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Characterizing forbidden clawless triples implying hamiltonian graphs

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Abstract

In this paper we characterize those forbidden triples of graphs, no one of which is a generalized claw, sufficient to imply that a 2-connected graph of sufficiently large order is hamiltonian.

1. Introduction

Given a family $\mathscr{F} = \{F_1, F_2, \ldots, F_k\}$ of graphs we say that a graph G is $\mathscr{F}$*-free* if G contains no induced subgraph isomorphic to any F_i, $(i = 1, 2, \ldots, k)$. For simplicity, we simply say that G is $F_1F_2 \cdots F_k$-free. In [1] Bedrossian characterized the pairs AB of connected graphs with the property that every 2-connected AB-free graph G is hamiltonian, and this result was extended in [5] by Faudree and Gould to a larger class of pairs of forbidden graphs that imply hamiltonicity if the graph G is of sufficiently large order. Note that if A or B is P_2, then no such connected graph G exists. Also, if A or B is P_3, say A, then every P_3B-free connected graph is complete (and so hamiltonian or K_2). To eliminate these trivial or vacuous situations, we will assume that A and B are connected graphs with at least three edges. It is also clear that if every 2-connected AB-free graph G of sufficiently large order is hamiltonian, the same is true if G is $A'B'$-free, where A' is a connected induced subgraph of A with at least three edges and B' is a connected induced subgraph of B with at least three edges. Thus, we have the following result of Faudree and Gould [5] and Bedrossian [1].

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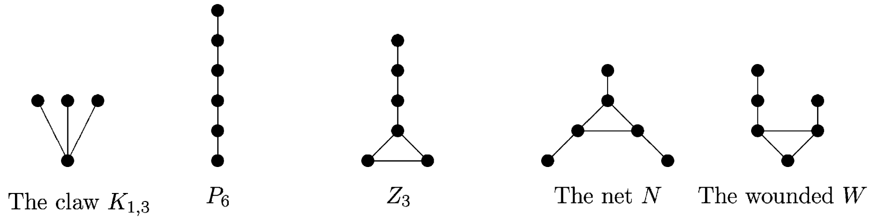

Fig. 1.

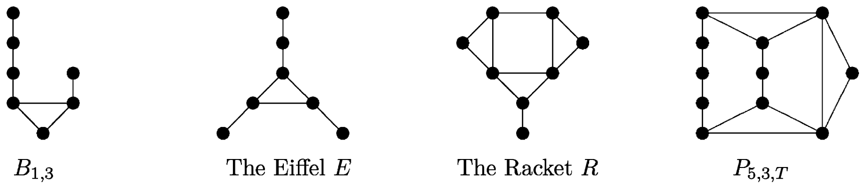

Fig. 2.

Theorem 1 (Bedrossian [1] and Faudree and Gould [5]). *Let A and B be connected graphs with at least three edges, and let G be a 2-connected graph of sufficiently large order. Then G being AB-free implies that G is hamiltonian if and only if $A = K_{1,3}$ and B is a connected induced subgraph of P_6, Z_3, W or N (see Fig. 1) with at least three edges.*

As indicated in Theorem 1, the *claw* $K_{1,3}$ occurs in every forbidden pair. This is not the case for forbidden triples. In this paper we characterize those forbidden triples XYZ of connected graphs, none of which is a generalized claw $K_{1,r}$, ($r \geqslant 3$), which imply that every 2-connected XYZ-free graph of sufficiently large order is hamiltonian. Again, we restrict our attention to triples XYZ such that each of X, Y, Z is a connected graph with at least three edges. Furthermore, since none of X, Y or Z is a claw, forbidding no pair of the graphs X, Y, Z is sufficient for hamiltonicity. The situation for all graphs in which one of X, Y or Z is a claw was recently completed in [3]. The case for forbidden triples that imply hamiltonicity for graphs of sufficiently large order still remains open. Before stating this result, some additional graphs must be described.

The generalized Bull $B_{1,3}$, the Eiffel E, and the Racket R are graphs that are displayed in Fig. 2. The graph P_{x_1,x_2,x_3} is obtained from two vertex disjoint triangles by joining corresponding vertices by a path P_{x_i} for $(1 \leqslant i \leqslant 3)$. Any of the paths P_{x_i} may be replaced by a triangle T. Again, see Fig. 2 for an example of a graph of this type.

Theorem 2 (Brousek [3]). *Let X and Y be connected graphs such that neither X nor Y is an induced subgraph of any of the graphs P_6, N, or W. Then,*

(i) *there exists 62 pairs XY of graphs such that the property "G is 2-connected and $K_{1,3}XY$-free" implies G is hamiltonian*, and
(ii) *$K_{1,3}XY$ is a maximal triple of forbidden subgraphs such that G being $K_{1,3}XY$-free implies G is hamiltonian if and only if*

$$K_{1,3}XY \in \{K_{1,3}DP_{3,3,3}, K_{1,3}EP_{T,T,T}, K_{1,3}P_7P_{T,T,T}, K_{1,3}B_{1,3}R\}.$$

We will call a triple XYZ *good* if each of X, Y and Z is a connected graph with at least three edges which is not a generalized claw and also for which every 2-connected XYZ-free graph of sufficiently large order is hamiltonian.

For convenience we introduce the following notation. Let $A(i,j,k)$ denote the graph obtained from a claw $K_{1,3}$ by subdividing the edges, i, j, and k times, respectively. Note that G_4 does not contain $C_3, C_4, P_7, A(3,0,0)$, or $A(2,1,0)$ as an induced subgraph. Hence, $X \preccurlyeq G_4$ and we can conclude that $X \preccurlyeq P_6$, $X \preccurlyeq A(2,0,0)$, or $X \preccurlyeq A(1,1,0)$. Therefore if XYZ is a good triple, we know that $X=P_4$, $X=P_5$, $X=P_6$, $X=A(1,0,0)$, $X=A(2,0,0)$ or $X=A(1,1,0)$; $Y=B_1$, B_2, or B_3; and $Z=K_{2,k}$ for some $k \geqslant 2$.

In this paper we will characterize those triples XYZ, none of which is a generalized claw, that are good. For convenience, if H is an induced subgraph of G, we will write $\preccurlyeq G$. If XYZ is a good triple, then certainly $X'Y'Z'$ is a good triple if $X' \preccurlyeq X$, $Y' \preccurlyeq Y$ and $Z' \preccurlyeq Z$, and X', Y' and Z' are connected graphs with at least three edges. If XYZ is a good triple, we will say that it is a *maximal good triple*, if it is a proper induced triple of no other good triple. As before, we say that XYZ is an induced triple. More specifically the following will be proved.

Theorem 3. *Let G be a 2-connected graph of sufficiently large order n, and let X, Y and Z be connected graphs with at least three edges, none of which is a generalized claw. Then G being XYZ-free implies that G is hamiltonian if and only if XYZ is one of the following triples*:

(i) $P_4B_2K_{2,\lceil (n+1)/2 \rceil}$, (ii) $P_4B_3K_{2,3}$, (iii) $P_5B_1K_{2,\lfloor n/3 \rfloor}$, (iv) $A(1,0,0)B_1K_{2,\lfloor n/2 \rfloor -2}$,
(v) $A(2,0,0)B_1K_{2,2}$, (vi) $A(1,1,0)B_1K_{2,2}$, (vii) $P_6B_1K_{2,2}$,
(viii) $P_5B_2K_{2,3}$,

or XYZ is an induced triple of one of these eight triples.

2. Examples

In this section we will describe examples that place restrictions on which triples XYZ can be good, and verify that the only possible good triples with none of the graphs being a generalized claw are those described in Theorem 3.

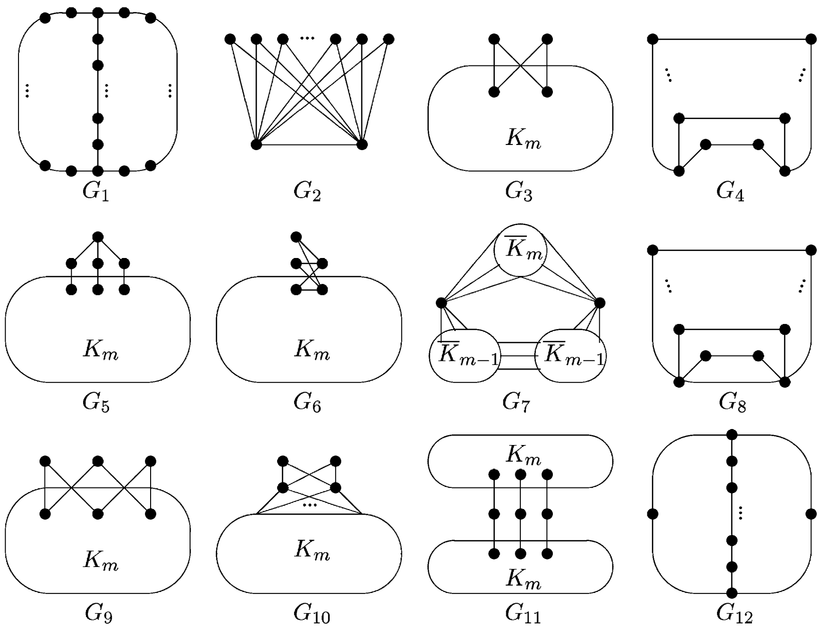

Fig. 3.

Suppose that XYZ is a good triple. Then every 2-connected nonhamiltonian graph of sufficiently large order contains at least one of X, Y and Z as an induced subgraph. We begin by considering the 2-connected nonhamiltonian graphs of Fig. 3.

Without loss of generality, assume that X is an induced subgraph of G_1. Since $X \neq K_{1,3}$ and X has at least three edges, it follows that $P_4 \preccurlyeq X \preccurlyeq S$, where S is a proper subdivision (i.e., a subdivision with at least four edges) of $K_{1,3}$. Since $K_{2,k}$ is nonhamiltonian for $k \geqslant 3$ and contains no induced P_4, and Z is not a generalized claw $K_{1,r}$, we may assume $Z = K_{2,k}$ for some $k \geqslant 2$. Finally, since G_2 and G_3 are nonhamiltonian and neither contains an induced P_4 or C_4 and Y is not a generalized claw, we conclude that $Y \preccurlyeq G_2$ and $Y \preccurlyeq G_3$, and thus $Y = B_1$, B_2 or B_3 (generalized books, see Fig. 4).

Let $A(i,j,k)$ denote the graph obtained from a claw $K_{1,3}$ by subdividing the edges i, j, and k times respectively. Note that G_4 does not contain C_3, C_4, P_7, $A(3,0,0)$, or $A(2,1,0)$ as an induced subgraph. Hence, $X \preccurlyeq G_4$ and we can conclude that $X \preccurlyeq P_6$, $X \preccurlyeq A(2,0,0)$, or $\preccurlyeq A(1,1,0)$. Therefore if XYZ is a good triple, we know that $X = P_4$, $X = P_5$, $X = P_6$, $X = A(1,0,0)$, $X = A(2,0,0)$ or $X = A(1,1,0)$; $Y = B_1$, $Y = B_2$, or $Y = B_3$; and $Z = K_{2,k}$ for some $k \geqslant 2$.

Since G_5 is $P_6B_2K_{2,2}$-free, and G_8 is $P_6B_1K_{2,3}$-free, any good triple XYZ with $X = P_6$ will have $Y = B_1$ and $Z = K_{2,2}$. Since G_9 is $P_5B_3K_{2,2}$-free, any good triple XYZ with $X = P_5$ will have $Y = B_2$ or B_1. Since G_6 is $P_5B_2K_{2,4}$-free, any good triple XYZ with

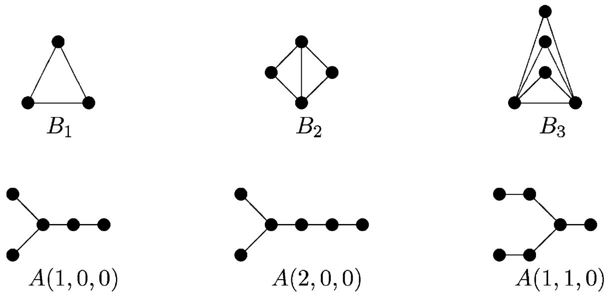

Fig. 4.

$X=P_5$ and $Y=B_2$ will have $Z=K_{2,3}$ or $Z=K_{2,2}$. Since G_7 is $P_5B_1K_{2,n/3+1}$-free, any good triple XYZ with $X=P_5$ and $Y=B_1$ will have $Z=K_{2,k}$ for $\leqslant k \leqslant \lfloor n/3 \rfloor$.

The graph G_{10} is $P_4B_3K_{2,4}$-free, so any good triple XYZ with $X=P_4$ and $Y=B_3$ will have $Z=K_{2,3}$ or $K_{2,2}$. The graphs $K_{(n+1)/2,(n-1)/2}$ for n odd and $K_{(n+2)/2,(n-2)/2}$ for n even are $P_4B_1K_{2,\lceil (n+1)/2 \rceil+1}$-free, so any good triple XYZ with $X=P_4$ and $Y=B_2$ will have $Z=K_{2,k}$ for $2 \leqslant k \leqslant \lceil (n+1)/2 \rceil$.

The graph G_{11} is $A(2,0,0)B_2K_{2,2}$-free, and the graph G_8 is $A(2,0,0)B_1K_{2,3}$-free, so any good triple XYZ with $X=A(2,0,0)$ will have $Y=B_1$ and $Z=K_{2,2}$. Similarly, G_{11} is $A(1,1,0)B_2K_{2,2}$-free, and the graph G_{12} is $A(1,1,0)B_1K_{2,3}$-free so any good triple XYZ with $X=A(1,1,0)$ will have $Y=B_1$ and $Z=K_{2,2}$. As before, G_{11} is $A(1,0,0)B_2K_{2,2}$-free. For odd n, the graph obtained from the complete bipartite graph $K_{(n+1)/2,(n-1)/2}$ by deleting a maximum matching is $K_{2,(n-3)/2}$-free as well as being $A(1,0,0)B_1$-free. The corresponding graph for n even gives an $A(1,0,0)B_2K_{2,(n-2)/2}$-free graph. Therefore, any good triple XYZ with $X=A(1,0,0)$ will have $Y=B_1$ and $Z=K_{2,k}$ for $2 \leqslant k \leqslant \lceil n/2 \rceil - 2$.

This confirms that the only possible good triples that do not contain any generalized claws are those listed in Theorem 3.

3. Characterizing non-claw forbidden triples

In this section we will determine all of the “maximal” good triples when no graph of the triple is a generalized claw, which will complete the proof of Theorem 3.

In what follows, $N(v)$ denotes the set of vertices adjacent to v (i.e., the *neighbors of* v) and $N[v]=N(v)\cup\{v\}$ is the *closed neighborhood of* v. Also, if $S \subseteq V(G)$, then the subgraph induced by S will be denoted by $\langle S \rangle$.

Theorem 4. *Let G be a 2-connected P_4B_2-free graph of order n. Then either $G=K_n$ or $G=K_{m,n-m}$ for some m satisfying $2 \leqslant m \leqslant n/2$.*

Proof. Since G is B_2-free, $N(v)$ is P_3-free for every vertex v of G. Consequently, each component of $N(v)$ is complete. If $N(v)$ is connected for each v, then $G=K_n$.

Assume, then, that some vertex v has a disconnected neighborhood with components $H_1, H_2, \ldots, H_t$, with $t \geqslant 2$. Since G is 2-connected, there exists a vertex $v_1 \in V(G) - N[v]$ that is adjacent to a vertex of H_1. Since G is P_4-free, v_1 is adjacent to every vertex of $H_2, H_3, \ldots, H_t$ and, consequently, to every vertex of H_1 as well. If $|V(H_i)| \geqslant 2$ for some i, then $B_2 \preccurlyeq G$. Thus, we may assume that $\langle N[v] \cup \{v_1\} \rangle = K_{2,m}$ and $N(v) \subseteq N(v_1)$. Suppose $N(v) \neq N(v_1)$; let $w \in N(v_1) - N(v)$, and let x and y be distinct vertices in $N(v)$. Then $\langle \{v, x, v_1, w\} \rangle = P_4$ unless $wx \in E(G)$. Similarly, w is adjacent to y, which implies that $B_2 \preccurlyeq G$. Thus no such vertex w exists and $N(v) = N(v_1)$. Continuing in this fashion we see that $G = K_{m,n-m}$, for $2 \leqslant m \leqslant n/2$. $\square$

Corollary 1. *Let G be a 2-connected $P_4B_2K_{2,\lceil (n+1)/2 \rceil}$- free graph of order n. Then G is hamiltonian.*

In [4] it was shown that every connected $K_{1,3}P_4$-free graph is traceable. This will be used in the proof of our next result.

Theorem 5. *Let G be a 2-connected P_4-free nonhamiltonian graph. Then, either $B_3 \preccurlyeq G$ or $K_{2,3} \preccurlyeq G$.*

Proof. By Theorem 1, $K_{1,3} \preccurlyeq G$. Suppose $\langle \{v, v_1, v_2, v_3\} \rangle = K_{1,3}$, where v is the center of the claw. We consider two cases.

Case 1: Suppose $N[v] = V(G)$. Since G is 2-connected, $V(G) - \{v\} = N(v)$ is connected. Since $\langle N(v) \rangle$ is also P_4-free, it follows by the result in [4] that either $N(v)$ is traceable or $K_{1,3} \preccurlyeq N(v)$. Since G is nonhamiltonian, we conclude that $K_{1,3} \preccurlyeq N(v)$, and so $B_3 \preccurlyeq G$.

Case 2: Suppose $N[v] \neq V(G)$. Let $x \in V(G) - N[v]$. If x is adjacent to one of v_1, v_2, v_3, then x is adjacent to all three, since $P_4 \not\preccurlyeq G$. Thus $K_{2,3} \preccurlyeq G$. If, on the other hand, x is adjacent to none of v_1, v_2, v_3, then since G is 2-connected, there is an $x' \in N(v) - \{v_1, v_2, v_3\}$ such that $xx' \in E(G)$. Since $P_4 \not\preccurlyeq G$, this implies that x' is adjacent to v_1, v_2 and v_3, and so $B_3 \preccurlyeq G$. $\square$

Corollary 2. *If G is a 2-connected $P_4B_3K_{2,3}$-free graph, then G is hamiltonian.*

Theorem 6. *If G is a 2-connected P_5-free bipartite graph of order n, then $K_{2,\lceil n/2 \rceil} \preccurlyeq G$.*

Proof. Let G be a 2-connected P_5-free bipartite graph of order n with partite sets V_1 and V_2 with $|V_1| \leqslant |V_2|$. Assume that $K_{2,\lceil n/2 \rceil} \not\preccurlyeq G$. Thus, at most one of the vertices of V_1 has degree $|V_2|$. Let w be a vertex of V_1 of largest degree and let x be a vertex of $V_1 - \{w\}$. Then $d(x) < |V_2|$, so there is a vertex $u \in V_2$ not adjacent to x. Since G is 2-connected, there is an x–u path P in G that does not contain w. Furthermore, since $P_5 \not\preccurlyeq G$, we may assume that P has length 3, say $P = (x, y, z, u)$. Since G is 2-connected, $d(x) \geqslant 2$. Let t be any vertex in $V_2 - \{u, y\}$ adjacent to x.

Then $Q=(u,z,y,x,t)$ is a path in G. Since x is not adjacent to u and $P_5 \not\preccurlyeq G$, it follows that t is adjacent to z. Thus, z is adjacent to every vertex of V_2 that is adjacent to x, as well as u, contradicting the choice of x. Consequently, G must be a complete bipartite graph, and thus $K_{2,\lceil n/2\rceil} \preccurlyeq G$. □

Theorem 7. *For n sufficiently large there are no 2-connected $P_5B_1K_{2,\lfloor n/3\rfloor}$-free graphs of order n.*

Proof. Suppose, to the contrary, that such a graph G exists. It follows from the previous theorem that G is not bipartite and, since $B_1=K_3 \not\preccurlyeq G$, that G contains an odd cycle of length at least 5. Let C be a shortest odd cycle in G. Then C is a 5-cycle since $P_5 \not\preccurlyeq G$; say $C=(v_0,v_1,v_2,v_3,v_4,v_0)$. If $x\in V(G)-V(C)$, then x is adjacent to at least one vertex of C; otherwise P_5 or B_1 is an induced subgraph of G. Furthermore, x is adjacent to exactly two vertices of C, and these vertices are at a distance 2 in C, say v_{i-1} and v_{i+1}, with the subscripts taken modulo 5. Therefore, x can replace v_i in the cycle C. Thus $V(G)$ can be partitioned into sets V_0,V_1,V_2,V_3,V_4 such that the vertices in each V_i are independent and are adjacent to all vertices in V_{i-1} and V_{i+1} (subscripts modulo 5). These properties of the sets V_i follow since neither P_5 nor B_1 is an induced subgraph of G. Since $K_{2,\lfloor n/3\rfloor} \not\preccurlyeq G$, it follows that exactly two of the "nonadjacent" sets V_i have $|V_i|=1$, say V_0 and V_2. Furthermore, $|V_1|<\lfloor n/3\rfloor$, $2\leqslant |V_2|<\lfloor n/3\rfloor-1$ and $2\leqslant |V_4|<\lfloor n/3\rfloor-1$. Then, however, $|V(G)|<n$, which produces a contradiction. Thus, no such G exists. □

Theorem 8 (Vacuously). *If G is a 2-connected $P_5B_1K_{2,\lfloor n/3\rfloor}$- free graph of sufficiently large order n, then G is hamiltonian.*

The tree $A(1,0,0)$ will play an important role in our next few results.

Theorem 9. *If G is a 2-connected $A(1,0,0)B_1$-free graph of sufficiently large order n, then either G is hamiltonian or $K_{2,\lceil n/2\rceil-2} \preccurlyeq G$.*

Proof. Assume first that G is not bipartite. Let C be an odd cycle of shortest length. It follows that C is induced, since any chord would result in a shorter cycle. Then either $V(C)=V(G)$ and G is hamiltonian or $V(G)-V(C)\neq\emptyset$. If G is not hamiltonian then there is a vertex $x\in V(G)-V(C)$ that is adjacent to vertices of C. Since G is B_1-free, x cannot be adjacent to consecutive vertices of C. However, since G is $A(1,0,0)$-free there cannot be consecutive vertices of C both of which are nonadjacent to x. This implies that x is adjacent to alternate vertices of C, which contradicts the fact that C has odd length. Thus, G is hamiltonian.

Assume next that G is bipartite but not hamiltonian. Let V_1 and V_2 be the partite sets of G with $|V_2|\geqslant|V_1|$. Then $|V_2|\geqslant n/2$ and, since G is 2-connected, $|V_1|\geqslant 2$. Since G is not hamiltonian, there is a vertex v of V_1 of degree at least 3. Consider any other vertex w of V_1. Suppose $N(v)\cap N(w)=\emptyset$ and choose v and w satisfying the above condition which are the closest. Consider a shortest v–w path $v = v_1, v_2, \ldots, v_m = w$. By the minimality of the v–w path, we have that $m = 5$ and then $v_3 \in V_1$ and $v_3 \neq v_1, v_m$. If v_3 has two nonadjacencies in $N(v)$, then $\langle N[v] \cup \{v_3\} \rangle$ contains $A(1,0,0)$ as an induced subgraph. Thus v_3 has at least two adjacencies in $N(v)$, and $\langle N[v] \cup \{w\} \rangle$ contains $A(1,0,0)$ as an induced subgraph, a contradiction. Hence, we may assume $N(v) \cap N(w) \neq \emptyset$, and let $z \in N(v) \cap N(w)$.

Note that w cannot have two nonadjacencies x and y in $N(v)$, since then $\langle \{x, y, z, v, w\} \rangle = A(1,0,0)$. Similarly, w cannot have two adjacencies x, y outside of $N(v)$ for otherwise $\langle \{x, y, z, v, w\} \rangle = A_1$.

Suppose $w_1, w_2 \neq v$ in V_1 each have an adjacency, x_1 and x_2, respectively, in $V_2 - N(v)$ and $x_1 \neq x_2$. Since w_1 and w_2 have at most one nonadjacency each in $N(v)$, there is at least one vertex z in $N(v)$ adjacent to both of them. But then $\langle \{z, v, w_1, w_2, x_1\} \rangle = A(1,0,0)$. It follows that $|V_2 - N(v)| \leqslant 1$, and thus, $K_{2,\lceil n/2 \rceil - 2} \preccurlyeq G$. $\square$

Corollary 3. *If G is a 2-connected $A(1,0,0)B_1K_{2,\lceil n/2\rceil -2}$-free graph of sufficiently large order, then G is hamiltonian.*

The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. Our next result deals with 2-connected nonhamiltonian graphs of sufficiently large order and large girth. For convenience we let $r = \lceil (k-1)/2 \rceil$ and $= \lfloor (k-1)/2 \rfloor$.

Theorem 10. *Let G be a 2-connected, nonhamiltonian graph of order n with $g(G) > k$ for ($k \geqslant 3$). Then for n sufficiently large, $A(r-2, r-2, k-2)$, $A(r-1, r-1, s-1)$, and P_{3r} are induced subgraphs of G.*

Proof. Let $F_0 = A(r-1, r-1, k-2)$, and let F_1 denote the graph obtained by identifying a vertex of each of two disjoint cycles of length at least $k+1$. Let F_2 be a graph obtained from a cycle C of length at least $3k+1$ by adding a new vertex v, together with three disjoint (except for v) paths from v to C in such a way that the endvertices of these paths are at distance at least $k-1$ from each other along C. Let F_3 be a graph that consists of adjacent vertices u and v together with three other disjoint u–v paths, each of length at least k. Let F_4 be the graph that consists of four disjoint (except for u and v) u–v paths joining nonadjacent vertices u and v, where each path has length at least $\lceil (k-1)/2 \rceil + 1$.

Clearly, if a graph G contains any one of F_i as an induced subgraph, for $0 \leqslant i \leqslant 4$, then G also contains each of $A(r-2, r-2, k-2)$, $A(r-1, r-1, s-1)$, and P_{3r} as an induced subgraph. We show, therefore, that if G is a 2-connected nonhamiltonian graph of order n with $g(G) > k$ which contains no F_i, for $0 \leqslant i \leqslant 4$, then n is a bounded function of k. This will complete the proof of the theorem. $\square$

Suppose first that G contains an induced cycle C of length at least $2k+1$. Let P be a shortest path joining nonconsecutive vertices u and v on C such that $|V(C) \cap V(P)| = 2$. Overall, for such choices of u and v on C, choose u and v so that they are as close on C as possible. Let C' be the resulting cycle consisting of P and the shortest u–v path on C. Clearly, C' is induced and thus an F_0 results as an induced subgraph, since $g(G) > k$. Hence, we may assume that G has no induced cycles of length at least $2k+1$.

This bound on the length of induced cycles of G implies that G has vertices of high degree, say at least $31k$. Pick a vertex v of degree at least $31k$ and let C be a shortest cycle of G containing v. Necessarily, $k < |V(C)| \leqslant 2k$. Label the vertices in $V(G) - V(C)$ that are adjacent to v as $v_1, v_2, \ldots, v_m$, where $m \geqslant 31k - 2$. Note that if v_i were adjacent to some two vertices of C, then $g(G) = 4$ and $k = 3$, and so the result follows immediately.

Let $S_0 = V(C) \cup \{v_1, \ldots, v_m\} - \{v_1\}$. Find a shortest path P_1 from v_1 to $\langle S_0 \rangle$ in G. Choose, if possible, such a P_1 that ends on C. Let $S_1 = S_0 \cup V(P_1)$. Find a shortest path P_2 from v_2 to $\langle S_1 \rangle$ in G. Choose such a path whose final vertex is closest to C in $\langle S_1 \rangle$. Let $S_2 = S_1 \cup V(P_2)$. Continue, in this fashion, to finally find a shortest path P_m from v_m to $\langle S_{m-1} \rangle$ in G, choosing such a path whose final vertex is closest to C in $\langle S_{m-1} \rangle$. Let $S_m = S_{m-1} \cup V(P_m)$.

If some P_j ends at a neighbor of v off of C, then G contains F_1 as an induced subgraph. If some P_j ends at a vertex in S_m at a distance at least 2 from C in $\langle S_m \rangle$, then again G contains F_1 as an induced subgraph. Thus, to avoid the contradiction that G contains F_1 as an induced subgraph we have that every P_j, for $1 \leqslant j \leqslant m$, ends on C, and is called a path of type I, or ends at a vertex at distance one from C in $\langle S_m \rangle$, and is called a path of type II.

If four paths of type I end at the same vertex of C, then G contains F_3 or F_4 as an induced subgraph. Therefore, the number of paths of type I is atmost $3|V(C)| \leqslant 6k$. Similarly, if five paths of type II end at the same vertex of C, then G contains F_3 or F_4 as an induced subgraph. Therefore, the number of paths of type II is atmost $4(6k) = 24k$. Thus, $d_G(v) \leqslant m+2 \leqslant 6k+24k+2 < 31k$, which produces a contradiction and completes the proof. □

Corollary 4. *Let G be a 2-connected $A(2,0,0)B_1K_{2,2}$-free graph of sufficiently large order, then G is hamiltonian.*

Corollary 5. *Let G be a 2-connected $A(1,1,0)B_1K_{2,2}$-free graph of sufficiently large order, then G is hamiltonian.*

In fact, it is easy to see that if G is a 2-connected $A(1,1,0)$, B_1, $K_{2,2}$-free graph of order n then $G = C_n$.

In fact, it is easy to see that if G is a 2-connected $A(1,1,0)B_1K_{2,2}$-free graph of sufficiently large order, then G must by a cycle.

Corollary 6. *Let G be a 2-connected $P_6B_1K_{2,2}$-free graph of sufficiently large order, then G is hamiltonian.*

Our next theorem gives the last maximal good triple for this section. Since the techniques used to prove Theorem 9 are similar in nature to those used in previous proofs, we will simply outline the proof.

Theorem 11. *If G is a 2-connected $P_5B_2K_{2,3}$-free graph of sufficiently large order, then G is $K_{1,3}$-free.*

Proof (Outline).

Suppose, to the contrary, that G has an induced claw. Let v be the center of the claw and v_1, v_2 and v_3 be the remaining vertices of the claw.

Fact 1: $\langle N(v)\rangle$ consists of disjoint complete subgraphs since G is B_2-free.

Fact 2: $\langle N(v)\rangle$ has at most one nontrivial component. If this were not the case, then $\langle N(v)\rangle$ would have two nontrivial components C_1 and C_2, and, since G is 2-connected, G would contain a (shortest) path P from C_1 to C_2 in $G - v$. But then G contains either an induced P_5 or an induced B_2.

Fact 3: $\langle N(v)\rangle$ has at most three components. If this were not the case, then $\langle N(v)\rangle$ would have at least four (complete) components C_1, C_2, C_3 and C_4, at most one of which, say C_1, is nontrivial. Again, looking at the shortest path P from C_1 to C_2 and a case-by-case analysis, we find that G contains either an induced P_5, B_2, or $K_{2,3}$.

Fact 4: $\langle N(v)\rangle = \langle\{v_1, v_2, v_3\}\rangle = K_{1,3}$. Otherwise, either $P_5 \leqslant G$, $B_2 \leqslant G$ or $K_{2,3} \leqslant G$.

Fact 5: One of v_1, v_2, v_3, say v_1, has a large collection of neighbors, say S, such that $\langle S\rangle$ is complete and, in G, each vertex of S is not adjacent to v_2 or v_3; otherwise, $K_{2,3} \leqslant G$ results.

Fact 6: No such graph G exists since G is 2-connected, for otherwise B_2 or P_5 would be an induced subgraph. □

It was shown in [2] that every 2-connected $K_{1,3}P_5$-free is hamiltonian.

Corollary 7. *If G is a 2-connected $P_5B_2K_{2,3}$-free graph of sufficiently large order, then G is hamiltonian.*

Proof. Such a graph is $K_{1,3}P_5$-free. □

This exhausts all of the possibilities, thus completing the proof of Theorem 3, and characterizing these triples.

4. Questions

Brousek [3] determined all triples XYZ with $X = K_{1,3}$ such that XYZ-free implies the graph is hamiltonian, but none of the pairs of the triple implies hamiltonian. Many such triples were excluded because of small examples. Hence, it is natural to ask the following question.

Question 1. *What are the triples XYZ with $X = K_{1,3}$ such that XYZ-free graphs of sufficiently large order are hamiltonian, but no pair of the triple XYZ has this property?*

The results of this paper determine all triples XYZ, none of which is a generalized claw, such that XYZ-free graphs of sufficiently large order are hamiltonian. Thus, as above, it is natural to ask the next question.

Question 2. *What are the triples XYZ, none of which is a generalized claw, such that all XYZ-free graphs are hamiltonian?*

The one class of forbidden triples XYZ that imply hamiltonicity in a 2-connected graph that has not been studied are those with $X = K_{1,r}$ with $r \geqslant 4$, a generalized claw. Thus, the following question is of interest.

Question 3. *What are the triples XYZ with $X = K_{1,r}$ for $r \geqslant 4$, such that all XYZ-free graphs (of sufficiently large order) are hamiltonian?*

An answer to the previous questions would give an answer to the following general question.

Question 4. *What are the triples XYZ such that all XYZ-free graphs (of sufficiently large order) are hamiltonian?*

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