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Results on degrees and the structure of 2-factors

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Abstract

The object of this paper is to review the general problem of using degree conditions to determine the structure of 2-factors in graphs. We shall discuss open problems and developments in this area and to a very limited extent, provide examples of the proof techniques used. We shall also consider some of the corresponding questions and development for digraphs. This is not intended as a complete survey, but rather an overview, indicating some new directions and open problems. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

All graphs considered in this paper are simple finite graphs unless otherwise stated. Let G be a graph. The minimum degree of G will be denoted by $\delta(G)$. A hamiltonian cycle of G is a cycle of G which spans $V(G)$, that is, it contains every vertex of G. The girth of G, denoted $g(G)$, is the length of a shortest cycle in G. We use the notation $\bar{G}$ to denote the complement of the graph G. For any graph G, F is a 2-factor of G if and only if F is a union of vertex disjoint cycles that span $V(G)$.

Throughout this paper we are motivated by the following natural questions.

Question 1. *What conditions on $\delta(G)$ (or degree conditions in general) are sufficient to ensure that G contains a 2-factor? Further, from these conditions can we determine the number of cycles in the 2-factor or the size of these cycles, or both?*

Clearly, hamiltonian cycles are 2-factors. Further, there are many results relating degree conditions and hamiltonian cycles. For example, two of the most well-known are stated below. Here, $\sigma_2(G) = \min\{\deg u + \deg v \mid u, v \in V(G),\ uv \notin E(G)\}$.

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Theorem 1 (Ore [27]). *Let G be a graph of order $n \geqslant 3$. If $\sigma_2(G) \geqslant n$, then G is hamiltonian.*

Theorem 2 (Dirac [14]). *Let G be a graph of order $n \geqslant 3$. If the minimum degree $\delta(G) \geqslant n/2$, then G is hamiltonian.*

However, we shall not concern ourselves here with hamiltonian results. The interested reader should see [22]. We shall instead concentrate on trying to determine the structure of general 2-factors. Terms not defined here can be found in [21].

2. More general conjectures and early results

The fundamental conjecture relating degree conditions and general subgraph containment is the following powerful conjecture due independently to Bollobás and Eldridge [7] and Catlin [9,10].

Conjecture 1. If G and H are graphs of order n such that $(\Delta(H)+1)(\Delta(\bar{G})+1) \leqslant n+1$, then H is a subgraph of G.

This conjecture has many interesting implications, however, we shall restrict our attention to the question at hand. For 2-factors, or more generally when $\Delta(H)=2$, this becomes:

Conjecture 2. If G and H are graphs of order n with $\Delta(H) \leqslant 2$ and $\Delta(\bar{G}) \leqslant (n-2)/3$, then H is a subgraph of G.

The bound in Conjecture 2 corresponds to that given in the following well-known result due to Corrádi and Hajnal [13].

Theorem 3. *Let G be a graph of order $n \geqslant 3k$ with $\delta(G) \geqslant 2k$, $(k \geqslant 1)$, then G contains the vertex disjoint union of k cycles.*

A long-standing conjecture due to Erdős would generalize the Corrádi–Hajnal result. Using the Regularity Lemma, Komlós et al. [19] have shown this conjecture holds for large n.

Conjecture 3. Let H be a graph of order $4k$ with $\delta(H) \geqslant 2k$, then H contains k vertex disjoint 4-cycles.

Another beautiful conjecture due to Alon and Yuster [4] was recently solved by Komlós et al. [19]. Their solution of the Erdős Conjecture is a special case of this result.

Theorem 4. *Let G be a graph of order n. For every graph H there is a constant K such that $\delta(G) \geqslant (1 - 1/\chi(H))n$ implies that there is a union of disjoint copies of H covering all but at most K vertices of G.*

Note that when H is a 2-factor, $\chi(H) \leqslant 3$ and the bound of $\delta(G) \geqslant 2n/3$ appears once again.

Catlin, in his Ph.D. thesis [9], investigated Conjecture 2 and in so doing bolstered the study of the structure of 2-factors. He announced the following more general result, also found independently by Sauer and Spencer [29]. The proof presented here is from Catlin's Thesis [9].

Theorem 5. *If G and H are graphs of order n such that $2\Delta(\bar{G})\Delta(H) < n$, then H is a subgraph of G.*

Proof. Given G and H satisfying the conditions of the statement, suppose H is an edge minimal graph that is not a subgraph of G. Then for any fixed edge $e = ww'$ in $E(H)$, $H - e$ is a subgraph of G. Let $\pi : V(H) \to V(G)$ be an embedding of $H - e$ into G. To find an embedding of H, we shall alter π by transposing $\pi(w)$ with another vertex z of G so that the resulting embedding still embeds $H - e$ and also maps e onto an edge of G, hence embedding H, a contradiction to our assumptions. The vertex z must preserve the adjacency structure of $\pi(w)$ and allow the missing edge e to also be embedded in G.

To find such a vertex, define $M(v) = \{v'' \in V(G)\colon \pi^{-1}(v)\pi^{-1}(v'') \in E(H - e)\}$. A successor of v is any vertex $v_1 \in V(G)$ such that for each $v'' \in M(v)$, v_1 is adjacent to v'' in G and $v_1 \neq v$. Let $S(v)$ be the set of all successors of v. We also define v to be a predecessor of v_1 if $v_1 \in S(v)$. Let $P(v_1)$ denote the set of all predecessors of v_1.

Let $v = \pi(w)$ and note that if $v_1 \in S(v) \cap P(v)$ and if $v_1 \neq v$, then v_1 is a candidate for the vertex z. For each vertex $x \in V(H)$, the map

$$\pi_{v_1}(x) = \begin{cases} \pi(x) & \text{if } \pi(x) \neq v \text{ or } v_1, \\ v_1 & \text{if } \pi(x) = v, \\ v & \text{if } \pi(x) = v_1. \end{cases}$$

A vertex x is not in $S(v)$ if x is adjacent in $\bar{G}$ to a vertex x' in $M(v)$. For any vertex $x' \in M(v)$, there are at most $\Delta(\bar{G})$ choices for x. Since $\deg_{H-e}(w) \leqslant \Delta(H) - 1$, we have $|M(v)| \leqslant \Delta(H) - 1$ choices of x'. Hence, at most $\Delta(\bar{G})[\Delta(H) - 1]$ vertices x are not in $S(v)$. Neighbors (in H) of any nonneighbor (in G) of v cannot be exchanged with v. There are at most $\Delta(\bar{G})\Delta(H) - 1$ such neighbors possible, since $x' = \pi(w')$ is a nonneighbor of v (in G) and w' has at most $\Delta(H) - 1$ neighbors in H different from v. Thus,

$$\begin{aligned} |P(v) \cap S(v)| &\geqslant |V(G)| - |V(G) - P(V)| - |V(G) - S(v)| \\ &\geqslant n - [\Delta(\bar{G})\Delta(H) - 1] - \Delta(\bar{G})[\Delta(H) - 1] \end{aligned}$$

$$= n - 2\Delta(H)\Delta(\bar{G}) + \Delta(\bar{G}) + 1$$
$$\geqslant 2 + \Delta(\bar{G}).$$

But at most $1 + \Delta(\bar{G})$ vertices are not adjacent in G to $\pi(w')$ Therefore, there is a $v_1 \in P(v) \cap S(v)$ that is adjacent to $\pi(w')$ in G. Thus, π_{v_1} is an embedding of H into G. □

The 2-factor version of Theorem 5 is the following corollary.

Corollary 6. *If G has order n and H is any 2-factor on n vertices and $4\Delta(\bar{G}) < n$, then H is a subgraph of G.*

We can conclude from this corollary that if G has minimum degree $\delta(G) \geqslant 3n/4$, then G contains any graph of order n and maximum degree two as a subgraph. Sauer and Spencer [29] also conjectured that the minimum degree condition could be lowered to $\delta(G) \geqslant 2n/3$. They also showed via a probabilistic argument that Theorem 5 is essentially best possible by proving the existence of graphs G and H of order n for which $\Delta(G)\Delta(H)$ is about $4n \log n$ and for which H is not a subgraph of G.

Catlin [9] also gave a slight improvement of Theorem 5 for the case of interest here, however, this result is still not best possible. His proof technique was similar to that of Theorem 5.

Theorem 7. *Let G and H be graphs of order n with $\Delta(H)=2$. If $\Delta(\bar{G}) \leqslant (2n-11)/7$, then H is a subgraph of G.*

Catlin [9,10] continued his assault on the 2-factor problem with the following:

Theorem 8. *If G has order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geqslant 3$ for each $i = 1, \ldots, k$ and $\delta(G) \geqslant 2n/3 + O(n^{1/3})$, then G contains k vertex disjoint cycles $C_1, \ldots, C_k$ of lengths $n_1, \ldots, n_k$, respectively.*

Catlin later improved this result by replacing $O(n^{1/3})$ by $O(1)$. However, it would be many years before Conjecture 2 would be completely settled. In the meantime, other results would be obtained. For example, for the case $k = 2$, the following strong result was obtained by El-Zahar [16].

Theorem 9. *Let G be a graph of order n and let $n_1 \geqslant 3$ and $n_2 \geqslant 3$ be two integers such that $n_1 + n_2 = n$. If the minimum degree $\delta(G) \geqslant \lceil n_1/2 \rceil + \lceil n_2/2 \rceil$, then G has two vertex disjoint cycles C_1 and C_2 of length n_1 and n_2, respectively.*

The key to the proof of Theorem 9 is the following lemma.

Lemma 1 (El-Zahar [16]). *Let G have order $n = n_1 + n_2$ and $\delta(G) \geqslant \lceil n_1/2 \rceil + \lceil n_2/2 \rceil$. Then there is a partition of G into subgraphs G_1 and G_2 such that one of the following* *conditions holds*:

(1) $|V(G_i)| = n_i$ *and* $\delta(G_i) \geqslant n_i/2$, $i = 1, 2$,
(2) G_1 *contains a path on* $n_i - 1$ *vertices*, $|V(G_2)| = n_j + 1$ *and* $\delta(G_2) \geqslant n_j/2 + 1$ *where* $\{i, j\} = \{1, 2\}$.

Proof of Theorem 9 (*Sketch*, *El−Zahar* [16]). If G has a partition satisfying condition (1) of Lemma 1, then the result follows easily from the classic hamiltonian result of Dirac (Theorem 2).

If instead (2) holds with $|V(G_1)| = n_1 + 1$ and $|V(G_2)| = n_2 + 1$, the idea is to find a vertex $w \in V(G_2)$ such that $G_1 + w$ is hamiltonian. Then again by Dirac's Theorem, $G_2 - w$ will also be hamiltonian.

Thus, if G_1 has a hamiltonian path from vertex a to vertex b and if

$$\deg_{G_1} a + \deg_{G_1} b < n_1 - 1 \tag{1}$$

then $\deg_{G_2} a + \deg_{G_2} b > n_2 + 1$, and hence, $aw, bw \in E(G)$ for some $w \in V(G_2)$. Thus, $G_1 + w$ is hamiltonian. Now we can assume Eq. (1) does not hold. Then by (1) of Lemma 1 and Ore's Theorem (Theorem 1), G_1 contains a hamiltonian cycle, call it C.

Now let $X = \{x \in V(G_1) \mid \deg_{G_1} x < n_1/2\}$. By considering the three cases: $|X| \geqslant 2$, $|X| = 1$, and $X = \emptyset$, the proof is completed. We consider here only the first of these cases.

Thus, suppose that $|X| \geqslant 2$. For any $x_1, x_2 \in V(X)$ we have that $\deg_{G_2} x_1 + \deg_{G_2} x_2 \geqslant n_2 + 2$. Thus, $x_1w, x_2w \in E(G)$ for some $w \in V(G_2)$. If x_1 and x_2 were adjacent on C, then $G_1 + w$ would be hamiltonian as required. Hence, assume no two vertices in X are adjacent on C and let p_i and s_i be the predecessor and successor of x_i ($i = 1, 2$), respectively, according to some orientation of C. Then we get a path $p_1C^-s_2x_2wx_1s_1C^+p_2$ where $p_1C^-s_2$ and $s_1C^+p_2$ denote subpaths of C, respectively, opposite to and in the same direction as the orientation. Since p_1, p_2 are not in X, this path contains a hamiltonian cycle by the proof of Theorem 1. □

In the same paper, El-Zahar conjectured that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k$ ($n_i \geqslant 3$) with minimum degree

$$\delta(G) \geqslant \left\lceil \frac{n_1}{2} \right\rceil + \left\lceil \frac{n_2}{2} \right\rceil + \cdots + \left\lceil \frac{n_k}{2} \right\rceil,$$

then contains k vertex disjoint cycles of length $n_1, n_2, \ldots, n_k$, respectively.

If El-Zahar's conjecture is true, then it follows that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k$ ($n_i \geqslant 3$) with $\delta(G) \geqslant 2n/3$, then G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$, of lengths $n_1, n_2, \ldots, n_k$, respectively. Recall Theorem 5 implies that El-Zahar's conjecture holds with $\delta(G) \geqslant 3n/4 - 1$.

Recently, Wang [31] has provided a slight strengthening to Theorem 9.

Theorem 10. *Let* G *be a graph of order* $n \geqslant 6$ *with* $\delta(G) \geqslant \lceil (n+1)/2 \rceil$. *Then for any two integers* s *and* t *with* $s \geqslant 3$, $t \geqslant 3$ *and* $s + t \leqslant n$, G *contains two vertex-disjoint* *cycles of lengths s and t, respectively, unless n, s and t are odd and* $G \cong K_{(n-1)/2,(n-1)/2} + K_1$.

Clearly $K_{(n-1)/2,(n-1)/2} + K_1$ does not contain two vertex disjoint odd cycles for any odd $n \geqslant 3$. If n is even, $K_{n/2,n/2}$ contains no odd cycles at all. Wang [31] also considered this situation.

Theorem 11. *Let G be a graph of order* $n \geqslant 8$ *with n even and* $\delta(G) \geqslant n/2$. *Then for any two even integers s and t with* $s \geqslant 4$, $t \geqslant 4$ *and* $s + t \leqslant n$, *G contains two vertex disjoint cycles of lengths s and t, respectively.*

In 1993, Aigner and Brandt [2] finally settled Conjecture 2.

Theorem 12. *Let G be a graph of order n with* $\delta(G) \geqslant (2n-1)/3$, *then G contains any graph H of order at most n with* $\Delta(H) = 2$.

The degree condition of Theorem 12 is best possible. To see this consider the complete tripartite graph $G = K_{t+1,t+1,t-1}$. This graph has order $n = 3t + 1$ and minimum degree $2t = (2n-2)/3$, but it fails to contain t vertex disjoint triangles.

Alon and Fischer [3] independently proved that if G has sufficiently large order n and minimum degree at least $2n/3$, then G contains any graph H with $\Delta(H) \leqslant 2$.

3. Relaxing the problem

Theorem 12 is very powerful as it guarantees the graph H contains all possible 2-factors. But Theorem 12 also requires a very high minimum degree. It is now natural to ask if we can obtain a little less in graphs where the minimum degree is not as high. Our new problem becomes:

Problem 1. *What minimum degree (or degree condition) is sufficient to guarantee a graph G contains a 2-factor consisting of a specified number k of cycles.*

In this study, both Theorem 3 and the following result on independent cycles have proven useful.

Theorem 13 (Justesen [17]). *If G is a graph of order* $n \geqslant 3k$ *such that* $\sigma(G) \geqslant 4k$, *then G contains k vertex disjoint cycles.*

Using Theorem 13 the following was shown in [8].

Theorem 14. *Let k be a positive integer and let G be a graph of order* $n \geqslant 4k$. *If* $\sigma_2(G) \geqslant n$, *then G has a 2-factor with exactly k vertex disjoint cycles.*

Note that Theorem 14 generalizes the classic hamiltonian result of Ore [27] for the case when $n \geqslant 4k$. The complete bipartite graph $K_{n/2,n/2}$ shows that this result is best possible. The following generalization of Theorem 2 is also clear.

Corollary 15 (Brandt et al. [8]). *Let k be a positive integer and let G be a graph of order $n \geqslant 4k$. If $\delta(G) \geqslant n/2$, then G has a 2-factor with exactly k vertex disjoint cycles.*

The next result gives a sufficient condition for a graph to have k disjoint cycles which are either triangles or 4-cycles. This result is also from Brandt et al. [8].

Theorem 16. *Let $s \leqslant k$ be two nonnegative integers and let G be a graph of order $n \geqslant 3s + 4(k - s)$. If $\sigma_2(G) \geqslant (n + s)/2$, then G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that*

$$|V(C_i)| = 3 \quad \textit{for } 1 \leqslant i \leqslant s,$$

$$|V(C_i)| \leqslant 4 \quad \textit{for } s+1 \leqslant i \leqslant k,$$

that is, the first s cycles are triangles and the others are either triangles or 4-cycles.

4. Special cases and restricted classes

In this section we consider some results on restricted classes of graphs. We say G is $\{H_1, \ldots, H_k\}$-free if G contains no subgraph isomorphic to any H_i, $i = 1, \ldots, k$. Each graph H_i is said to be forbidden in G. We begin with a special case of a more general result from Egawa and Ota [15].

Theorem 17. *If G is a connected $K_{1,3}$-free graph with $\delta(G) \geqslant 4$, then G contains a 2-factor.*

Egawa and Ota [15] extended this approach to $K_{1,r}$-free graphs.

Theorem 18. *Let G be a connected $K_{1,r}$-free graph ($r \geqslant 3$) with*

$$\delta(G) \geqslant \left\lceil \frac{r^2}{8(r-1)} + \frac{3r-6}{2} + \frac{r-1}{8} \right\rceil,$$

then G has a 2-factor.

Acree [1] found several results where the Corrádi–Hajnal condition (from Theorem 3) could be used in conjunction with forbidden subgraphs to obtain 2-factor results. The graph Z_2 is formed by identifying a vertex of a triangle with an end vertex of a path of length 2.

Theorem 19 (Acree [1]). *If G is a 2-connected $\{K_{1,3}, Z_2\}$-free graph of order $n \geqslant 3k$ $(k \geqslant 1)$ such that $\delta(G) \geqslant 2k$, then G contains a 2-factor consisting of exactly k disjoint cycles.*

A graph G is said to be locally connected if for each vertex $x \in V(G)$, the graph induced by the neighborhood of x, $N(x) = \{w \in V(G) | xw \in E(G)\}$, is connected.

Theorem 20 (Acree [1]). *If G is a connected, locally connected $K_{1,3}$-free graph of order $n \geqslant 3k$ with $\delta(G) \geqslant 2k$, then G contains a 2-factor consisting of exactly k disjoint cycles.*

Turning to another restricted class of graphs, let $G = (V_1, V_2; E)$ be a bipartite graph. We say G is balanced if $|V_1| = |V_2|$. Amar [5] obtained the following:

Theorem 21. *If G is a balanced bipartite graph of order $2n$ with $\deg u + \deg v \geqslant n+2$ for any $u \in V_1$ and $v \in V_2$, then for any $n_1 \geqslant 2$, $n_2 \geqslant 2$ with $n_1 + n_2 = n$, G contains two vertex disjoint cycles of lengths $2n_1$ and $2n_2$.*

Wang [32] obtained a bipartite result reminiscent of El-Zahar's Theorem.

Theorem 22. *If G is a balanced bipartite graph of order $2n$ with $n = n_1 + \cdots + n_k$ and $\delta(G) \geqslant n_1 + n_2 + \cdots + n_{k-1} + n_k/2$, then G contains k disjoint cycles of lengths $2n_1, 2n_2, \ldots, 2n_k$, respectively.*

Moon and Moser [26] obtained the following well-known hamiltonian result.

Theorem 23. *Let G be a balanced bipartite graph of order $2n$. If $\delta(G) \geqslant (n+1)/2$, then G is hamiltonian.*

This result was generalized in [11].

Theorem 24. *Let k be a positive integer and let G be a balanced bipartite graph of order $2n$ where $n \geqslant \max\{52, 2k^2+1\}$. If $\delta(G) \geqslant (n+1)/2$, then G contains a 2-factor with exactly k cycles.*

Finally, Las Vergnas [18] determined a condition sufficient to insure a hamiltonian cycle that contains all edges of a perfect matching.

Theorem 25. *Let G be a balanced bipartite graph of order $2n$. If $\deg u + \deg v \geqslant n+2$ for every pair of nonadjacent vertices u and v from different parts, then each perfect matching of G is contained in a hamiltonian cycle.*

In [12], the following 2-factor result related to Theorem 25 was obtained.

Theorem 26. *Let k be a positive integer and G a balanced bipartite graph of order $2n$ where $n \geqslant 9k$. If $\delta(G) \geqslant (n+2)/2$, then for every perfect matching M, G contains a 2-factor with exactly k cycles including every edge of M.*

5. Digraphs

It is natural to ask similar questions for digraphs. This has been done to some degree and a variety of results have been obtained. Kotzig [20] showed regular multi-digraphs contain 2-factors. A great deal of recent work has centered on special classes of digraphs where connectivity rather than degree conditions become critical. A very reasonable approach would be to consider the special class of tournaments, that is, complete graphs where each edge receives a direction. Thomassen (see [30]) raised the problem of finding a 2-factor consisting of exactly two cycles. The cycles of such a 2-factor are called complementary cycles. The following result is due to Reid [28].

Theorem 27. *Every 2-connected tournament on $n \geqslant 6$ vertices contains two complementary cycles of lengths 3 and $n-3$, respectively, unless the tournament is T_7^1 (see Fig. 2).*

If for each integer t, $3 \leqslant t \leqslant n-3$ a digraph D of order n contains two complementary cycles of lengths t and $n-t$, then we say that D is complementary pancyclic. Song [30] used induction to extend Reid's Theorem.

Theorem 28. *Every 2-connected tournament on $n \geqslant 6$ vertices is complementary pancyclic unless it is isomorphic to T_7^1.*

It is natural now to consider digraphs that are close to tournaments structurally. We say a digraph is semicomplete if for any two vertices x and y, there is at least one arc (directed edge) between them. Clearly tournaments are semicomplete. Recall that the out-neighbors of a vertex x are those vertices which receive a directed arc from x, while the in-neighbors of x are those vertices which send an arc into x. A digraph D is locally semicomplete if the graphs induced by both the out-neighbors, denoted $N^+(x)$, and in-neighbors, denoted $N^-(x)$, of every vertex x form a semicomplete digraph. The closed neighborhood of x is $N(x) \cup \{x\} = N[x]$. Let $\deg^+ x = |N^+(x)|$ and $\deg^-(x) = |N^-(x)|$. For convenience, let $T' = \{T_6^1, T_6^2, T_6^3, T_7^1, T_7^2\}$ (see Figs. 1 and 2). Note that each digraph in T' is 2-connected and locally semicomplete. Further, note that none is cycle complementary.

A digraph is termed round if we can label its vertices $v_0, \ldots, v_{n-1}$ such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+\deg^+(v_i)}\}$ and $N^-(V_i) = \{v_{i-\deg^-(v_i)}, \ldots, v_{i-1}\}$, where all subscripts are taken modulo n. Let R_n^2 be a 2-regular round, local tournament on n-vertices. We define

$$R' = \{R_n^2 \mid n \text{ is odd and } n \geqslant 7\}.$$

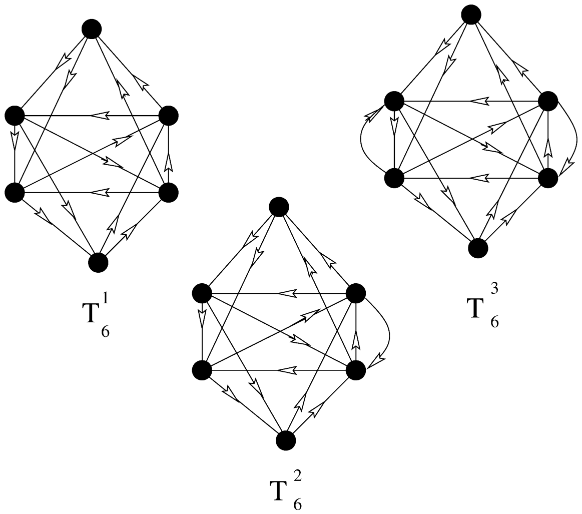

Fig. 1. Some exceptional digraphs.

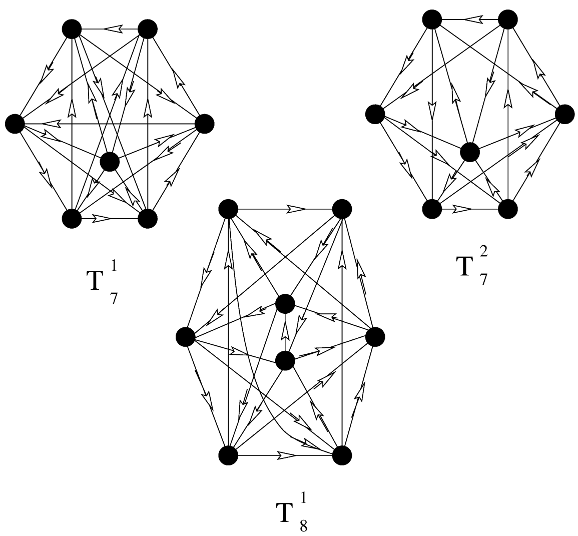

Fig. 2. Other exceptional digraphs.

A digraph is strong if there is a directed path between any two vertices. Bang-Jensen [6] showed that strong locally semicomplete digraphs are hamiltonian, extending earlier work on tournaments. As a result of this, a semicomplete digraph D is cycle complementary if and only if it has a cycle C such that $D - V(C)$ is strong. Guo and Volkman [25] proved that even more is possible.

Theorem 29 (Guo and Volkman [25]). *If D is a 2-connected locally semicomplete digraph on $n \geqslant 6$ vertices, then D contains a $g(D)$ cycle C such that $D - V(C)$ is strong and the closed neighborhood of C is $V(D)$, unless D is a member of $T' \cup T_8^1 \cup R'$.*

Corollary 30 (Guo and Volkman [24]). *Let D be a 2-connected locally semicomplete digraph on $n \geqslant 8$ vertices. Then D is not cycle complementary if and only if D is 2-regular (that is, each vertex has outdegree and indegree 2) and n is odd.*

Guo [23] proposed a question similar to our original question on graphs.

Problem 2. *Let k be a positive integer. What is the least integer $f(k)$ such that all but a finite number of $f(k)$-connected locally semicomplete digraphs contain a 2-factor with exactly k cycles?*

Clearly, $f(1) = 1$ from the result of Bang-Jensen mentioned earlier. Corollary 30 shows that $f(2) = 2$. In fact, Guo [23] conjectures the following:

Conjecture 4. Let D be a k-connected locally semicomplete digraph on at least $3k$ vertices. Then D contains a 2-factor consisting of exactly k cycles, each of length at least 3, unless D is a member of a finite family of k-connected locally semicomplete digraphs.

Guo and Volkman [25] continued to extend their earlier work on complementary cycles to complementary m-pancyclic digraphs. The next result also generalizes Song's Theorem.

Theorem 31. *If D is a 2-connected locally semicomplete digraph on $n \geqslant 6$ vertices, then D is complementary $g(D)$-pancyclic, unless D is isomorphic to a member of $T' \cup \{T_8^1\} \cup R'$.*

Corollary 32 (Guo and Volkman [25]). *If D is a 2-connected, chordal locally semicomplete digraph on at least six vertices, then D is complementary pancyclic unless D is isomorphic to one of $\{T_6^1, T_6^2, T_6^3, T_7^1\}$.*

Corollary 33 (Guo and Volkman [25]). *Let D be a 2-connected locally semicomplete digraph on at least six vertices. If D has a minimum separating set S such that $D-S$ is semicomplete, then D is complementary pancyclic unless D is isomorphic to a member of $\{T_6^1, T_6^2, T_6^1\}$.*

Theorem 34 (Guo and Volkman [25]). *Let D be a 2-connected locally semicomplete digraph on n vertices. If D has a k-cycle C with $3 \leqslant k \leqslant n/2 - 1$, such that $D - V(C)$ is strong and the closed neighborhood of C is $V(D)$, then D is complementary k-pancyclic.*

We conclude with a problem and conjecture both from Guo [23].

Problem 3. *Let $k \geqslant 1$ be an integer. What is the least integer $h(k)$ such that all but a finite number of $h(k)$-connected locally semicomplete digraphs contain a 2-factor consisting of k vertex disjoint cycles of lengths $n_1, \ldots, n_k$ where $n_i \geqslant g(D)$ for $i = 1, \ldots, k$* and $\sum_{i=1}^{k} n_i = n$?

Conjecture 5 (Guo [23]). For all k, $h(k) = f(k)$ where $f(k)$ is as defined in Problem 2.

Acknowledgements

This paper is dedicated to the memory of Paul Catlin.

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