Discrete Mathematics 231 (2001) 135–141

DISCRETE MATHEMATICS

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Edge disjoint monochromatic triangles in 2-colored graphs

P. Erdős[a], R.J. Faudree[b,*,1], R.J. Gould[c,2], M.S. Jacobson[d,3], J. Lehel[d,4]

[a] *Mathematical Institute, Hungarian Academy of Sciences, Budapest, Hungary*
[b] *Department of Mathematical Sciences, Memphis State University, Memphis, TN 38152, USA*
[c] *Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA*
[d] *Department of Mathematics, University of Louisville, Louisville, KY 40292, USA*

Received 14 July 1999; revised 12 July 2000; accepted 7 August 2000

Abstract

Let $N(n,k)$ be the minimum number of pairwise edge disjoint monochromatic complete graphs K_k in any 2-coloring of the edges of a K_n. Upper and lower bounds on $N(n,k)$ will be given for $k \geqslant 3$. For $k=3$, exact values will be given for $n \leqslant 11$, and these will be used to give a lower bound for $N(n,3)$. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

For any positive integer $k \geqslant 2$, the Ramsey number $r(k)$ is the largest positive integer n such that if the edges of a complete graph K_n are 2-colored, there is no monochromatic K_k. The existence of such numbers was verified by Ramsey in [4]. Thus, for any $n > r(k)$, there will be monochromatic K_k's in any 2-coloring of the edges of the K_n. How many such monochromatic K_k's will there be? For the case, $k=3$, Goodman's result in [2] implies that every 2-coloring of the edges of a K_n has at least $\binom{n}{3} - \lfloor n/2 \lfloor ((n-1)/2)^2 \rfloor \rfloor$ monochromatic triangles. Here another measure, the number of *edge disjoint* monochromatic K_k's is considered, which motivates the following definition.

Definition 1. Let $N(n,k)$ be the minimum number of pairwise edge disjoint monochromatic complete subgraphs K_k in any 2-coloring of the edges of a K_n.

* Corresponding author.
E-mail address: rfaudree@memphis.edu (R.J. Faudree).

[1] Research is partially supported by ONR Grant No. N00014-94-J-1085.
[2] Research is partially supported by ONR Grant No. N00014-97-1-0499.
[3] Research is partially supported by ONR Grant No. N00014-94-J-1098.
[4] On leave from the Automation and Computing Research Institute of the Hungarian Academy of Sciences.

PII: S0012-365X(00)00312-5

Since there are $\binom{n}{2}$ edges in K_n, one would expect that the number of pairwise disjoint monochromatic complete subgraphs K_k is a 2-coloring of the edges of a K_n (for n sufficiently large) would be cn^2 for some appropriate c. In fact the following is true.

Theorem 1. $\lim_{n\to\infty} N(n,k)/n(n-1)$ *exists and equals* $\sup_n N(n,k)/n(n-1)$.

Note that if $c_k = \sup_n N(n,k)/n(n-1)$, then for any given $\varepsilon > 0$, there is an $m = m(\varepsilon)$ such that $N(m,k) \geqslant (1-\varepsilon)c_k m(m-1)$. Then for n sufficiently large, the edges of K_n can be packed with $\binom{n}{2}/\binom{m}{2} - o(n^2) \geqslant (1-\varepsilon)\binom{n}{2}/\binom{m}{2}$ edge disjoint K_m's by Theorem 6. Therefore for n sufficiently large,

$$N(n,k) \geqslant (1-\varepsilon)\binom{n}{2}\Big/\binom{m}{2}(1-\varepsilon)c_k m(m-1) \geqslant (1-\varepsilon)^2 c_k n(n-1).$$

This verifies Theorem 1, so the question that remains is to determine $\lim_{n\to\infty} N(n,k)/n(n-1)$.

Concerning the case $k=3$ consider the 2-coloring of K_n determined by $K_{\lceil n/2\rceil} \cup K_{\lfloor n/2\rfloor}$ being the graph of the first color and $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ being the graph of the second color. In this coloring there are no monochromatic triangles in the second color, and there are approximately

$$2\frac{\binom{n/2}{2}}{3} = \frac{n^2}{12} + o(n^2).$$

edge disjoint triangles in the first color depending on how close to a complete Steiner Triple System each of the complete graphs $K_{\lceil n/2\rceil}$ and $K_{\lfloor n/2\rfloor}$ have. This example led to the following conjecture of Erdős.

Conjecture 1. If n is sufficiently large, then

$$N(n,3) = \frac{n^2}{12} + o(n^2).$$

By determining $N(n,3)$ for small values of n, in fact for $n \leqslant 11$, an outline of the proof of the following result will be presented in Section 3.

Theorem 2. *For* $n \geqslant 3$,

$$3n^2/55 + o(n^2) \leqslant N(n,3) \leqslant n^2/12.$$

In the example that led to the conjecture of Erdős, the number of monochromatic triangles is very unbalanced with all of them being in just one of the colors. We may ask the related question of what is the minimum number of edge disjoint triangles K_3 that are all at the same color. More formally, let $N'(n,k)$ be the maximum of the minimum number of edge disjoint complete graphs K_k in either color 1 or color 2 in any 2-coloring of the edges of a K_n. For $n = 5m$, consider the graph G of order n obtained from a cycle C_5 by replacing each vertex of the C_5 by a $\bar{K}_m$, each edge by a $K_{m,m}$, and then adding an arbitrary selection of one-half of the edges from each of the copies of K_m. Thus, the graph G has $5m^2+5\binom{m}{2}/2$ edges and at most $5\binom{m}{2}/2=n^2/20+o(n^2)$ edge disjoint triangles. The complementary graph $\bar{G}$ has the same properties, so this gives a 2-coloring of the edges of a K_n such that the number of monochromatic edge disjoint triangles in either of the colors is at most $n^2/20$, and this lead to the following conjecture of Jacobson.

Conjecture 2. If n is sufficiently large, then

$$N'(n,3)=\frac{n^2}{20}+o(n^2).$$

We cannot verify this conjecture in general, but some support can be given for small orders. Unfortunately, the same approach that was used in the lower bound for the Erdős conjecture does not apply. The best lower bound comes from Theorem 2, and the fact that one-half of the monochromatic triangles must be in one of the colors. This gives the following weak result.

Theorem 3. *For $n \geqslant 3$,*

$$3n^2/110+o(n^2) \leqslant N'(n,3) \leqslant n^2/20.$$

For $k \geqslant 4$, the nature of $N(n,k)$ and also the nature of $N'(n,k)$ is different. In fact, there appears to be a difference between the case $k=4$ and the $k \geqslant 5$ cases. For $k=4$, the following will be verified using the obvious general fact that $N'(n,k) \geqslant N(n,k)/2$.

Theorem 4. *For n sufficiently large,*

$$n^2/204+o(n^2) \leqslant N(n,4) \leqslant 2N'(n,4) \leqslant n^2/36+o(n^2).$$

The next result gives upper and lower bounds for $N(n,k)$ and $N'(n,k)$ in terms of the Ramsey number $r(k)$. This result and the previous result will be proved in Section 2.

Theorem 5. *For $k \geqslant 5$ a fixed integer and n sufficiently large,*

$$\frac{n^2}{4kr(k)}+o(n^2) \leqslant N(n,k) \leqslant 2N'(n,k) \leqslant \frac{n^2}{2r(k)}+o(n^2).$$

2. $N(n,k)$ for $k \geqslant 4$

There is a natural generalization of the bipartite coloring used in the example for the number of monochromatic K_3's for arbitrary K_k's. Consider the 2-coloring of the edges of a K_n when $n=(k-1)m$, where the first color graph is the disjoint union of complete graphs $(k-1)K_m$ and the second color graph is the complete $(k-1)$-partite graph $K_{m,m,\dots,m}$. The only monochromatic K_k's occur in the first color, and the number is approximately

$$(k-1)\frac{\binom{m}{2}}{\binom{k}{2}} = \frac{n^2}{k(k-1)^2} + \mathrm{o}(n^2),$$

since the edges of each K_m can be partitioned into approximately $\binom{m}{2}/\binom{k}{2}$ edge disjoint K_k's. However, for $k \geqslant 5$, there is an example that gives a sharper upper bound. Consider the case when $n = m \cdot r(k)$. Partition the vertices of K_n into $r(k)$ parts each with m vertices. Arbitrarily color the edges in each of the parts either color 1 or color 2, subject only to the restriction of using each color the same number of times. By the definition of the Ramsey number, there is a 2-coloring of the edges of a $K_{r(k)}$ such that there is no monochromatic K_k. Extend this coloring to K_n by coloring all the edges between 2 of the $r(k)$ parts of the K_n with the same color as the corresponding edge of $K_{r(k)}$ is colored. Thus, all of the edges between two parts in the K_n will have the same color, and there will be no monochromatic K_k with at most one vertex in each part. The number of monochromatic K_k's is then at most the number of edges in the $r(k)$ different complete graphs K_m. Hence, because of the balance in the number of edges in each color,

$$N(n,k) \leqslant 2N'(n,k) \leqslant r(k)\binom{m}{2} = \frac{n^2}{2r(k)} + \mathrm{o}(n^2).$$

Note that for $k=3$, we have $n^2/2r(k) = n^2/10 > n^2/12 = n^2/(k(k-1)^2)$, and for $k=4$, $n^2/2r(k) = n^2/34 > n^2/36 = n^2/(k(k-1)^2)$. However, for $k \geqslant 5$, we have $n^2/2r(k) < n^2/(k(k-1)^2)$, so the latter coloring gives a better upper bound on the number of edge disjoint monochromatic K_k's.

To obtain the lower bound, we need a well-known result about partitioning the edges of a large complete graph K_n into smaller complete graphs K_k. We will state only the special form that we need of a much more general theorem The more general result is due to Fort and Hedlund [1], Hanani [3], and Schönheim [5].

Theorem 6 (Fort and Hedlund [1], Hanani [3] and Schönheim [5]). *If k is a fixed integer and n is sufficiently large, then the edges of K_n can be partitioned into $\binom{n}{2}/\binom{k}{2} - \mathrm{o}(n^2)$ edge disjoint copies of K_k along with a collection of at most $\mathrm{o}(n^2)$ edges.*

Proof of Theorems 4 and 5. We have already verified the upper bounds for Theorems 4 and 5. To prove the lower bound, use the previous result and partition the edges of a large K_n into approximately $\binom{n}{2}/\binom{2r(k)}{2}$ edge disjoint copies of a $K_{2r(k)}$. Consider an arbitrary 2-coloring of the edges of K_n. By the definition of $r(k)$, there must be a monochromatic K_k in a $K_{2r(k)}$. Delete $k-1$ of the vertices of this monochromatic K_k, and repeat this procedure. As long as there are more than $r(k)$ vertices, there will be a monochromatic K_k. Thus, there will be at least $r(k)/(k-1)$ edge disjoint monochromatic K_k's in each of the $K_{2r(k)}$'s. This implies that the 2-colored K_n will contain at least

$$\frac{\binom{n}{2}}{\binom{2r(k)}{2}} \frac{r(k)}{(k-1)} + o(n^2) = \frac{n^2}{4(k-1)r(k)} + o(n^2)$$

edge disjoint monochromatic K_k's. Thus, we have proved Theorems 4 and 5. □

In the proof of either Theorem 4 or Theorem 5 above one may try to partition the edges of K_n into complete graphs other than $K_{2r(k)}$, say for example $K_{4r(k)}$, to see if this produces more monochromatic K_k's. However, it does not, so some other proof technique would have to be used to improve the lower bound in Theorem 4 or Theorem 5, if it is not sharp.

3. The case of triangles

The outline of the proof of Theorem 2 is to first determine $N(n,3)$ for small values of n, in particular for $n \leqslant 11$. Then the result for $n = 11$ will be used to give a general lower bound for $N(n,3)$. To do this, we need to determine the maximum number of edge disjoint K_3's in a K_n for small values of n. Let $t(n)$ denote the maximum number of edge disjoint triangles in K_n. Of course by Theorem 6, we know that $t(n)$ is approximately $\binom{n}{2}/3$, and in fact by a result in [6] it is exactly $n(n-1)/6$ for all values of $n \geqslant 6$ such that $n \equiv 1, 3 \pmod 6$. It is strightforward to verify the following.

Lemma 1. *If $t(n)$ is the maximum number of edge disjoint triangles in a K_n, then the following table gives the values of $t(n)$, for $3 \leqslant n \leqslant 13$.*

n	3	4	5	6	7	8	9	10	11	12	13
t	1	1	2	4	7	8	12	13	17	20	26

In determining an upper bound for the values of the function $N(n,3)$, a canonical 2-coloring of K_n is useful. If $n = p + q$, then consider the 2-coloring of K_n such that the graph induced by the first color is $K_p \cup K_q$, and the second color graph is the complete bipartite $K_{p,q}$. All of the triangles will be in the first color, and so $N(n,3) = N(p+q,3) \leqslant t(p) + t(q)$. Thus, by Lemma 1, an appropriate choice of p and q gives the following upper bounds for $N(n,3)$.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
p	2	3	4	4	5	5	6	5	6	7	8	8	8	9	10	10	10	11
q	4	4	4	5	5	6	6	8	8	8	8	9	10	10	10	11	12	12
$N(n,3) \leqslant$	1	2	2	3	4	6	8	10	12	15	16	20	21	25	26	30	33	37

With the information in the previous table, the proof of the following theorem only requires verifying the lower bound for $N(n,3)$. This is a long, tedious, but not difficult task. The details will be not be included in the paper, but are in an appendix that can be obtained from the authors.

Theorem 7. *For* $3 \leqslant n \leqslant 11$, *the values of* $N(n,3)$ *are given in the following table.*

n	3	4	5	6	7	8	9	10	11
$N(n,3)$	0	0	0	1	2	2	3	4	6

The previous result allows us to verify Theorem 2, which gives a lower bound approximation to the conjecture of Erdős. Since $N(11,3)/\binom{11}{2} = 3/55$, $\sup_n N(n,3)/n(n-1) \geqslant 3/55$ and $3n^2/55 + o(n^2) \leqslant N(n,3)$.

Recall that Conjecture 2 deals with the function $N'(n,3)$, which is the number of edge disjoint triangles in one of the colors in any 2-coloring of the edges of a K_n. Thus, clearly $N'(n,3) \geqslant N(n,3)/2$, but it may be larger. Using the results of the previous theorem, we can prove the following for small values of n. The details will not be included, but again appear in an appendix that can be obtained from the authors.

Theorem 8. *For* $3 \leqslant n \leqslant 10$, *the values for* $N'(3,n)$ *are given in the following table.*

n	3	4	5	6	7	8	9	10
$N'(n,3)$	0	0	0	1	1	2	2	3

The previous result does not give an improved lower bound for $N'(n,3)$, since the dominant color could vary from subgraph to subgraph. However, this does show that the ‘blownup’ C_5 coloring is an extremal example for $N'(n,3)$ for small values of n. Theorem 2 follows directly from the lower bound in Theorem 1 and the example associated with Conjecture 2.

4. Questions

The obvious questions are to confirm the conjectures of Erdős and Jacobson. A different technique will have to be used to get the exact result. Of course, determining $N(n,3)$ for n larger than 11 would probably increase the lower bound in Theorem 1. For example, if K_k has a Steiner Triple System, and it is verified that $N(2k,3) = 2k(k-1)/3$, then this would give a lower bound on $N(n,3)$ of at least $n^2/(12 + (6/(k-1)))$. With the use of a computer, it would not be difficult to determine $N(n,3)$ for values of n larger than 11.

Acknowledgements

The authors would like to thank both the referees and the editor for their helpful comments.

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