Journal of Combinatorial Theory, Series B **83**, 213–220 (2001)
doi:10.1006/jctb.2001.2048, available online at http://www.idealibrary.com on IDEAL®

Partitioning Vertices of a Tournament into Independent Cycles

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Received October 7, 1997; published online July 24, 2001

Let k be a positive integer. A strong digraph G is termed k-connected if the removal of any set of fewer than k vertices results in a strongly connected digraph. The purpose of this paper is to show that every k-connected tournament with at least $8k$ vertices contains k vertex-disjoint directed cycles spanning the vertex set. This result answers a question posed by Bollobás. © 2001 Elsevier Science

This article will generally follow the notation and terminology defined in [1]. A digraph is called *strongly connected* or *strong* if for every pari of vertices u and v there exists a directed path from u to v and a directed path from v to u. Let k be a positive integer. A digraph G is *k-connected* if the removal of any set of fewer than k vertices results in a strong digraph. A tournament with n vertices will also be called an *n-tournament*.

It is well-known that every tournament contains a hamiltonian path and every strong tournament contains a hamiltonian cycle. Reid [2] proved that if T is a 2-connected n-tournament, $n \geqslant 6$, that is, T is not the 7-tournament that contains no transitive subtournament with 4 vertices (i.e., the quadratic residue 7-tournament), then T contains two vertex-disjoint cycles

[1] Research partially supported under N.S.F. Grant DMS 0070059.
[2] Research partially supported under O.N.R. Grant N00014-97-1-0499.

0095-8956/01 $35.00

spanning $V(T)$. In fact, he showed that one cycle can be taken to be a triangle. This result established an affirmative answer (for $r=s=1$) to the following problem asked by Thomassen (see [3]): If r and s are positive integers, does there exist a (least) positive integer $m=m(r,s)$ so that all but a finite number of m-connected tournaments can be partitioned into an r-connected subtournament and an s-connected subtournament? Song [4] was able to show that if T is a 2-connected n-tournament with $n \geqslant 6$ then the vertices of T can be partitioned into two cycles of lengths s and $n-s$ for any integer s with $3 \leqslant s \leqslant n-3$, unless T is the 7-tournament described above. The following problem was posed by Bollobás (see [2]) for tournaments.

Problem 1. If k is a positive integer, what is the least integer $g(k)$ so that all but a finite number of $g(k)$-connected tournaments contain k vertex-disjoint cycles that span $V(T)$?

Reid observed that $g(k)$ exists and $g(k) \leqslant 3k-4$ for $k \geqslant 2$ as follows: Recall that $g(1)=1$ and $g(2)=m(1,1)=2$. If T is $(g(k-1)+3)$-connected, then the removal of a triangle leaves a $g(k-1)$-connected tournament that can be expressed as $k-1$ nontrivial vertex-disjoint cycles; that is, $g(k) \leqslant g(k-1)+3$. Thus, $g(3) \leqslant 5$, and, in general, $g(k) \leqslant 3k-4$. The following example shows that $g(k) \geqslant k$.

Let $n \geqslant 3k$. Let T be an n-tournament with $V(T)=\{v_1, v_2, ..., v_n\}$, where v_i dominates v_j for all $1 \leqslant i \leqslant j \leqslant n$ except when $1 \leqslant i \leqslant k$ and $n-k+1 \leqslant j \leqslant n$ (in which case v_j dominates v_i). If S is any set of fewer than k vertices, then $T-S$ is strongly connected; that is, T is k-connected. Clearly, any nontrivial cycle in T must use an arc $\overrightarrow{v_j v_i}$ for some $1 \leqslant i \leqslant k$ and some $n-k+1 \leqslant j \leqslant n$, so that T contains at most k vertex-disjoint cycles.

The main result of this article, stated below, shows that $g(k)=k$.

THEOREM 1. *Every k-connected n-tournament T with $n \geqslant 8k$ contains k vertex-disjoint cycles that span $V(T)$.*

In [4], Song posed the following problem.

Problem 2. If k is a positive integer, what is the least integer $f(k)$ so that all but a finite number of $f(k)$-connected tournaments contain k vertex-disjoint cycles of lengths $n_1, n_2, ..., n_k$ where $n=n_1+n_2+\cdots+n_k$ and $n_i \geqslant 3$ for all $i=1, 2, ..., k$?

Clearly, $f(1)=g(1)=1$. Song showed that $f(2)=g(2)=2$. Clearly, $f(k) \geqslant g(k)$ holds for every k. Song conjectured that $f(k)=g(k)$.

Let T be a tournament. The arc set of T will be denoted by $E(T)$. If $\overrightarrow{uv}$ is an arc in T, then *udominates* v and v *is dominated* by u. A set $A \subseteq V(T)$ *dominates* a set $B \subseteq V(T)$ if every vertex of A dominates every vertex of B. If $A = \{x\}$, we say that x *dominates* B. For any $X \subseteq V(T)$, let $T[X]$ denote the subtournament induced by X.

Let T be a tournament and let C be a cycle in T. For every vertex $v \in V(C)$, let v_C^+ denote the successor of v on C and let v_C^- denote the predecessor of v on C. If no confusion arises, v^+ and v^- will be used to denote v_C^+ and v_C^-, respectively. Let X be a cycle or a path of T and let u and v be two vertices on X (u, v are in that order along X if X is a path). We define $X[u, v]$ as the subpath of X from u to v. For any $u \notin V(C)$, if u is dominated by a vertex $x \in V(C)$ and u dominates x^+, then $ux^+C[x^+, x]\, xu$ is a cycle longer than C. In this case, we say that u *can be inserted into* C. So, if u cannot be inserted into a cycle C, then either u dominates $V(C)$ or $V(C)$ dominates u. In the case, we call C an *out-cycle* of u while in the second case we call C an *in-cycle* of u. The following lemma will be used in the proof of Theorem 1.

Lemma 1. *Every k-connected tournament with $n \geqslant 5k-3$ vertices and $k \geqslant 2$ contains k vertex-disjoint cycles.*

Proof. To the contrary, let k ($\geqslant 2$) be the smallest positive integer such that there is a k-connected tournament T with $n \geqslant 5k-3$ vertices, which does not contain k vertex-disjoint cycles. By the minimality of k and the fact that every strong tournament has a cycle, T contains $k-1$ vertex-disjoint cycles. Since every cycle of length at least 4 contains a chord, T contains $k-1$ vertex-disjoints triangles, say, $T_1, T_2, ..., T_{k-1}$. Let $H = T - \bigcup_{i=1}^{k-1} V(T_i)$. Since H does not contain a cycle, H is a transitive tournament. Let $P = v_1 v_2 \cdots v_m$ be the unique hamiltonian path in H. Since H is transitive, then $\overrightarrow{v_i v_j} \in E(T)$ for any $1 \leqslant i < j \leqslant m$.

Let $F = \{v_1, v_2, ..., v_k\}$ and $B = \{v_{m-k+1}, v_{m-k+2}, ..., v_m\}$. Since $m \geqslant (5k-3) - 3(k-1) = 2k$, then $F \cap B = \varnothing$. Since T is k-connected, there exist k vertex-disjoint paths $P_1, P_2, ..., P_k$ from B to F. Clearly, these paths plus the appropriate arcs from F to B form k vertex-disjoint cycles. ∎

Proof of Theorem 1. Let T be a k-connected tournament with $n \geqslant 8k$ vertices. Since $8k \geqslant 5k-3$, T contains k vertex-disjoint cycles by Lemma 1. Let $C_1, C_2, ..., C_k$ be k vertex-disjoint cycles of T such that $\sum_{i=1}^{k} |V(C_i)|$ is maximum. Let $\mathscr{C} = \{C_1, C_2, ..., C_k\}$. To the contrary, then, we may assume that $\sum_{i=1}^{k} |V(C_i)| < n$. Let $H = T - \bigcup_{i=1}^{k} V(C_i)$. Since H is a tournament, H has a hamiltonian path. Let $P = v_1 v_2 \cdots v_m$ be a hamiltonian path in H. The linear order of $v_1, v_2, ..., v_m$ will play a role in our proof.

For each $v_i \in V(H)$ $(1 \leqslant i \leqslant m)$ and each $C_\ell \in \mathscr{C}$ $(1 \leqslant \ell \leqslant k)$, since v_i cannot be inserted into C_ℓ, C_ℓ is either an in-cycle of v_i or an out-cycle of v_i. We partition $\mathscr{C}$ into two sets $\mathscr{I}_i$ and $\mathscr{O}_i$ for each $i = 1, 2, ..., m$ as

$$\mathscr{I}_i = \{ C_\ell \mid C_\ell \text{ is an in-cycle of } v_i \},$$

$$\mathscr{O}_i = \{ C_\ell \mid C_\ell \text{ is an out-cycle of } v_i \}.$$

For any two vertices v_i, $v_j \in V(H)$ and a cycle $C_\ell \in \mathscr{C}$, if $i < j$ and C_ℓ is an out-cycle of v_j, then C_ℓ is also an out-cycle of v_i; otherwise, let x and x^+ be two consecutive vertices on C_ℓ. The cycle $P[v_i, v_j]\, C_\ell[x^+, x]\, v_i$ is longer than C_ℓ which leads to a contradiction of the maximality of $\sum_{s=1}^{k} |V(C_s)|$. Thus, $\mathscr{O}_j \subseteq \mathscr{O}_i$. As a consequence,

$$\mathscr{O}_m \subseteq \mathscr{O}_{m-1} \subseteq \cdots \subseteq \mathscr{O}_1 \qquad \text{and} \qquad \mathscr{I}_m \supseteq \mathscr{I}_{m-1} \supseteq \cdots \supseteq \mathscr{I}_1.$$

CLAIM 1. *If S is a strong subtournament of H, then $\mathscr{I}_i = \mathscr{I}_j$ and $\mathscr{O}_i = \mathscr{O}_j$ for any two vertices v_i and $v_j \in V(S)$.*

Proof. Suppose, to the contrary, that there is a cycle $C_\ell \in \mathscr{C}$ such that $C_\ell \in \mathscr{I}_i$ and $C_\ell \in \mathscr{O}_j$. Let $P[v_i, v_j]$ be a path in S connecting v_i and v_j and let x be an arbitrary vertex on C_ℓ. Then, the cycle $P[v_i, v_j]\, C[x^+, x]\, v_i$ is longer than C, a contradiction. ∎

We will show that there exist k vertex disjoint cycles which contain all vertices of $\bigcup_{i=1}^{k} V(C_i)$ and v_m, which produces a contradiction. For convenience, let $\mathscr{I} = \mathscr{I}_m$, $\mathscr{O} = \mathscr{O}_m$, and $H^* = H - v_m$.

CLAIM 2. *$\sum_{C_i \in \mathscr{O}} |V(C_i)| \geqslant k$ and $\sum_{C_j \in \mathscr{I}} |V(C_j)| \geqslant k$.*

Proof. Let S be the strong component containing v_m in H. (Note that S could be $\{v_l\}$.) Since $P = v_1 v_2 \cdots v_m$ is a hamiltonian path in H, $V(H) - V(S)$ dominates $V(S)$. By Claim 1, $\bigcup_{C_i \in \mathscr{I}} V(C_i)$ dominates $V(S)$. Also, S is the strong component of v_m in $T[V(H) \cup (\bigcup_{C_j \in \mathscr{I}} V(C_j))] = T - \bigcup_{C_i \in \mathscr{O}} V(C_i)$. Thus, $\sum_{C \in \mathscr{O}_m} |V(C)| \geqslant k$ is k-connected.
Since v_m dominates $\bigcup_{C_i \in \mathscr{O}} V(C_i)$, $V(H)$ dominates $\bigcup_{C_i \in \mathscr{O}} V(C_i)$. As S is the strong component of v_m in $T[V(H) \cup (\bigcup_{C_j \in \mathscr{O}} V(C_j))] = T - \bigcup_{C_i \in \mathscr{I}} V(C_i)$, we see that $\sum_{C_i \in \mathscr{I}_m} |V(C_i)| \geqslant k$. ∎

Without loss of generality, we may assume that $\sum_{C_i \in \mathscr{I}} |V(C_i)| \geqslant \sum_{C_j \in \mathscr{O}} |V(C_j)|$. Otherwise, we may reverse the directions of all arcs of T and exchange the roles of v_1 and v_m and consider $\mathscr{O}_1$. Since $\mathscr{O}_1 \supseteq \mathscr{O}_m$, $\sum_{C_i \in \mathscr{O}_1} |V(C_i)| \geqslant \sum_{C_j \in \mathscr{I}_1} |V(C_j)|$.

Since $|V(T)| = n \geqslant 8k$, we have that

$$\sum_{C_i \in \mathscr{I}} |V(C_i)| + |V(H^*)| \geqslant 4k.$$

Define

$$R = \left\{ y \in \bigcup_{C_i \in \mathscr{I}} V(C_i) : \overrightarrow{xy} \in E(T) \text{ for some } x \in \bigcup_{C_j \in \mathscr{O}} V(C_j) \right\}$$

and

$$U = \bigcup_{C_i \in \mathscr{I}} V(C_i) - R.$$

That is, any $y \in R$ is dominated by some vertices in $\bigcup_{C_j \in \mathscr{O}} V(C_j)$ and any $u \in U$ dominates all vertices in $\bigcup_{C_j \in \mathscr{O}} V(C_j)$ for all $u \in U$.

CLAIM 3. *For each $C_i \in \mathscr{I}$, $|V(C_i) \cap R| \leqslant 3$ and equality holds only when C_i is a triangle.*

Proof. Let $x \in C_s \in \mathscr{O}$ and $y \in V(C_t) \cap R$ such that $\overrightarrow{xy} \in E(T)$. If $\overrightarrow{y^- z} \in E(T)$ for some $z \in V(C_t) - \{y, y^-\}$, the cycles $v_m C_s[x^+, x]\, C_t[y, z^-]\, v_m$ and $C_t[z, y^-]\, z$ plus the remaining $k-2$ cycles of $\mathscr{C}$ contradict the maximality of $\sum_{i=1}^{k} |V(C_i)|$. Hence, $V(C_t) - \{y, y^-\}$ dominates y^-. Suppose w is another vertex in $R \cap V(C_t)$. Similarly, we have that $V(C_t) - \{w, w^-\}$ dominates w^-. If w and y are not two consecutive vertices on C_t, then w^- and y^- dominate each other, a contradiction. Thus, every two vertices in $R \cap V(C_t)$ must be consecutive vertices on C_t. Consequently, $|R \cap V(C_t)| \leqslant 3$ and the equality holds only when C_t is a triangle. ▮

Since $\sum_{C \in \mathscr{I}} |V(C)| + |H^*| \geqslant 4k$ and $|R \cap V(C_i)| \leqslant 3$ for each $C_i \in \mathscr{I}$, then $|U \cup H^*| = |U| + |V(H^*)| \geqslant k$ follows. Since T is k-connected and

$$\mathscr{O} = \mathscr{O}_m \subseteq \mathscr{O}_{m-1} \cdots \subseteq \mathscr{O}_1,$$

there exist k vertex-disjoint paths, $P_i[x_i, y_i]$ $(i = 1, 2, ..., k)$, such that x_i is in some cycle in $\mathscr{O}$ and $y_i \in U \cup V(H^*)$ and all internal vertices of the path are in $R \cup \{u\}$. Furthermore, we can assume that all internal vertices of the path $P_i[x_i, y_i]$ are in R. Otherwise, suppose that $v_m \in V(P_i[x_i, y_i])$ for some $i = 1, ..., k$. Let u be the predecessor of v_m on $P_i[x_i, y_i]$ and w be the successor of v_m on $P_i[x_i, y_i]$. We can suppose that u is in $\mathscr{I}$ and b is in H^*. So the arc uw belongs to T, and thus v_m can be omitted in the path $P_i[x_i, y_i]$.

For each $P_i[x_i, y_i]$, we define a *hop* to be two consecutive vertices u and u^+ on $P_i[x_i, y_i]$ such that u and u^+ are not consecutive vertices on the same cycle of $\mathscr{I}$. Let h_i be the number of hops on $P_i[x_i, y_i]$. We choose k vertex-disjoint paths $P_1[x_1, y_1]$, $P_2[x_2, y_2]$, ..., $P_k[x_k, y_k]$ such that:

1. For each i, $x_i \in \bigcup_{C_i \in \mathscr{O}} V(C_i)$, $y_i \in U \cup V(H^*)$, and all internal vertices are in R.
2. Under Condition 1, $\sum_{i=1}^{k} h_i$ is minimum.
3. Under Conditions 1 and 2, $\sum_{i=1}^{k} |V(P_i[x_i, y_i])|$ is maximum.

A cycle $C_i \in \mathscr{I}$ is called a *used in-cycle* with respect to $P_1[x_1, x_k]$, ..., $P_k[x_k, y_k]$ if C_i contains some vertices in $\bigcup_{\ell=1}^{k} V(P_\ell(x_\ell, y_\ell])$, otherwise it is called an *unused in-cycle*. Similarly, a cycle $C_i \in \mathscr{O}$ is called a *used out-cycle* if it contains some vertices in $\{x_1, x_2, ..., x_k\}$, otherwise it is called an *unused out-cycle*. All used in-cycles and out-cycles are called *used cycles* and all unused in-cycles and out-cycles are called *unused cycles*.

CLAIM 4. *For each used in-cycle C_j, $V(C_j) - \bigcup_{i=1}^{k} V(P_i[x_i, y_i]) \subseteq R$.*

Proof. Suppose, to the contrary, that there is a vertex $u \in U \cap (V(C_j) - \bigcup_{i=1}^{k} V(P_i[x_i, y_i]))$. Let u^* be the first vertex in $\bigcup_{i=1}^{k} V(P_i(x_i, y_i])$ along C_j in the reverse direction from u. Suppose that $u^* \in V(P_i(x_i, y_i])$. Let $P_i^* = P_i[x_i, u^*]\ C_j[u^*, u]$. If $u^* \neq y_i$, the number of hops on P_i^* is less than h_i, a contradiction to the minimality of $\sum_{i=1}^{k} h_i$. If $u^* = y_i$, the number of hops on $P_i^* = h_i$, but P_i^* is longer than $P_i[x_i, y_i]$, a contradiction to the maximality of $\sum_{i=1}^{k} |V(P_i[x_i, y_i])|$. ∎

For each $i = 1, ..., k$, let C_i^{in} be the cycle in $\mathscr{I}$ containing y_i and C_i^{out} be the cycle in $\mathscr{O}$ containing x_i. Starting from x_i, let x_i^* be the first vertex along cycle C_i^{out} in the reverse direction from x_i, such that $(x_i^*)^- \in \{x_1, x_2, ..., x_k\}$. For each $i = 1, 2, ..., k$, let

$$Q_i = C_i^{out}[x_i^*, x_i]\ P_i[x_i, y_i].$$

Clearly, all vertices in used out-cycles are in $\bigcup_{i=1}^{k} V(Q_i[x_i^*, y_i])$. By Claim 4, we choose k vertex-disjoint paths $Q_1[x_1, y_1]$, $Q_2[x_2, y_2]$, ..., $Q_k[x_k, y_k]$ such that

1. For each $i = 1, 2, ..., k$, $x_i \in \bigcup_{C_i \in \mathscr{O}} V(C_i)$, $y_i \in U \cup V(H^*)$, and all internal vertices are in $\bigcup_{C_j \in \mathscr{C}} V(C_j)$.
2. For each used in-cycle C_j, $V(C_j) - \bigcup_{i=1}^{k} V(Q_i[x_i, y_i]) \subseteq R$.
3. For each used out-cycle C_j, $V(C_j) \subseteq \bigcup_{i=1}^{k} V(Q_i[x_i, y_i])$.
4. Under the above three conditions, $\sum_{i=1}^{k} |V(Q_i[x_i, y_i])|$ is maximum.

Let r be the number of unused cycles with respect to $Q_1[x_1, y_1]$, $Q_2[x_2, y_2]$, ..., $Q_k[x_k, y_k]$. Let S be the set of vertices in used cycles but not in $\bigcup_{i=1}^{k} V(Q_i(x_i, y_i])$. Then, from Statements 2 and 3 above, $S \subseteq R$. Note that $\{y_1, y_2, ..., y_k\} \subseteq U \cup V(H^*)$ dominates $\bigcup_{C_i \in \mathscr{O}} V(C_i)$. In particular, we have $\overrightarrow{y_i x_j} \in E(T)$ for all $i = 1, 2, ..., k$ and $j = 1, 2, ..., k$.

If $S = \emptyset$, let

$$C_1^* = Q_1[x_1, y_1]\, y_1 v_m x_1,$$

$$C_i^* = Q_i[x_i, y_i], \qquad \text{for} \quad i = 2, ..., k-r-1,$$

and

$$C_{k-r}^* = Q_{k-r}[x_{k-r}, y_{k-r}]\, Q_{k-r+1}[x_{k-r+1}, y_{k-r+1}] \cdots$$
$$\cdots Q_{k-1}[x_{k-1}, y_{k-1}]\, Q_k[x_k, y_k]\, x_{k-r}.$$

Let $\mathscr{C}^*$ be the set containing the above cycles and all unused cycles. Clearly, $\mathscr{C}^*$ contains exactly k vertex-disjoint cycles, and the union of the vertex sets of these cycles contains all vertices in $\bigcup_{i=1}^{k} V(C_i)$ and v_m, a contradiction to the maximality of $\sum_{i=1}^{k} |V(C_i)|$.

Thus, we conclude that $S \neq \emptyset$. Let $Q[w_1, w_q] = w_1 w_2 \cdots w_q$ be a hamiltonian path in $T[S]$.

Claim 5. *w_1 dominates $\{y_1, y_2, ..., y_k\}$.*

Proof. Suppose, to the contrary and without loss of generality, that $\overrightarrow{y_1 w_1} \in E(T)$. Let

$$C_1^* = Q_1^*[x_1, y_1]\, Q[w_1, w_q]\, v_m x_1,$$

$$C_2^* = Q_i[x_i, y_i]\, x_i, \qquad \text{for} \quad i = 2, ..., k-r-1,$$

and

$$C_{k-r}^* = Q_{k-r}[x_{k-r}, y_{k-r}]\, Q_{k-r+1}[x_{k-r+1}, y_{k-r+1}] \cdots$$
$$\cdots Q_{k-1}[x_{k-1}, y_{k-1}]\, Q_k[x_k, y_k]\, x_{k-r}.$$

In the same manner as before, these cycles lead to a contradiction of the maximality of $\sum_{i=1}^{k} |V(C_i)|$. ∎

Claim 6. *w_1 dominates $\bigcup_{i=1}^{k} V(Q_i[x_i, y_i])$.*

Proof. Suppose, to the contrary, that there is a vertex $u \in V(Q_i[x_i, y_i])$ such that $\overrightarrow{uw_1} \in E(T)$. Since $\overrightarrow{w_1 y_i} \in E(T)$, there are two consecutive vertices u_i and u_i^+ on $Q_i[x_i, y_i]$ such that $\overrightarrow{u_i w_1} \in E(T)$ and $\overrightarrow{w_1 u_i^+} \in E(T)$. Path $Q_i[x_i, u_i]\, w_1 Q_i[u_i^+, y_i]$ plus the other $k-1$ paths contradict the maximality of $\sum_{i=1}^k |V(Q_i[x_i, y_i])|$. ∎

Since $w_1 \in R$, there is a vertex $x \in \bigcup_{C_j \in \mathcal{O}} V(C_j)$ which dominates w_1. From Claim 6, x must be on an unused out-cycle C_s since $\bigcup_{i=1}^k V(Q_i[x_i, y_i])$ contains all vertices in all used out-cycles. Let x^+ be the successor of x on C_s. We construct $k-r+1$ cycles as follows.

$$C_1^* = Q_1[x_1, y_1]\, C_s[x^+, x]\, Q[w_1, w_q]\, v_m x_1,$$
$$C_i^* = Q_i[x_i, y_i]\, x_i, \qquad \text{for} \quad i = 2, ..., k-r,$$

and

$$C_{k-r+1}^* = Q_{k-r+1}[x_{k-r+1}, y_{k-r+1}]\, Q_{k-r+2}[x_{k-r+2}, y_{k-r+2}] \cdots$$
$$\cdots Q_k[x_k, y_k]\, x_{k-r+1}^*.$$

These $k-r+1$ cycles and $r-1$ remaining unused cycles lead to a contradiction of the maximality of $\sum_{i=1}^k |V(C_i)|$, which completes the proof of Theorem 1.

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