The Structure of $\{K_{1,3}, Z_2\}$-free Graphs

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Abstract

A graph H is said to be $\{F, G\}$-free if it contains neither an induced copy of F nor an induced copy of G. In this paper we describe all possible structures of $\{K_{1,3}, Z_2\}$-free connected graphs.

1 Introduction

In this paper we consider only connected simple graphs. The graph Z_2 is shown below in Figure 1.

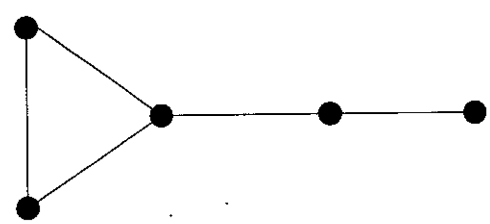

Figure 1: The graph Z_2.

We say that a graph G is $\{K_{1,3}, Z_2\}$-*free* if it does not contain any induced copy of $K_{1,3}$ or of Z_2 as an induced subgraph. The four-vertex star $K_{1,3}$ will also be called the *claw*. Whenever we list

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the vertices of an induced claw, the only vertex of degree three will always be the first vertex in the list.

If the vertices u and v are adjacent in G, we write $u \sim v$; if u and v are nonadjacent in G, we write $u \not\sim v$. For any vertex v in G we denote the neighborhood of v by $N(v)$. If $A \subset V(G)$, then we use the symbol $N_A(v)$ to represet the set $N(v) \cap A$ and the symbol $< A >$ to represent the subgraph of G induced by A. The connectivity of G is denoted by $\kappa(G)$. For definitions and notation not explained in this paper, see [1].

This paper tightens and expands upon Fuller's earlier results on the structure of $\{K_{1,3}, Z_2\}$-free graphs of connectivity at most three.

Theorem 1 *[2] Let G be a $\{K_{1,3}, Z_2\}$-free connected graph which is neither complete nor a tree such that $\kappa(G) = 1$. Then G is a member of one of the families of graphs shown in Figure 2.*

Theorem 2 *[2] Let G be a $\{K_{1,3}, Z_2\}$-free, connected graph which is neither complete nor a cycle such that $\kappa(G) = 2$. Then G is a member of one of the families of graphs shown in Figure 3.*

Theorem 3 *[2] Let G be a $\{K_{1,3}, Z_2\}$-free, connected graph that is not complete such that $\kappa(G) = 3$. Then G is a member of one of the families of graphs shown in Figures 4 and 5.*

We will use the following theorem by Shepherd [3] to tighten Theorems 2 and 3.

Theorem 4 *[3] A connected graph G is claw-free if and only if for every minimal cut set S and every $v \in S$, $\langle N(v) - S \rangle$ is either a single vertex or the disjoint union of two complete graphs.*

In the main body of the paper we will use Fuller's results to describe all $\{K_{1,3}, Z_2\}$-free graphs of connectivity at least four.

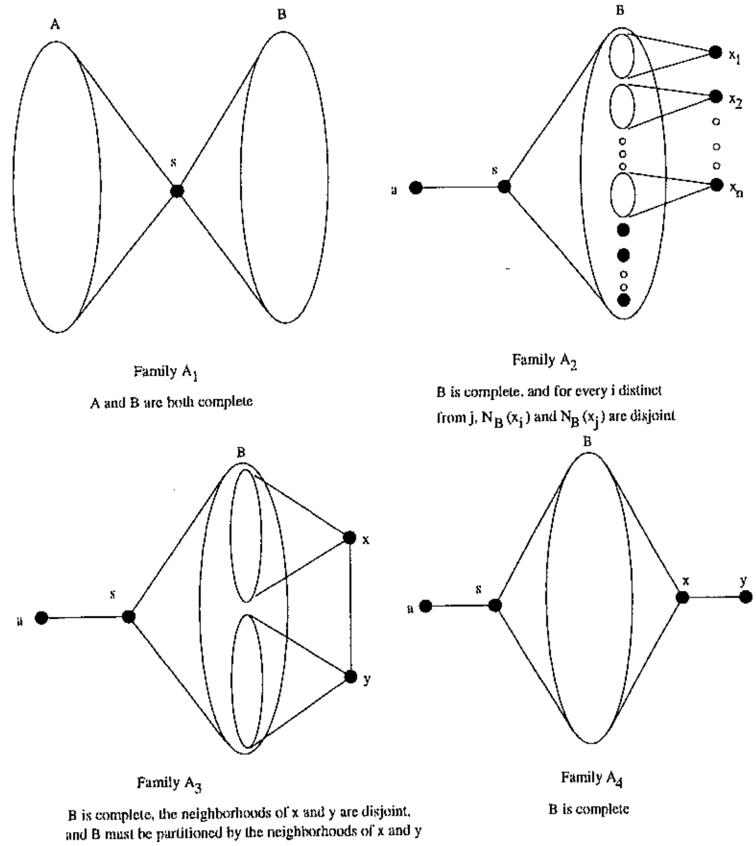


Figure 2: $\{K_{1,3}, Z_2\}$-free Families of Connectivity One.

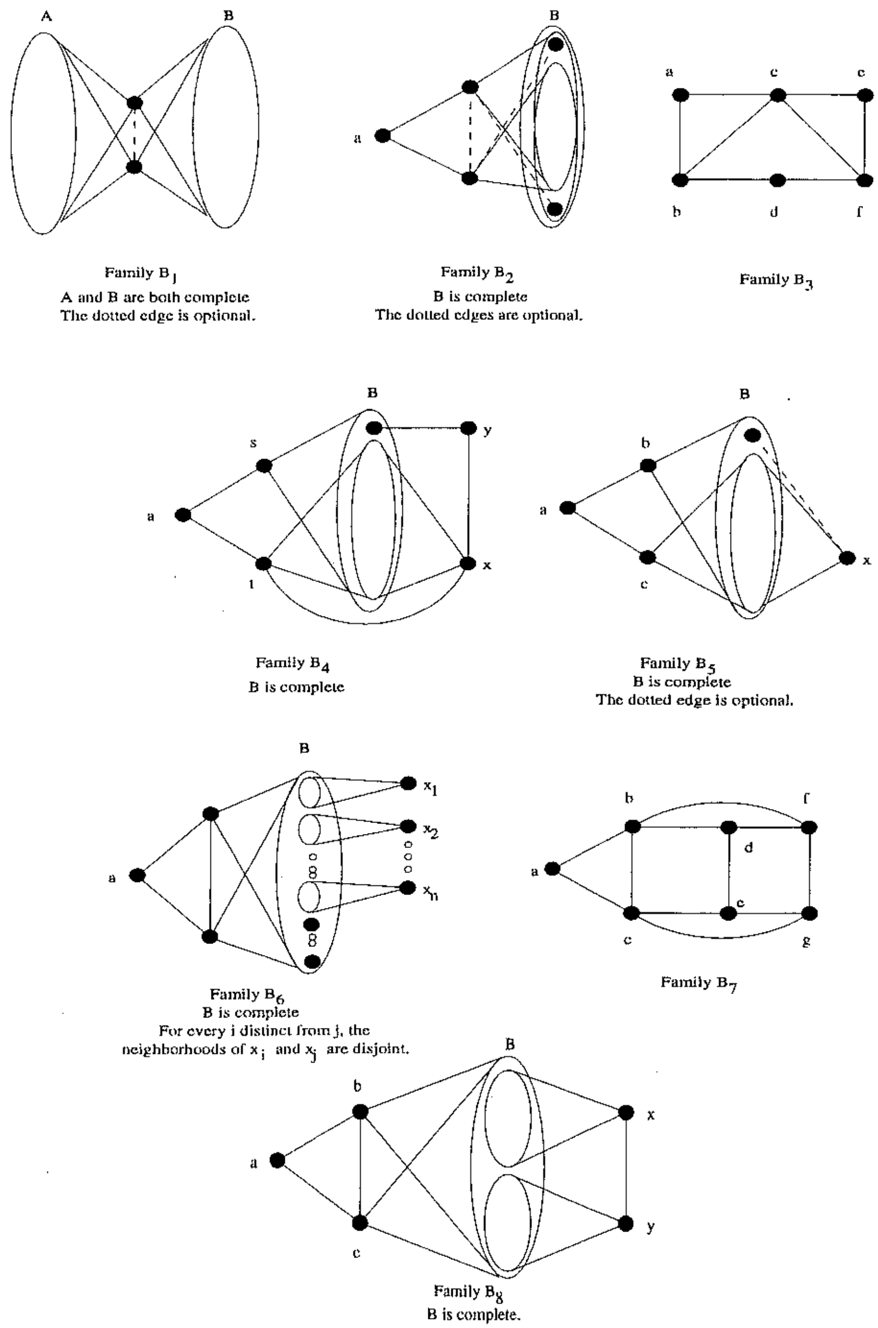
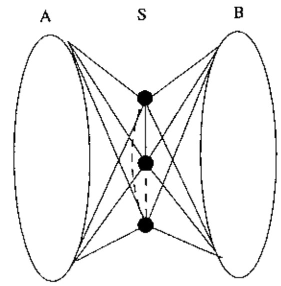
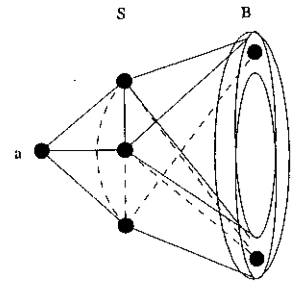


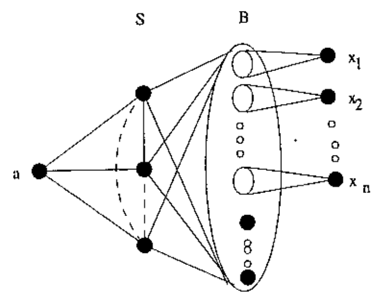
Figure 3: Families of graphs from Theorem 2.



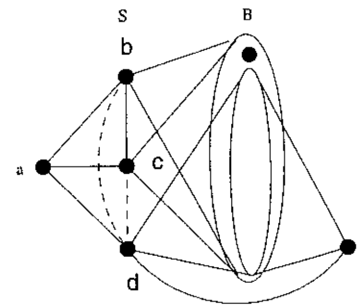
Family D_1
A and B are complete.
The dotted edges are optional.



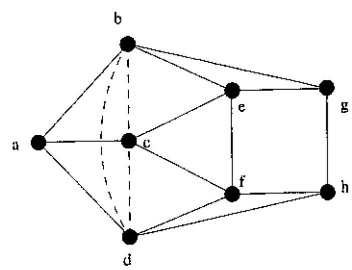
Family D_2
B is complete.
The dotted edges are optional.



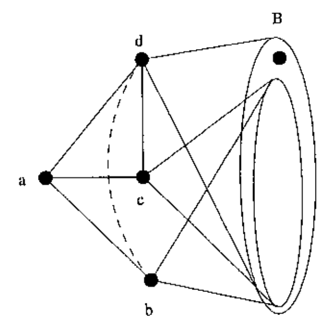
Family D_3
B is complete.
The dotted edges are optional.



Family D_4
B is complete.
The dotted edges are optional.



Family D_5
At least one of the dotted edges must occur.



Family D_6
B is complete.
The dotted edge is optional.

Figure 4: Families of graphs from Theorem 3.

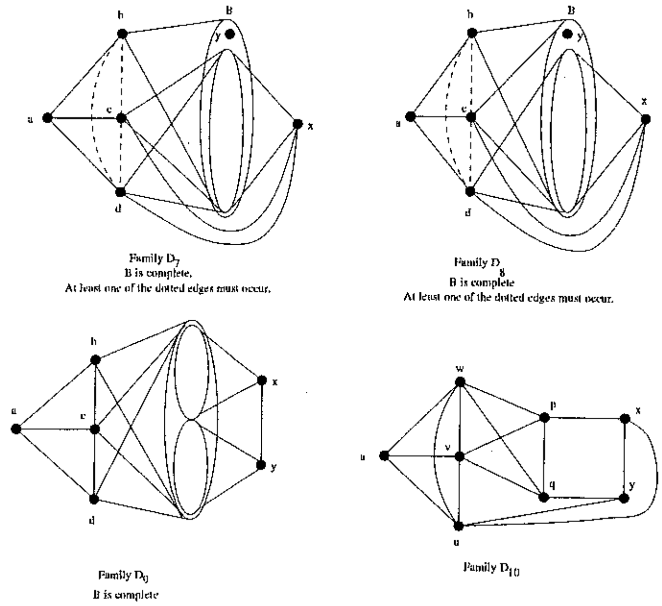


Figure 5: More families of graphs from Theorem 3.

2 $\{K_{1,3}, Z_2\}$-free Graphs of Connectivity at Most Three

In this section we strengthen Theorems 2 and 3, lessening the number of families that need to be considered. We achieve this result by a series of lemmas.

Lemma 1 *Family B_8 from Figure 3 is not Z_2-free.*

Lemma 2 *The only member of family D_4 from Figure 4 that is $\{K_{1,3}, Z_2\}$-free is the one where $< S >$ is complete.*

Proof: Suppose that $< S >$ is not complete. Let b and c be the two vertices of S adjacent to every vertex of B and let d be the remaining vertex of S. Suppose that $e \notin N_B(d)$ and let x be as labelled in Figure 4. Suppose that neither b nor c is adjacent to d. Then $< b, c, a, d, x > \cong Z_2$. If either $b \sim d, c \not\sim d$ or $c \sim d, b \not\sim d$, then $< e, b, c, d, x > \cong Z_2$. Therefore b and c must both be adjacent to d, so $< S >$ must be complete. □

Lemma 3 *The only member of family D_5 from Figure 4 that is $\{K_{1,3}, Z_2\}$-free is the one where $b \sim d$, $b \not\sim c$ and $c \not\sim d$.*

Proof: Let the vertices of the graph be labelled $\{a, b, c, d, e, f, g, h\}$ as shown in Figure 4. If none of the dotted edges are used then we have a claw $< a, b, c, d >$. Therefore we must use at least one of the dotted edges.

Suppose that b is adjacent to c. Then $< c, a, b, f, h > \cong Z_2$. Similarly, if c is adjacent to d then $< a, d, c, e, g > \cong Z_2$.

Therefore b must be adjacent to d and c cannot be adjacent to either b or d. □

Lemma 4 *Family D_6 from Figure 5 is not $K_{1,3}$-free.*

Lemma 5 *Family D_7 from Figure 5 is not $\{K_{1,3}, Z_2\}$-free unless the subgraph $< S > = < b, c, d >$ is complete.*

Proof: Let a, b, c, d, x and y be as labelled in the figure. If $c \not\sim d$, then for any vertex z in $N_B(c)$ we have $< z, c, d, y > \cong K_{1,3}$. Thus c must be adjacent to d. Suppose that neither c nor d is adjacent to b. Then $< c, d, a, b, y > \cong Z_2$. Since this cannot be the case, suppose that exactly one of $\{c, d\}$ is adjacent to b. Then $< x, c, d, b, y > \cong Z_2$. Therefore S must be complete if G is $\{K_{1,3}, Z_2\}$-free. □

Lemma 6 *Family D_8 from Figure 5 is not claw-free.*

Lemma 7 *Family D_9 from Figure 5 is not Z_2-free.*

Lemma 8 *Family D_{10} from Figure 5 contains a copy of Z_2.*

Lemma 1 gives us the following improvement of Theorem 2:

Lemma 9 *If G is $\{K_{1,3}, Z_2\}$-free, G is not a cycle, and $\kappa(G) = 2$, then G is in one of Families B_1 through B_7.*

Moreover, Lemmas 2 through 8 let us amend Theorem 3.

Lemma 10 *If G is $\{K_{1,3}, Z_2\}$-free, $\kappa(G) = 3$, and G is not complete, then G is in one of Families D_1 through D_5 or is in Family D_7.*

3 $\{K_{1,3}, Z_2\}$-free Graphs of Connectivity at Leas Four

Here we extend the results in Lemma 10 to determine the structures of all possible $\{K_{1,3}, Z_2\}$-free graphs having connectivity at least four.

Theorem 5 *Let G be a $\{K_{1,3}, Z_2\}$-free connected graph of connectivity at least four. Then either G is complete or G is a member of one of the families shown in Figures 6 and 7.*

Our proof uses induction on the connectivity of the graph. We begin with an outline of the method of proof and a description of the families of graphs shown in Figures 6 and 7 before moving to the proof itself.

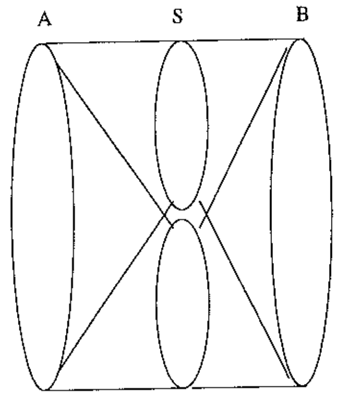
We start with a graph G of connectivity $k \geq 4$ and consider a cut set S of cardinality k. Let v be any vertex in S.

Let $H = G - \{v\}$. Then H is also a $\{K_{1,3}, Z_2\}$-free graph, and it has connectivity $k-1$. Thus H is in one of families D_1 through D_5 in Figure 4, in family D_7 from Figure 5, or in one of families E_1 through E_7 in Figures 6 and 7 and has $S - \{v\}$ as a minimal cut set. Using the structure of H, Theorem 4, and the fact that G is $\{K_{1,3}, Z_2\}$-free, we can determine the possible adjacencies of v.

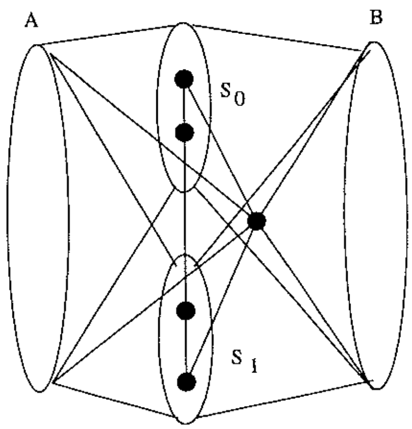
In family E_1, $\langle S \rangle$ can be partitioned into two complete subgraphs. Moreover, every vertex of S is adjacent to every vertex of A and to every vertex of B. In family E_2, again every vertex in S is adjacent to every vertex of A and to every vertex of B. There are two complete subgraphs S_0 and S_1 within S, as well as a set S_2 of vertices such that for each s in S_2 we have $S_0 \not\subseteq \langle N_S(s) \rangle$ and $S_1 \not\subseteq \langle N_S(s) \rangle$. In addition, we require that $\langle S - N_S(s) \rangle$ be complete.

In family E_3, every vertex in S is adjacent to all but at most one vertex of B. Let S_b be the set of vertices of S that are not adjacent to the vertex b in B. Then $\langle S_b \rangle$ is complete, since if $x, y \in S_b$ and $z \in B - \{b\}$ we have a claw $\langle z, b, x, y \rangle$. There are also two complete subgraphs S_0 and S_1 in S such that every vertex in S_0 and every vertex in S_1 is adjacent to every vertex of B. Family E_4 is like family E_3, but it has a subgraph $\langle \{a\} \cup S_0 \cup S_1 \cup S_2 \cup B \rangle$ that belongs in family E_2 rather than in family E_1.

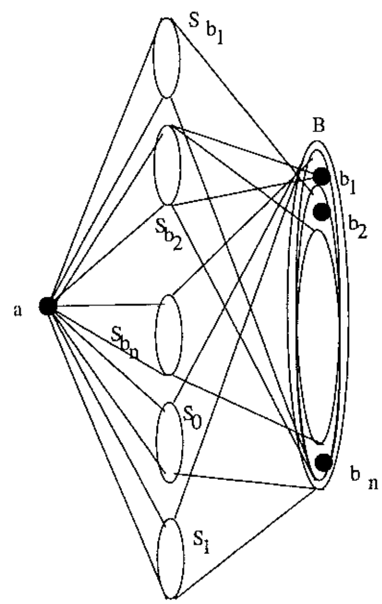
In family E_5, $\langle S \rangle$ is complete and every vertex in S is adjacent to a. We let x be adjacent to all but one vertex of B. It is possible to



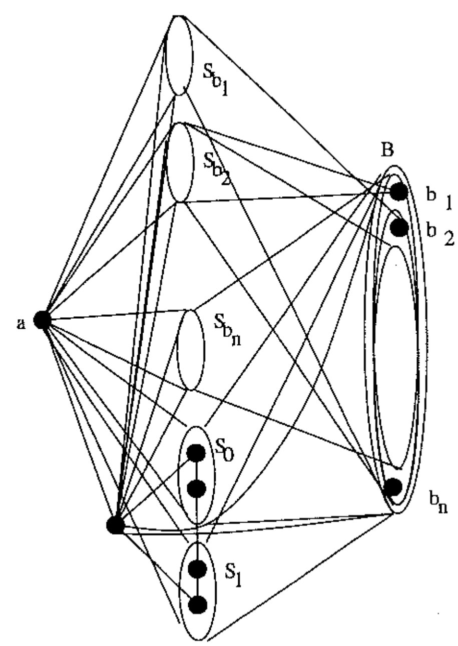
Family E_1
A and B are complete.



Family E_2
A and B are complete.



Family E_3
B is complete.



Family E_4
B is complete.

Figure 6: $\{K_{1,3}, Z_2\}$-free Families.

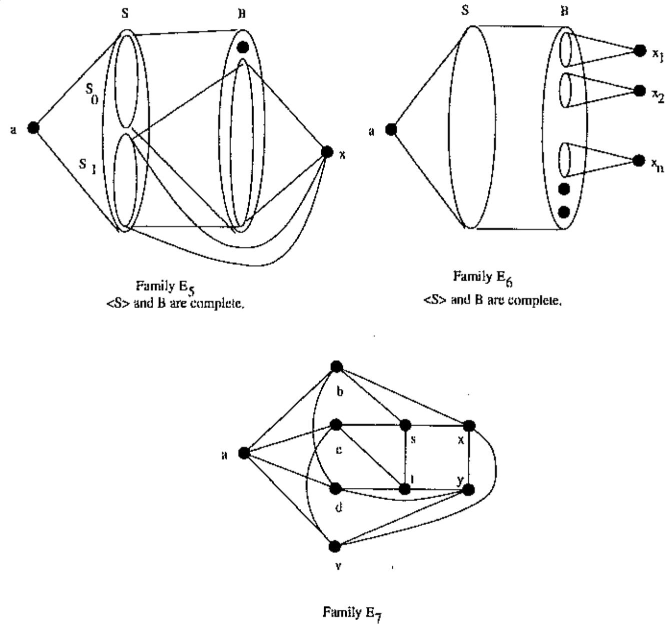


Figure 7: More $\{K_{1,3}, Z_2\}$-free Families.

partition S into two sets of vertices: S_0, the set of vertices adjacent to every vertex of B and S_1, the set of vertices adjacent to every vertex of $N[x]$. In family E_6, $\langle S \rangle$ is complete and every vertex in S is adjacent to a and to every vertex in B. For every $1 \leq i \leq n$, $N(x_i) \geq k$ and for every $1 \leq i, j \leq n, i \neq j$, we have $N(x_i) \cap N(x_j) = \emptyset$.

Proof: We will proceed by induction on $\kappa(G)$. If $\kappa(G) = 1, 2$, or 3, then G falls into one of the families in Theorem 1 or in Lemmas 9 or 10.

First note that any graph in family D_1 from Figure 4 could be considered a member of family E_1 in Figure 6; any graph in family D_2 from Figure 4 could be considered a member of family E_3 in Figure 6; any graph in family D_3 from Figure 4 could be considered a member of family E_6 in Figure 7; and any member of family D_4 from Figure 4 or family D_7 from Figure 5 could be considered a member of family E_5 in Figure 7. Therefore, we can assume that every graph of connectivity $k-1$ falls in one of Families E_1 through E_7 of Figures 6 and 7 unless it is in family D_5 in Figure 4.

Now, suppose that G is a k-connected, $\{K_{1,3}, Z_2\}$-free graph with connectivity $\kappa(G) = k \geq 4$, G is not complete, S is a cut-set of G such that $|S| = k$ and $v \in S$. Consider the subgraph $H =< G - \{v\} >$ of G. Note that H has connectivity $k-1$, so it is either a complete graph, in family D_5 from Figure 4, or is in one of Families E_1 through E_7 from Figure 6.

CASE 1. Suppose that H is a complete graph. If v is not adjacent to every vertex of H, then it is not in a minimal cut set S of G. Therefore, G must be a complete graph as well.

CASE 2. Suppose that H is a member of family E_1 from Figure 6.

Claim 1 *Unless $|V(A)| = 1$ and $|V(B)| > 1$, v must be adjacent to every vertex of A and to every vertex of B.*

Suppose that $|V(A)| = |V(B)| = 1$. Let $A = \{a\}$ and $B = \{b\}$. Then v must be adjacent to both a and b in order to be a member of the minimal cut set S.

Suppose that $|V(A)| = |V(B)| = 2$. Then v must be adjacent to at least one vertex of A and one vertex of B in order to be a member of the cut set S. If v is adjacent to exactly one vertex of A and one vertex of B, then v must be adjacent to some vertex of $S - \{v\}$ since otherwise we have a cut set of cardinality two consisting of the two vertices adjacent to v. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, and let $v \sim a_1$ and $v \sim b_1$. Assume that there is some $s \in S$ such that $v \sim s$. Then $\langle s, a_2, v, b_2 \rangle$ is a claw unless v is adjacent to either a_2 or b_2. Suppose without loss of generality that $v \sim a_2$, $v \not\sim b_2$. Then $\langle A \cup B \cup \{v\} \rangle \cong Z_2$, which contradicts our assumption that G is Z_2-free. Therefore, v must be adjacent to every vertex of A and to every vertex of B.

Assume that $|V(A)| \geq 2$ and that $|V(B)| > 2$. Again, v must be adjacent to at least one vertex of A and at least one vertex of B in order to be a member of the cut set S. Let a_1 be a vertex in A that is adjacent to v and let b_1 be a vertex in B that is adjacent to v. If there are two vertices $b_2, b_3 \in B$ such that $v \not\sim b_2$ and $v \not\sim b_3$, then $\langle b_2, b_3, b_1, v, a_1 \rangle \cong Z_2$, which is a contradiction of our assertion that G is $\{Z_2\}$-free. Therefore, v must be adjacent to all but at most one vertex of B. Similarly, v must be adjacent to all but at most one vertex of A. Because v is adjacent to all but at most one vertex of B and $|V(B)| > 2$, v is adjacent to at least two vertices of B. Let b_1 and b_2 be vertices of B that are adjacent to v, and assume there exist vertices a_1 and a_2 in A such that $v \sim a_1$ and $v \not\sim a_2$. Then $\langle b_1, b_2, v, a_1, a_2 \rangle$ is a copy of Z_2 in G, which contradicts our assumption that G is Z_2-free. Thus v must be adjacent to every vertex of A. Now since v must be adjacent to at least two vertices in A, we can use similar reasoning to show that v must be adjacent to every vertex of B. This concludes the proof of our claim.

If $\langle S - \{v\} \rangle$ is complete, then G is a member of family E_1. Otherwise there exist two disjoint nonempty subsets S_1 and S_2 of $S - \{v\}$ such that $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are complete and $S_1 \cup S_2 = S - \{v\}$. If v is adjacent to every vertex from either S_1 or S_2, we are once again in family E_1.

Recall that $|S - \{v\}| \geq 3$. Suppose without loss of generality that $|S_1| = 1$ or that $|S_2| = 1$. If v is adjacent to the vertex in S_1, then G is a member of family E_1 and we are done. Suppose instead that v is not adjacent to the vertex in S_1, and that it is adjacent to some, but not all, of the vertices in S_2. Then every vertex in $S_2 - N(v)$ must be adjacent to the vertex in S_1, since otherwise we have a claw centered at a. But since $< S_2 >$ is complete and every vertex in $S_2 - N(v)$ is adjacent to the vertex in S_1, we can partition $S \cup \{v\}$ into the complete subgraphs $< S_1 \cup (S_2 - N(v)) >$ and $< N_{S_2}(v) \cup \{v\} >$ and we are again in family E_1.

Suppose that $|S_1| \geq 2$ and $|S_2| \geq 2$. Again, if v is adjacent to every vertex in S_1 or to every vertex in S_2, then G is in family E_1 and we are done. Suppose instead that v is not adjacent to every vertex in S_1 and it is not adjacent to every vertex in S_2, and suppose that one of the cliques, say S_1, is maximal. Note that $\langle S - N(v) \rangle$ must be complete since otherwise we have a claw centered at a. Since $|S_1| \geq 2$ and $|S_2| \geq 2$, we know there exist vertices $s_{1,1}$ in $S_1 - N(v)$, $s_{1,2}$ in $S_1 \cap N(v)$, $s_{2,1}$ in $S_2 - N(v)$, and $s_{2,2}$ in $S_2 \cap N(v)$ such that $s_{1,2} \sim s_{1,1}$, $s_{1,1} \sim s_{2,1}$, and $s_{2,1} \sim s_{2,2}$. Thus, there is an induced cycle containing exactly these five vertices.

Suppose there is another vertex u in S_1. We first claim that u must be adjacent to four of the five vertices in the cycle induced by $\{v, s_{1,2}, s_{1,1}, s_{2,1}, s_{2,2}\}$. In order to avoid $\langle u, s_{1,1}, s_{1,2}, s_{2,2}, s_{2,1} \rangle \cong Z_2$, u must be adjacent to either $s_{2,1}$ or to $s_{2,2}$. Suppose that u is adjacent to $s_{2,2}$. Then s is adjacent either to v or to $s_{2,1}$ since otherwise

$\langle s_{2,2}, v, s_{2,1}, u\rangle \cong K_{1,3}$. Either way, u is adjacent to four of the five vertices in the cycle. Suppose instead that u is not adjacent to $s_{2,2}$. Then u must be adjacent to $s_{2,1}$ and, similarly, u must be adjacent to v. Similarly, any vertex in $S_2 - \{s_{2,1}, s_{2,2}\}$ must have the same property.

This puts us in family E_2.

CASE 3 Suppose that H is a member of family E_2 as shown in Figure 6. Then v is adjacent to every vertex of A and to every vertex of B by the same reasoning given in Case 2.

Suppose that v is adjacent to every vertex of S_0. Consider $S_2 - N(v)$. Since G is claw-free, v must be adjacent to every vertex in S_1 that is not adjacent to every vertex of $S_2 - N(v)$. Therefore, G is in family E_2. Our reasoning is similar if v is adjacent to every vertex of S_1.

Suppose that $S_0 \not\subseteq N(v)$ and that $S_1 \not\subseteq N(v)$. Then $\langle S - N(v)\rangle$ must be complete in order to avoid a copy of $K_{1,3}$. Therefore, G is a member of family E_2.

CASE 4. Suppose that H is a member of family E_3 as shown in Figure 6. Then v is adjacent to a since it is a member of the minimal cut set S, and v is adjacent to at least one vertex of B for the same reason. In order to avoid a copy of Z_2 as shown in Figure 8, v must be adjacent to all but at most one vertex of B. Suppose there is

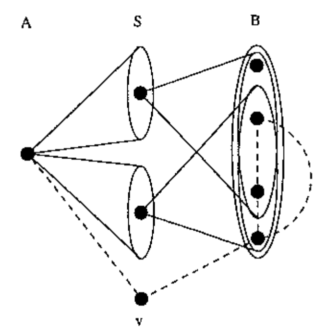


Figure 8: The Z_2 is shown by the dotted lines.

some vertex w in B such that $v \not\sim w$. Let S_w be the set of vertices in S that are not adjacent to w. Suppose that s is a vertex of S_w that is not adjacent to v. Then for any vertex $x \in B$ such that $x \neq w$, we have $\langle x, w, v, s\rangle \cong K_{1,3}$, which contradicts our assumption that G

is claw-free. Thus v must be adjacent to every vertex of S_w, which implies that G is a member of family E_3.

Suppose that v is adjacent to every vertex in B. Consider the set of vertices in S that are adjacent to every vertex in B. Let $< S_0 >$ be the largest complete subgraph of the graph induced by these vertices, and let S_1 contain all the vertices remaining in that set. Note that $< S_1 >$ must be complete. Then by the same reasoning given in Case 1, either v is adjacent to every vertex of S_0 or S_1, or if we let S' be the set of vertices in S adjacent to every vertex of B, then $\langle \{a\} \cup S' \cup B \rangle$ is in family E_2, in which case we are in Family E_4.

CASE 5. Suppose that H is a member of family E_4 as shown in Figure 6. Then the argument is similar to that shown in Cases 3 and 4.

CASE 6. Suppose that H is a member of family E_5 as shown in Figure 7. Then v must be adjacent to a since v is a member of the minimal cut set S and v must be adjacent to either x or to some vertex of B. Recall that S_0 is the set of vertices in S that are adjacent to every vertex of B, that S_1 is the set of vertices in S that are adjacent to every vertex of $\{x\} \cup N_B(x)$ and that $\kappa(H) \geq 3$. Suppose that $|V(B)| = 2$. Then $|N_B(x)| = 1$. Let b_1 be the vertex in $N_B(x)$ and $b_2 = B - N_B(x)$. Since $\kappa(H) \geq 3$, either $|S_0| > 1$ or $|S_1| > 1$. If $|S_0| = 1$, then $S_0 \cup \{b_1\}$ is a cut set of G since it disconnects b_2 from the rest of the graph. Since $|S_0 \cup \{b_1\}| < |S|$, this contradicts our assumption that S is a smallest cut set of H, so $|S_0| \geq 2$. But now $S_1 \cup \{b_1\}$ is a cut set because its removal disconnects x, and S cannot be the smallest cut set of H because $|S_1 \cup \{b_1\}| < |S|$. Therefore, $|V(B)| > 2$.

Subcase 6.1 Suppose that $v \not\sim x$. Then v must be adjacent to all but at most one vertex of B in order to avoid a copy of Z_2 as shown in Figure 8. Moreover, that vertex cannot be in $N_B(x)$ since otherwise we have a copy of Z_2 as shown in Figure 9. But then v must be adjacent to every vertex of B in order to avoid a claw centered at one of the vertices in $N_B(x)$. Let w be the vertex in B that is not adjacent to x, and let $b \in N_B(x)$. Note that v must be adjacent to every vertex of S_0 since otherwise for some $s \in S_0$ we have a claw $\langle b, v, s, x \rangle$. If there is some $s \in S$ such that $s \sim x$ but $s \not\sim v$, we have $\langle s, x, b, w, v \rangle \cong Z_2$, which contradicts our assumption that G is $\{Z_2\}$-free. Therefore, v is adjacent to every vertex of $S - \{v\}$ and so

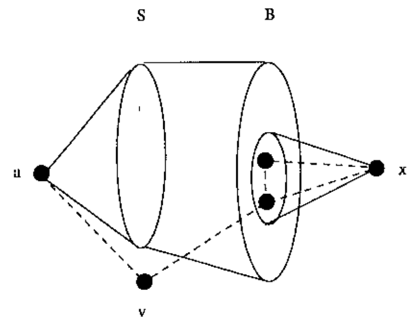


Figure 9: The Z_2 is shown by the dotted lines.

$< S >$ must be complete and G is in family E_5.

Subcase 6.2 Suppose that $v \sim x$. Then v must be adjacent to all but at most one vertex of $N_B(x)$ in order to avoid a copy of Z_2 as shown in Figure 10. Note that v must be adjacent to all but at most

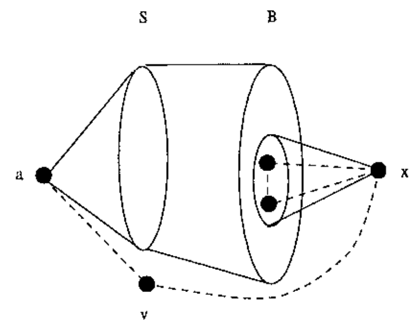


Figure 10: The Z_2 is shown by the dotted lines.

one vertex of B in order to avoid a copy of Z_2 like the one shown in Figure 8. Moreover, v cannot be adjacent to the vertex w in $B - N_B(x)$ since such an adjacency would give us a claw $\langle v, a, b, x\rangle$. This implies that v is adjacent to x and to every vertex in $N_B(x)$, but not to w. Thus, v must be adjacent to every vertex s in S that is adjacent to x, since otherwise for every $b \in N_B(x)$, we have $\langle b, w, s, v\rangle \cong K_{1,3}$. But this means that $< S >$ is complete because otherwise we have a copy of Z_2 as shown in Figure 11. Therefore, G is a member of family E_5.

CASE 7. Suppose that H is a member of family E_6. Note that v must be adjacent to a. Consider the following two subcases.

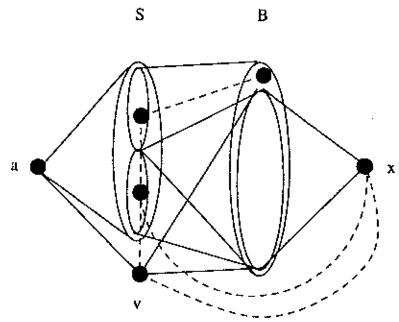


Figure 11: The Z_2 is shown by the dotted lines.

Subcase 7.1 Assume that v is not adjacent to any vertex x_i, $1 \leq i \leq n$. Note that $|N(x_i)| \geq k-1 \geq 3$ or we would have a smaller cut set than S. Then v must be adjacent to some vertex in B. Note that v must be adjacent to all but at most one vertex of B since otherwise we have a Z_2 as shown in Figure 12. If that one vertex is

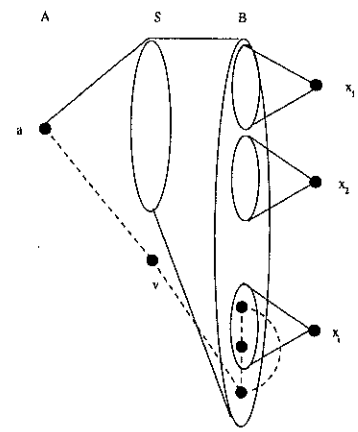


Figure 12: The Z_2 is shown by the dotted lines.

in some $N_B(x_i)$, then we have a copy of Z_2 as shown in Figure 13. Therefore, assume we have some vertex $z \in B$ such that $v \not\sim z$, and for all i, $z \not\sim x_i$. Then for every i, and for every $y_i \in N(x_i)$, there is a claw $\langle y_i, v, z, x_i \rangle$. Thus, v must be adjacent to every vertex in B.

Suppose that there is some vertex $u \in S - \{v\}$ such that $u \not\sim v$. If $b_i \in N(x_i)$, then $\langle b_i, x_i, u, v \rangle \cong K_{1,3}$. Therefore, $\langle S \rangle$ must be complete and so G is a member of family E_6.

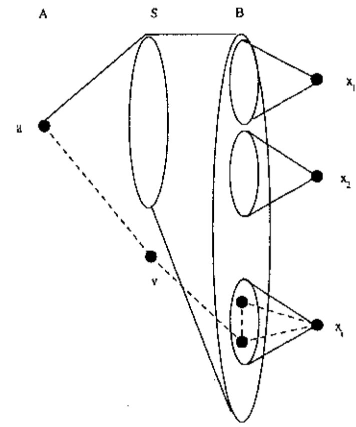


Figure 13: The Z_2 is shown by the dotted lines.

Subcase 7.2 Suppose that there is some i such that $v \sim x_i$. Then v cannot be adjacent to any $x_j, j \neq i$, and v cannot be adjacent to any vertex in $B - N(x_i)$ by Theorem 4. Now we proceed in a manner similar to that used in Case 6. If v is not adjacent to any vertex in B, then we have a Z_2 as shown in Figure 14 and if v is adjacent to a vertex in B, we have a Z_2 as shown in Figure 15. Therefore, $v \not\sim x_i$ for any i.

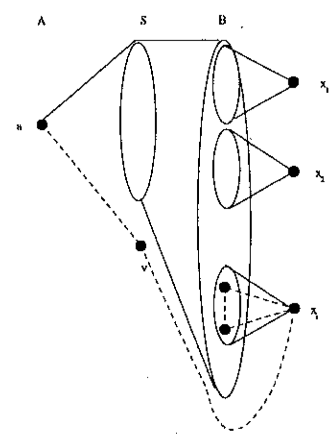


Figure 14: The Z_2 is shown by the dotted lines.

CASE 8. Suppose that H is a member of family D_5 in Figure 4. Note that v must be adjacent to a since it is a member of the minimal cut set S. Similarly, v must be adjacent to x, y, s, or t.

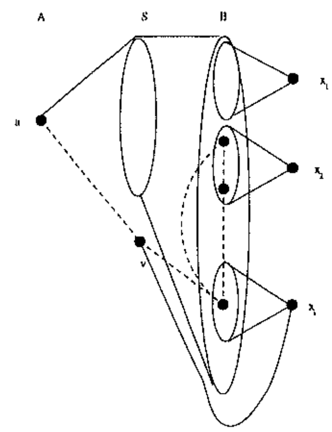


Figure 15: The Z_2 is as shown by the dotted lines.

Subcase 8.1 Suppose that v is adjacent to x. Then v must be adjacent to either b or y in order to avoid a claw.

Suppose first that v is not adjacent to y. Then $v \sim b$ in order to avoid a claw and v must be adjacent to t because otherwise $\langle v, a, b, s, t \rangle \cong Z_2$. But then $\langle v, a, t, x \rangle$ is a claw, which contradicts our assumption that v is claw-free. Thus, v must be adjacent to y.

Assume that v is adjacent to y. Then v must be adjacent to c since otherwise $\langle x, y, v, a, c \rangle \cong Z_2$. Note that v cannot be adjacent to s or t because such an adjacency would create a claw and that v cannot be adjacent to either b or d since such an adjacency would create a claw. Therefore, v is a member of family E_7.

Subcase 8.2 Suppose that v is adjacent to y. By symmetry, this subcase is exactly the same as Subcase 8.1.

Subcase 8.3 Suppose that v is not adjacent to either x or y, but is adjacent to s. Then v must be adjacent to t in order to avoid a claw. Moreover, v must either be adjacent to c or be adjacent to both b and d in order to avoid claws. Suppose first that $v \sim c$. Then $\langle v, c, t, x, y \rangle \cong Z_2$. Therefore, $v \not\sim c$, which implies that v is adjacent to b and to d. But then $\langle a, v, d, y, x \rangle \cong Z_2$. Thus, v must be adjacent to both x and y.

Subcase 8.4 Suppose that v is not adjacent to either x or y, but is adjacent to t. By symmetry, this subcase is the same as Subcase 8.3.

CASE 9. Suppose that H is a member of family E_7 in Figure 7. Then by reasoning similar to that found in Subcase 8.3, v cannot be adjacent to s, t, x, or y. Therefore, H cannot be a member of family E_7.

Because we have looked at all possible choices for H, this concludes our proof of Theorem 5. □

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