $\delta(G) = 2k - 3$ .

For  $2(n + 2)/5 \leq k \leq n/2$ , consider the graph that consists of three parts: copies of  $K_{k/2}$ ,  $K_{n-3k/2}$  and  $K_k - C_k$ , where the vertices of the “missing” cycle are indexed in the natural order and  $k$  is even. The vertices of  $K_{n-3k/2}$  are as evenly as possible divided into two sets  $A$  and  $B$ . All vertices of  $A$  are adjacent to the even indexed vertices of  $C_k$  and all vertices of  $B$  are adjacent to the odd indexed vertices of  $C_k$ . In addition, there is a specified set of  $5k/2 - n - 2$  consecutive vertices of  $C_k$  that are adjacent to all vertices of  $A$  and  $B$ . Then  $G$  is not  $k$ -ordered hamiltonian since there is no hamiltonian cycle containing the vertices of  $C_k$  in order, and  $\delta(G) = n/2 + 3k/4 - 3$ .

For  $n/2 < k \leq 2n/3$ , consider the graph  $K_n - C_k$ . Then  $G$  is not  $k$ -ordered hamiltonian since there is no hamiltonian cycle containing the vertices of  $C_k$  in order, and  $\delta(G) = n - 3$ .

For  $2n/3 < k \leq n$ , consider the graph  $K_n - (k/2)K_2$ , where  $k$  is even, and the vertices of the “missing” matching are labeled as  $x_1, x_2, \dots, x_k$  in the natural order. Then  $G$  is not  $k$ -ordered hamiltonian since there is no hamiltonian cycle containing  $x_1, x_2, \dots, x_k$  in order, and  $\delta(G) = n - 2$ .

We can summarize the previous discussion as follows. Let  $\delta(n, k)$  be the smallest integer  $m$  for which any graph of order  $n$  with minimum degree at least  $m$  is  $k$ -ordered hamiltonian. We then have the following theorem.

**Theorem 3.1.** For positive integers  $k, n$  with  $3 \leq k \leq n$  we have

- (i)  $\delta(n, k) = \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$ , for  $k \leq (n + 3)/11$ ;
- (ii)  $\delta(n, k) > \frac{n}{2} + \frac{k}{2} - 2$ , for  $(n + 3)/11 < k \leq n/3$ ;
- (iii)  $\delta(n, k) \geq 2k - 2$ , for  $n/3 < k < 2(n + 2)/5$ ;
- (iv)  $\delta(n, k) = \lceil n/2 + \frac{3k-9}{4} \rceil$ , for  $2(n + 2)/5 \leq k \leq n/2$ ;
- (v)  $\delta(n, k) = n - 2$ , for  $n/2 < k \leq 2n/3$ ; and
- (vi)  $\delta(n, k) = n - 1$ , for  $2n/3 < k \leq n$ .

Figure 1 indicates the relationship between the exact values known for  $\delta(n, k)$  and the bounds provided by the examples.

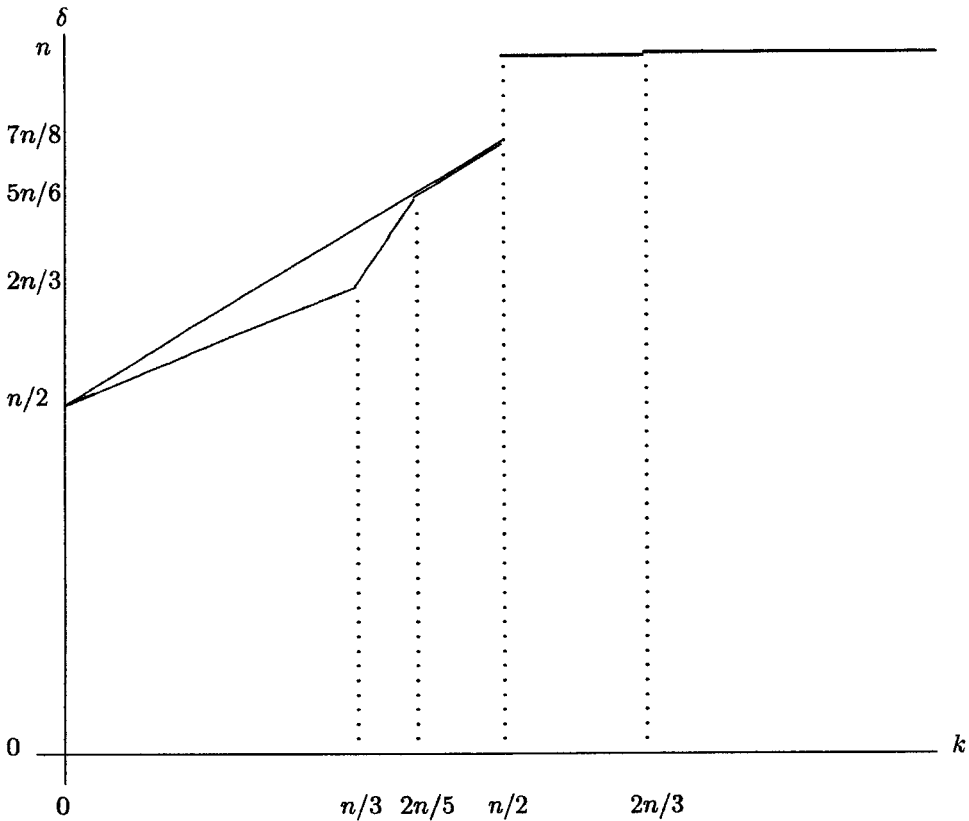


FIGURE 1. Bounds for  $\delta(n, k)$ .

## ACKNOWLEDGMENT

Alexandr V Kostochka thanks 99-01-00581 of the Russian Foundation for Fundamental Research and a Dutch-Russian Cooperative Grant.

## REFERENCES

- [1] G. Dirac, Some theorems on abstract graphs, Proc London Math Soc 2 (1952), 69–81.
- [2] J. Faudree, R. Faudree, R. Gould, M. Jacobson, and L. Lesniak, On  $k$ -ordered graphs, J Graph Theory, to appear.
- [3] H. Kierstead, G. Sárközy, S. Selkow, On  $k$ -ordered hamiltonian graphs, J Graph Theory 32 (1999), 17–25.
- [4] L. Lovász and M. D. Plummer, Matching Theory, Akadémiai Kiadó, Budapest (1986).
- [5] L. Ng and M. Schultz,  $k$ -Ordered hamiltonian graphs,  $k$ -Ordered hamiltonian graphs, J Graph Theory 2 (1997), 45–57.
- [6] O. Ore, Note on Hamilton circuits, Amer Math Monthly 67 (1960), 55.