2-Factors and Forbidden Subgraphs

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Abstract

Every 2-factor of a graph G consists of a spanning collection of vertex disjoint cycles. In particular, a hamiltonian cycle is an example of a 2-factor consisting of precisely one cycle. A characterization has been given of all pairs of forbidden subgraphs that imply a 2-connected graph of order $n \geq 10$ is hamiltonian. We generalize this idea by examining some pairs of forbidden subgraphs that imply a 2-connected graph of order $n > 3k + 15$ contains a 2-factor consisting of k disjoint cycles.

1 Introduction

The use of forbidden subgraphs to obtain classes of graphs possessing special properties has long been studied. For instance, a characterization has been given of all pairs of forbidden subgraphs that imply a 2-connected graph of order $n \geq 10$ is hamiltonian [4]. We generalize this idea by examining some pairs of forbidden subgraphs that imply a 2-connected graph of order $n > 3k+15$ contains a 2-factor consisting of k disjoint cycles. This continues a line of investigation generalizing results on hamiltonian graphs to results on 2-factors. In each case, the conditions sufficient to imply the graph is hamiltonian are actually sufficient to imply it contains a wide range of 2-factors. See for example, [1] or [3]. All graphs in this paper will be simple finite graphs with vertex set $V(G)$ and edge set $E(G)$. For terms or notation not defined here, see [2].

The graph H is called a *subgraph* of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set $S \subseteq V(G)$, we define the *subgraph induced by* S, denoted $\langle S \rangle$, to be the subgraph of G with vertex set S and edge

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set $\{uv \in E(G) | u, v \in S\}$. If a graph G contains no induced subgraph isomorphic to H, we say G is H-free and we call H a *forbidden subgraph* of G.

We say a subgraph H *spans* the graph G if $V(H) = V(G)$. The subgraph H of G is said to be a *2-factor of G* if H spans G and for every $v \in V(H)$, $\deg_H v = 2$. A trivial consequence of the definition is that every 2-factor of a graph G consists of a spanning collection of vertex disjoint cycles. In particular, a hamiltonian cycle is an example of a 2-factor consisting of precisely one cycle.

The following theorem, found in [3], will be useful to us. The result is that in graphs that do not contain the induced subgraph $K_{1,3}$, we can always find a collection of disjoint triangles and thus a collection of disjoint cycles.

Theorem 1 *Let G be a $K_{1,3}$-free graph of order n, and $k \geq 2$ an integer. If $n > 3k + 15$ and $\delta(G) \geq max(3, k)$, then G contains k disjoint triangles.*

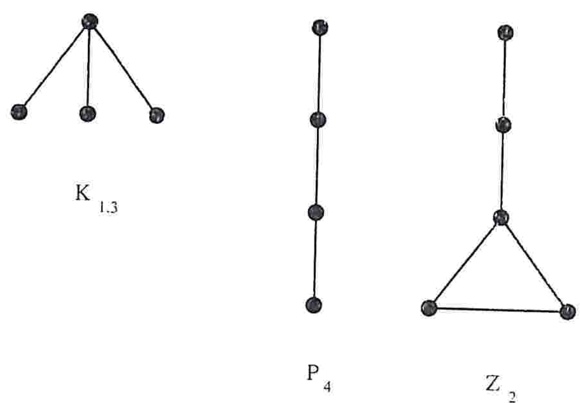

Figure 1: Important forbidden subgraphs.

In Theorem 2 we forbid the graph Z_2 in addition to $K_{1,3}$ and in Theorem 3 we forbid the graph P_4 in addition to $K_{1,3}$. We will show that by forbidding these additional subgraphs in a graph G we can now always find a range of 2-factors in G.

2 Theorems

Theorem 2 *If G is a 2-connected, $K_{1,3}$-free, Z_2-free graph of order $n > 3k + 15$ such that $\delta(G) \geq max(3, k)$, then G contains a 2-factor consisting of k disjoint cycles.*

Proof: We know G is hamiltonian which gives us the case when $k = 1$. We will consider $k \geq 2$. By Theorem 1, we know that G contains k disjoint triangles and hence k disjoint cycles. We choose k such cycles $\{C_1, ..., C_k\}$ such that they include a maximum number of vertices of G. We place an orientation on each of the cycles C_i, $i = 1, ..., k$, and for $w \in V(C_i)$, we let $w^-(w^+)$ be the predecessor (successor) of w on the cycle. Now, suppose that $\bigcup_{i=1}^{k} V(C_i) \neq V(G)$. Then there exists $x \in V(G) \setminus \bigcup V(C_i)$ and $v \in V(C_i)$ for some $i = 1, ..., k$ such that $xv \in E(G)$. Since G is 2-connected there exists an $x - v^-$ path that does not contain v. Let P^- be the shortest such path. Similary, let P^+ be the shortest $x - v^+$ path that does not contain v. Now, from all such x, v, P^-, P^+, choose x, v and $P \in \{P^-, P^+\}$ such that P is as short as possible. Clearly we may assume that $xv^-, xv^+ \notin E(G)$, else we may extend C_i through x, contradicting the maximality of our cycle system. Thus $< \{v, v^-, v^+, x\} > \cong K_{1,3}$, unless we have $v^-v^+ \in E(G)$. We will assume, without loss of generality, that $P = P^-$ and thus let $P = xx_1x_2...x_tv^-$. By our choice of P, $x_i \neq v^+$ for $i \in \{1, 2, ..., t\}$, else we could have chosen a shorter path. Suppose $x_1 \in V(G) \setminus \bigcup V(C_i)$. Then $< \{v^-, v^+, v, x, x_1\} > \cong Z_2$, unless we have at least one of $x_1v^-, x_1v^+, x_1v \in E(G)$. If x_1v^- or $x_1v^+ \in E(G)$, then we may extend C_i through x and x_1. If $x_1v \in E(G)$, then we contradict our choice of P. Thus $x_1 \in V(C_i)$ or $x_1 \in V(C_j), j \neq i$.

Case 1 *Suppose $x_1 \in V(C_i)$.*

If $x_1v^- \in E(C_i)$, then $vxx_1...v^+v^-v$ extends C_i. If $x_1v^+ \in E(C_i)$, then $vxx_1...v^-v^+v$ extends C_i. Thus $x_1^+, x_1^-, v^+, v^- \in V(C_i)$ are all distinct. Furthermore, $< \{x_1, x, x_1^+, x_1^-\} > \cong K_{1,3}$, unless we have $x_1^+x_1^- \in E(G)$. Now we see that $< \{v^-, v^+, v, x, x_1\} > \cong Z_2$, unless we have at least one of x_1v^-, $x_1v^+, x_1v \in E(G)$. If $x_1v^- \in E(G)$, then $vxx_1v^-...x_1^+x_1^-...v^+v$ extends C_i. If $x_1v^+ \in E(G)$. then $vxx_1v^+...x_1^-x_1^+...v^-v$ extends C_i. Thus $x_1v \in E(G)$. If $x_1^-v^+ \in E(C_i)$, then $vxx_1x_1^-x_1^+...v^-v^+v$ extends C_i. Then $< \{x_1, x, v, v^+, v^{++}\} > \cong Z_2$, unless we have at least one of $v^{++}x_1, v^{++}x$, $v^{++}v \in E(G)$. If $v^{++}x_1 \in E(G)$, then $vxx_1v^{++}...x_1^- x_1^+...v^-v^+v$ extends C_i. If $v^{++}x \in E(G)$, then $vxv^{++}...x_1^-x_1\ x_1^+...v^-v^+v$ extends C_i. Thus $v^{++}v \in E(G)$. Also, $< \{x_1^-, x_1^+, x_1, v, v^+\} > \cong Z_2$, unless we have at least one of vx_1^+, vx_1^-, v^+x_1, $v^+x_1^+, v^+x_1^- \in E(G)$. If $vx_1^+ \in E(G)$, then $vxx_1x_1^-...v^+\ v^-...x_1^+v$ extends C_i. If $vx_1^- \in E(G)$, then $vxx_1x_1^+...\ v^-v^+...x_1^-v$ extends C_i. If $v^+x_1 \in E(G)$, then $vxx_1v^+...x_1^-x_1^+\ ...v^-v$ extends C_i.

If $v^+x_1^+ \in E(G)$, then $vxx_1x_1^- ... v^+ \; x_1^+ ... v^-v$ extends C_i. If $v^+x_1^- \in E(G)$, then $vxx_1x_1^+ ... v^-v^+ \; x_1^- ... v^{++}v$ extends C_i. Thus in all cases we extend C_i, a contradiction, and Case 1 is complete. □

Case 2 *Suppose* $x_1 \in V(C_j), j \neq i$.

Subcase 1 *Suppose* $C_j \not\cong K_3$.

We see that $< \{v, v^-, v^+, x\} > \cong K_{1,3}$, unless $v^-v^+ \in E(G)$, and similarly $< \{x_1, x_1^+, x_1^-, x\} > \cong K_{1,3}$, unless $x_1^+x_1^- \in E(G)$. Then $< \{v^-, v^+, v, x, x_1\} > \cong Z_2$, unless we have one of x_1v^-, $x_1v^+, x_1v \in E(G)$. If $x_1v^- \in E(G)$ we can extend our cycle system by letting $C_i' = v^-x_1xvv^+ ... v^-$ and $C_j' = x_1^-x_1^+ ... x_1^-$. So, $x_1v^- \notin E(G)$ and similarly $x_1v^+ \notin E(G)$. Consequently, it must be the case that $x_1v \in E(G)$. Then $< \{v^-, v^+, v, x_1, x_1^+\} > \cong Z_2$, unless we have at least one of $x_1v^-, x_1v^+, x_1^+v, x_1^+v^-, x_1^+v^+ \in E(G)$. If $x_1v^- \in E(G)$, then let $C_i' = vxx_1v^- ... v^+v$ and $C_j' = x_1^+x_1^- ... x_1^+$. If $x_1v^+ \in E(G)$, then let $C_i' = vxx_1v^+ ... v^-v$ and $C_j' = x_1^+x_1^- ... x_1^+$, and we have extended our cycle system. If $x_1^+v \in E(G)$, then $< \{v, v^-, x_1^+, x\} > \cong K_{1,3}$, unless we have at least one of $xx_1^+, x_1^+v^- \in E(G)$. If $xx_1^+ \in E(G)$, then we can easily extend C_j through x. Thus we may assume that $x_1^+v^- \in E(G)$. Also, $\{v, v^+, x_1^+, x\} > \cong K_{1,3}$, unless we have $x_1^+v^+ \in E(G)$. Then $< \{v^-, v^+, x_1^+, x_1, x\} > \cong Z_2$ which is a contradiction that arises from assuming that $x_1^+v \in E(G)$. So, it must now be the case that either $x_1^+v^+ \in E(G)$ or $x_1^+v^- \in E(G)$. By symmetry we may assume, without loss of generality, that $x_1^+v^+ \in E(G)$. As before, $< \{v^-, v^+, v, x_1, x_1^-\} > \cong Z_2$, unless we have at least one of $v^-x_1^-, v^+x_1^- \in E(G)$. If $v^-x_1^- \in E(G)$, then let $C_i' = v^- ... v^+x_1^+ ... x_1^-v^-$ and $C_j' = vxx_1v$, extending the cycle system. Thus $v^+x_1^- \in E(G)$. Then $< \{x_1^-, x_1^+, v^+, v, x\} > \cong Z_2$, unless at least one of $vx_1^-, vx_1^+ \in E(G)$. Say, without loss of generality, that $x_1^-v \in E(G)$. Then $< \{v, x, v^-, x_1^-\} > \cong K_{1,3}$ which contradicts our assumption that our graph G is $K_{1,3}$-free. □

Subcase 2 *Suppose* $C_j \cong K_3$.

We may assume $C_i \cong K_3$, or reversing the roles of v and x_1, we are back to Subcase 2.1. We know that x is adjacent to both C_i and C_j. We first consider the case that x is adjacent to another cycle C_k, $k \neq i, j$. So, there exists $y \in V(C_k) \cap N(x)$ and $C_k \cong K_3$, or we are back to Subcase 2.1. Then $< \{x, v, x_1, y\} > \cong K_{1,3}$, unless, without loss of generality, $vx_1 \in E(G)$. Then $< \{v^-, v^+, v, x_1, x_1^-\} > \cong Z_2$, unless at least one of $x_1v^-, x_1v^+, x_1^-v, x_1^-v^-, x_1^-v^+ \in E(G)$. If $x_1v^- \in E(G)$, then either $< \{x_1, v^-, x_1^-, x\} > \cong K_{1,3}$ or we can extend either C_i or C_j through x, or $v^-x_1^- \in E(G)$. Now, $< \{x_1, v^-, x_1^+, x\} > \cong K_{1,3}$, unless $v^-x_1^+ \in E(G)$.

Then $< \{x_1^+, x_1^-, v^-, v, x\} > \cong Z_2$, unless we have, without loss of generality, $vx_1^- \in E(G)$. Now, either $< \{v, v^+, x, x_1^-\} > \cong K_{1,3}$ or we can extend a cycle through x, or $v^+x_1^- \in E(G)$. Now let $C_i' = v^-v^+x_1^-x_1^+v^-$ and $C_j' = vxx_1v$. This is a contradiction to the maximality of our cycle system that arises from the assumption that $x_1v^- \in E(G)$. With similar arguments we conclude that $x_1v^+, vx_1^- \notin E(G)$. Thus we may say, without loss of generality, that $x_1^-v^- \in E(G)$. Again, $< \{v^-, v^+, v, x_1, x_1^+\} > \cong Z_2$, unless we have at least one of $v^+x_1^+, v^-x_1^+ \in E(G)$. If $v^+x_1^+ \in E(G)$, then let $C_i' = v^+v^-x_1^-x_1^+v^+$ and $C_j' = vxx_1v$. Thus $v^-x_1^+ \in E(G)$. Then either $< \{x_1^-, x_1^+, v^-, v, x\} > \cong Z_2$ or we can extend a cycle through x, unless $vx_1^+ \in E(G)$. Then $< \{v, x_1^+, v^+, x\} > \cong K_{1,3}$, and again we are done, unless $x_1^+v^+ \in E(G)$. Finally, let $C_i' = v^-v^+x_1^+x_1^-v^-$ and $C_j' = vxx_1v$. Thus we have extended our maximum cycle system, contradicting the assumption that x is adjacent to some cycle C_k, $k \neq i, j$. Hence, it must be the case that x is not adjacent to any other cycle. In other words, $N(x) \cap \bigcup V(C_i) = \{v, x_1\}$. Since $\delta(G) \geq 3$, then x must have a neighbor $y \in V(G) \setminus \bigcup V(C_i)$. Then we have $< \{v^-, v^+, v, x, y\} > \cong Z_2$ unless $yv \in E(G)$. Similarly, $yx_1 \in E(G)$. Now, again because $\delta(G) \geq 3$, v^+ must have a neighbor besides v and v^-. This neighbor cannot be x or y. Suppose there exists $w \in N(v^+) \setminus \{\bigcup V(C_i), y, x\}$. Then $< \{y, x, v, v^+, w\} > \cong Z_2$, unless $vw \in E(G)$. But now we let $C_i' = vv^-v^+wv$, which returns us to Subcase 2.1. Consequently, $N(v^+) \subseteq \bigcup V(C_i)$. Suppose first that $N(v^+) \cap V(C_j) \neq \emptyset$. Then $v^+x_1 \in E(G)$. If not, then, without loss of generality, $v^+x_1^- \in E(G)$. But then $< \{y, x, x_1, x_1^-, v^+\} > \cong Z_2$ unless $v^+x_1 \in E(G)$. As a result of the fact that $v^+x_1 \in E(G)$, we get that $< \{x_1, x, v^+, x_1^+\} > \cong K_{1,3}$ unless $x_1^+v^+ \in E(G)$ and $< \{x_1, x, v^+, x_1^-\} > \cong K_{1,3}$ unless $x_1^-v^+ \in E(G)$. In addition, $< \{y, x, x_1, v^+, v^-\} > \cong Z_2$ unless $v^-x_1 \in E(G)$. Then let $C_i' = v^-vyxx_1v^-$ and $C_j' = v^+x_1^-x_1^+v^+$, extending our maximum cycle system and contradicting our assumption. Hence, it must be the case that $N(v^+) \cap V(C_j) = \emptyset$. So, v^+ must have a neighbor $w \in V(C_k)$ such that $k \neq i, j$. Let w^+ and w^- be the neighbors of w on C_k. Now v^+w^+ and v^+w^- cannot be edges in G or we can extend C_k through v^+ and let $C_i' = vyxv$. In this way we have included x and y in our cycle system losing only v^-, thus contradicting the assumption of maximality. This forces $w^-w^+ \in E(G)$ or $< \{w, v^+, w^-, w^+\} > \cong K_{1,3}$. Similarly, v^-w^- and v^-w^+ cannot be edges in our graph G, or we can extend C_k through both v^- and v^+ and let $C_i' = vyxv$, again a contradiction. But this forces $wv^- \in E(G)$ or we have $< \{w^-, w^+, w, v^+, v^-\} > \cong Z_2$. If $C_k \not\cong K_3$, we are returned to Subcase 2.1 by letting $C_i' = vv^-wv^+v$, so we assume $C_k \cong K_3$. Finally we see that $< \{w^+, w^-, w, v^-, v\} > \cong Z_2$ unless vw^-, vw^+, or $vw \in E(G)$. The remaining conclusions are drawn from some previous results and the fact that x has no adjacencies on the cycle C_k. If $vw^- \in E(G)$, then $<$

$\{v, x, v^-, w^-\} >\cong K_{1,3}$. If $vw^+ \in E(G)$, then $< \{v, x, v^-, w^+\} >\cong K_{1,3}$. If $vw \in E(G)$, then $< \{w^+, w^-, w, v, x\} >\cong Z_2$. Thus, we have contradicted the assumption that G is $\{K_{1,3}, Z_2\}$-free and hence, the theorem is proved. □

Theorem 3 *If G is a 2-connected, $K_{1,3}$-free, P_4-free graph of order $n > 3k + 15$ such that $\delta(G) \geq max(3, k)$, then G contains a 2-factor consisting of k disjoint cycles.*

Proof: We know G is hamiltonian which gives us the case when $k = 1$. We will consider $k \geq 2$. By Theorem 1 we know that G contains k disjoint triangles and hence k disjoint cycles. We choose k such cycles $\{C_1, ..., C_k\}$ such that they include a maximum number of vertices of G. We place an orientation on each of the cycles C_i, $i = 1, ..., k$, and for $w \in V(C_i)$, we let $w^-(w^+)$ be the predecessor (successor) of w on the cycle. Now, suppose that $\bigcup_{i=1}^{k} V(C_i) \neq V(G)$. Then if we let $R = V(G) \setminus \bigcup V(C_i)$ we know, by connectivity, that there exists $x \in R$ and $v \in V(C_i)$, for some $i \in \{1, ..., k\}$, such that $xv \in E(G)$. Now, for $w \in V(C_i)$, $w \neq v$, we define S_w to be the segment of the cycle C_i from v^+ to w^-. Suppose there is a path P from the vertex x to $w \in V(C_i)$, $w \neq v$, such that the interior of P is contained in R. (We will say that such a path is in R). We want to choose such a path P so that there are no paths in R that originate at x and end at a vertex in S_w. In addition, from among all such paths, we choose P to be as short as possible. Clearly $vw \notin E(C_i)$ or we can extend C_i through x. Suppose $xw \notin E(G)$. Then let w_1 and w_2 be the first two vertices encountered when travelling along the path P from w to x. (Note that w_2 could be x) Then $< \{w^-, w, w_1, w_2\} >\cong P_4$ unless we have one of $w^-w_1, w^-w_2, ww_2 \in E(G)$. If $w^-w_1, w^-w_2 \in E(G)$, we have a path in R from x to a vertex in S_w, a contradiction. If $ww_2 \in E(G)$, we have shortened the path from x to w, another contradiction. Consequently, $xw \in E(G)$. Clearly we may assume that $xv^-, xv^+ \notin E(G)$, else we may extend C_i through x. Thus $< \{v, v^-, v^+, x\} >\cong K_{1,3}$ unless $v^-v^+ \in E(G)$. If $wv^- \in E(C_i)$, then $vxw...v^+v^-v$ extends C_i. If $wv^+ \in E(C_i)$, then $vxw...v^-v^+v$ extends C_i. Thus, the vertices w^+, w^-, v^+, v^- are all distinct. Furthermore, $< \{w, w^-, w^+, x\} >\cong K_{1,3}$, unless we have $w^+w^- \in E(G)$. If $v^+w^- \in E(C_i)$, then $vxww^-w^+...v^-v^+v$ extends C_i. So, v^{++} and w^- are distinct. Thus, $< \{x, v, v^+, v^{++}\} >\cong P_4$ unless one of $xv^+, xv^{++}, vv^{++} \in E(G)$. But $xv^+, xv^{++} \notin E(G)$ by our choice of P, so it must be the case that $vv^{++} \in E(G)$. By a similar argument, we get that $vy \in E(G)$ for all $y \in S_w$. But now, $vxww^+...v^-v^+...w^-v$ extends the cycle C_i, a contradiction. So there are no paths from x to C_i in R except for the edge xv. By the 2-connectedness of our graph G, we know there must be a path from x to at least one other cycle C_j, $j \neq i$, that does not contain the

vertex v. Take the shortest such path $P = xx_1x_2...x_rz$ such that $x_i \in R$ for all $i \in \{1, ..., r\}$ and $z \in V(C_j)$. If $|V(P)| \geq 4$, we must get a shorter path in the process of insuring that we do not have an induced P_4. Either way we arrive at a contradiction. So, if $xz \notin E(G)$, then $P = xx_1z$. Thus, $< \{x, x_1, z, z^-\} >\cong P_4$ unless one of $xz^-, x_1z^- \in E(G)$. If $xz^- \in E(G)$ we get a shorter path from x to C_j and if $x_1z^- \in E(G)$ we can extend C_j through x, both contradictions. Consequently, $xz \in E(G)$. So, x has exactly one neighbor on each of l different cycles, $l \geq 2$. The remainder of the proof is considered in two cases, based on whether or not x has any adjacencies off of the cycles.

Case 1 *The vertex x has at least one adjacency $y \in R$.*

Let $xv \in E(G)$ for $v \in V(C_i)$ and for $i \in \{1, ..., k\}$. Then $< \{y, x, v, v^-\} >\cong P_4$ unless one of $xv^-, yv^-, yv \in E(G)$. We know $xv^- \notin E(G)$, and if $yv^- \in E(G)$ then $vv^+...v^-yxv$ extends the cycle C_i. So, it must be the case that $yv \in E(G)$. Recall that x has exactly one neighbor on each of l different cycles, $l \geq 2$. Using the same argument for each neighbor of x in $\bigcup V(C_i)$ we can see that $N(x) \cap \bigcup V(C_i) = N(y) \cap \bigcup V(C_i)$.

Subcase 1 *There exists C_i such that $N(x) \cap V(C_i) \neq \emptyset$ and $|V(C_i)| \geq 4$.*

We will assume, without loss of generality, that $i = 1$. Let $v \in V(C_1)$ such that $xv \in E(G)$. Then, since x cannot be adjacent to any other element of $V(C_1)$, $< \{x, v, v^+, v^{++}\} >\cong P_4$ unless $vv^{++} \in E(G)$. By a similar argument, we can show that v is actually adjacent to every vertex of C_1. Now suppose we have $a, b \in V(C_1)$ such that $ab \notin E(G)$. Then $< \{v, x, a, b\} >\cong K_{1,3}$, a contradiction, which means that C_1 is a clique. Similarly, all cycles that are adjacent to x must be cliques. We know x is adjacent to at least one other cycle and, without loss of generality, we will assume x is adjacent to C_2. Let $w \in V(C_2)$ such that $xw \in E(G)$. Then $< \{w^-, w, x, v\} >\cong P_4$ unless one of $vw^-, vw \in E(G)$. If $vw^- \in E(G)$, we let $C_1' = v^-v^+...v^-$ and $C_2' = vxyww^+...w^-v$, extending the cycle system and giving us a contradiction. So, it must be the case that $vw \in E(G)$. Now we have that $< \{v^+, v, w, w^-\} >\cong P_4$ unless one of $v^+w^-, v^+w \in E(G)$. Suppose $v^+w^- \in E(G)$. Then $< \{v^-, v, w, w^+\} >\cong P_4$ unless one of $v^-w^+, v^-w, vw^+ \in E(G)$. If $v^-w^+ \in E(G)$, let $C_1' = vxywv$ and $C_2' = v^-w^+...w^-v^+...v^-$. If $v^-w \in E(G)$, let $C_1' = vxyv$ and $C_2' = v^-ww^+...w^-v^+...v^-$. If $vw^+ \in E(G)$, let $C_1' = wxyw$ and $C_2' = vw^+...w^-v^+...v^-v$. In all cases we get a contradiction by extending the cycle system through x and y. Hence, it follows that $v^+w^- \notin E(G)$ and $v^+w \in E(G)$. But then $< \{w, w^-, v^+, x\} >\cong K_{1,3}$, another contradiction. □

We must now assume that any cycle adjacent to x is a triangle.

Subcase 2 *If* $N(x) \cap V(C_i) \neq \emptyset$, *then* $C_i \cong K_3$.

Let $v, w \in N(x) \cap \bigcup V(C_i)$. Recall that $y \notin \bigcup V(C_i)$ but that xy, yv, and $yw \in E(G)$. Assume, without loss of generality, that $v \in V(C_1)$ and $w \in V(C_2)$. Thus $< \{v, x, w, w^-\} > \cong P_4$ unless one of $vw, vw^- \in E(G)$. If $vw^- \in E(G)$, then $< \{v, x, w^- v^+\} > \cong K_{1,3}$ unless $v^+w^- \in E(G)$ and $< \{x, v, w^-, w^+\} > \cong P_4$ unless $vw^+ \in E(G)$. If we let $C_1' = xywx$ and $C_2' = vw^+w^-v^+v^-v$, then we have extended our cycle system giving us a contradiction. Therefore, it must be the case that $vw^- \notin E(G)$ and $vw \in E(G)$. In addition, we can show by a similar argument that $wv^+ \notin E(G)$. But that means that $< \{v^+, v, w, w^-\} > \cong P_4$ unless $v^+w^- \in E(G)$. Now, $< \{v^-, v^+, w^-, w^+\} > \cong P_4$ unless one of $v^-w^-, v^+w^+, v^-w^+ \in E(G)$. If $v^-w^+ \in E(G)$, we let $C_1' = vxywv$ and $C_2' = v^-v^+w^-w^+v^-$, extending our cycle system and again giving us a contradiction. So either $v^-w^- \in E(G)$ or $v^+w^+ \in E(G)$. Without loss of generality, assume that $v^-w^- \in E(G)$. We have shown that $v^-w^+, vw^- \notin E(G)$. So, $< \{v, v^-, w^-, w^+\} > \cong P_4$ unless $vw^+ \in E(G)$. But then we let $C_1' = xywx$ and $C_2' = vw^+w^-v^+v^-v$, which again extends our cycle system and gives us a contradiction. So in all cases where x has an adjacency outside the cycle system, we are able to extend the cycle system which contradicts the maximality of the cycle system. □

Case 2 *Suppose* $N(x) \cap R = \emptyset$.

Since $\delta(G) \geq k$, x must have exactly one neighbor on each of the k cycles. Let $v \in N(x) \cap V(C_1)$ and $w \in N(x) \cap V(C_2)$. Now since $\delta(G) \geq 3$ and $d(x) = k$ we know that $k \geq 3$. Thus, if no neighbors of x are joined by an edge, we have an induced $K_{1,3}$ centered at x. We can then assume, without loss of generality, that $vw \in E(G)$. Now, if $v^+w^+ \notin E(G)$ then $< \{v^+, v, w, w^+\} > \cong P_4$ unless one of $vw^+, wv^+ \in E(G)$. But then $< \{v, x, w^+, v^+\} > \cong K_{1,3}$ if $vw^+ \in E(G)$ and $< \{w, x, v^+, w^+\} > \cong K_{1,3}$ if $wv^+ \in E(G)$. In each case we get a contradiction and so we conclude that $v^+w^+ \in E(G)$. Similarly, $v^-w^- \in E(G)$. If $C_1 \cong K_3$ and $C_2 \cong K_3$, then let $C_1' = vwxv$ and $C_2' = v^-w^-w^+v^+v^-$, which extends our cycle system, a contradiction. So, without loss of generality, assume that $|V(C_1)| \geq 4$. We see that $< \{x, v, v^+, w^+\} > \cong P_4$ unless $vw^+ \in E(G)$. But, we can now let $C_1' = v^-v^+ ... v^-$ and $C_2' = xvw^+ ... w^-wx$, which again extends our cycle system. Thus, we have in all cases contradicted the assumption of maximality of our cycle system and hence, the theorem is proved. □

References

[1] S. Brandt, G. Chen, R.J. Faudree, R.J. Gould and L. Lesniak. On the number of cycles in a 2-factor, J. Graph Theory, Vol. 24, No. 2 (1997), 165–173.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Chapman and Ha London, 1996.

[3] G. Chen, J.R. Faudree, R.J. Gould and A. Saito, Cycles in 2-facto of claw-free graphs, Discussiones Mathematicae - Graph Theory, appear.

[4] R.J. Faudree and R.J. Gould, Characterizing forbidden pairs for ham tonian properties, Discrete Math 173(1997) 45–60.