Hamiltonian Connected Graphs Involving Forbidden Subgraphs

Guantao Chen
Dept of Math/CS
Georgia State Univeristy
Atlanta, GA 30303

Ronald J. Gould
Dept of Math/CS
Emory University
Atlanta, GA 30322

December 9, 1998

Abstract

We consider the pair of graphs S and T such that every 3-connected $\{S, T\}$-free graph is hamiltonian connected. Such pairs are called hamiltonian connected pairs. We show three new hamiltonian connected pairs. Combing our results and a result of Shepherd, we conclude that if $\{S, T\}$ is a pair of graphs such that every 2-connected $\{S, T\}$-free graph is hamiltonian then every 3-connected $\{S, T\}$-free graph is hamiltonian connected.

1 Introduction

Let H be a graph. A graph G is said H-free if G does not contain H as an induced subgraph. More generally, given a family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ of graphs, we say that a graph G is $\mathcal{F}$-free if G contains no induced subgraph isomorphic to any H_i, $i = 1, 2, \ldots, k$. We call the graphs in $\mathcal{F}$ *forbidden subgraphs.* The use of forbidden subgraphs to obtain classes of graphs possessing special properties has long been a common graphical technique, especially for hamiltonicity. If P is a hamiltonian property (like traceable, hamiltonian, pancyclic, etc.), let $\kappa(P)$ denote the least connectivity possible in a graph with property P. In the last few years, the problem of determine the all pairs of connected graphs $\{H_1, H_2\}$ such that any $\kappa(P)$-connected $\{H_1, H_2\}$ -free graph will possess hamiltonian property P has been investigated. The idea was introduced by Bedrossian in[1] who considered it for Hamiltonian and pancyclic graphs. Following him, Faudree, Gould, Ryjacek, Schiermeyer [5, 6, 7]investigated it for traceable, panconnected, and cycle extendable. They also consider the problem when P is hamiltonian connected. However, only little progress has been made along this line. The purpose of this paper is to show that three important pairs for hamiltonian graphs are also pairs for hamiltonian connected graphs. Some of the graphs most commonly involving in forbidden families for hamiltonian properties are shown in Figure 1. It has been pointed out [5] that the star $K_{1,3}$, called the *claw*, must be a part of these forbidden families.

Figure 1

One of the earliest forbidden subgraph results dealing with hamiltonian properties is the following result due to Duffus, Gould, and Jacobson [4]. The graph $K_{1,3}$ and N are shown in Figure 1.

Theorem 1 *Let G be a $\{K_{1,3}, N\}$-free graph. Then*

1. if G is connected , then G is traceable and
2. if G is 2-connected, then G is hamiltonian.

Shepherd [9] showed that a result similar to Theorem 1 holds.

Theorem 2 *If G is a 3-connected $\{K_{1,3}, N\}$-free graph, then G is hamiltonian connected.*

Regarding hamiltonian graphs, the following results have also been obtained.

Theorem 3 *[3] If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph, then G is hamiltonian.*

Theorem 4 *[8] If G is a 2-connected $\{K_{1,3}, Z_2\}$-free graph, then G is hamiltonian.*

Theorem 5 *[1] If G is a 2-connected $\{K_{1,3}, W\}$-free graph, then G is hamiltonian.*

Theorem 6 *[6] If G is a 2-connected $\{K_{1,3}, Z_3\}$-free graph of order $n \geq 10$, then G is hamiltonian.*

A characterization of all pairs that imply a 2-connected graph is hamiltonian was accomplished in [1]. However, graphs of small order were used in the proof to eliminate certain graphs, namely Z_3. Later Theorem 6 was verified and the following result was obtained by Faudree and Gould.

Theorem 7 *[5] Let R and S be connected graphs ($R, S \neq P_3$) and G a 2-connected graph of order $n \geq 10$. Then G is (R, S) -free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is a subgraph of N, P_6, W, Z_3.*

Inspired by the above results, we obtain the following three results in this paper.

Theorem 8 *If G is a 3-connected $\{K_{1,3}, P_6\}$-free graph, then G is hamiltonian connected.*

Theorem 9 *If G is a 3-connected $\{K_{1,3}, W\}$-free graph, then G is hamiltonian connected.*

Theorem 10 *If G is a 3-connected $\{K_{1,3}, Z_3\}$-free graph, then G is hamiltonian connected.*

We place the three proofs of these ressults in sections 3, 4, and 5 after we prove a useful lemma.

The terminology and notation of this paper will generally follow closely that of [2]. All graphs considered here are simple and undirected. The vertex set of a graph G will be denoted by $V(G)$ while the edge set is denoted by $E(G)$. Let $v \in V(G)$ and H be a subgraph (or a vertex subset) of G. The neighborhood of v in H, denoted by $N_H(v)$, is the set of vertices in H adjacent to v in G. If $A \subseteq V(G)$, we define $N_H(A) = \cup_{v \in A} N_H(v)$. Let A and B be two disjoint vertex sets (or vertex disjoint subgraphs) of G. We define $E(A, B)$ be the set of edges with one endvertex in A and the other one in B. If H is a connected induced subgraph of G and $x, y \in N(H) \cup V(H)$, then we use xHy to denote a shortest $x - y$ path in G with all internal vertices in H.

Let $P = P[u, v]$ be a path in G from u to v including both u and v. We define $P(x, y]$, $P[x, y)$, $P(x, y)$ in the obvious manner. For any $w \in V(P[u, v])$, we use w^{+i} to denote the $i-$th successor of w (when it exists) along P in the direction from u to v while w^{-i} for the i-th predecessor of w along P (when it exists) along P in the reversed direction. When $i = 1$, we write $w^+ = w^{+1}$ and $w^- = w^{-1}$. For any two vertices x and y of P which are in that order along the path P, we let $P[x, y]$ denote the subpath from x to y while $P[y, x]$ denote the same path with a reversed orientation. If H is a connected component of $G - V(P)$ and there is a $u - v$ path $Q[u, v]$ which is longer than $P[x, y]$ and $V(Q[x, y]) \subseteq V(P[x, y]) \cup V(H)$, we say $P[x, y]$ can be extended by H. The path $P[u, v]$ is called a *maximal* $x - y$ path if there does not exist a path $Q[u, v]$ such that $V(Q[u, v]) \supset V(P[u, v])$. Clearly, any longest $u - v$ path is a maximal $x - y$ path.

2 A Lemma

Lemma 2.1 *Let G be a 3-connected $K_{1,3}$-free graph and let x and z be two vertices of G. If for every maximal $x-z$ path $P[x,z]$ and every component H of $G-V(P[x,z])$ the equality $N(H)\cap V(P[x,z]) = \{x,y,z\}$ holds, where y is uniquely determined by $P[x,z]$ and H, then G has an $x-z$ hamiltonian path.*

Proof: Suppose, to the contrary, G does not contain an $x-y$ hamiltonian path. Let $P[x,z]$ be a maximal $x-z$ path and H be a component of $G-V(P[x,z])$. Then, $N(H)\cap V(P[x,z]) = \{x,y,z\}$, where $y \in V(P(x,z))$. Let $z^* \in V(H)$ such that $zz^* \in E(G)$. By the maximality of $P[x,z]$, $z^*z^- \notin E(G)$. Since G is $K_{1,3}$-free, for each $a \in V(P(x,y))$ if $az \in E(G)$ then $az^- \in E(G)$. Hence, if $a, b \in V(P(x,y))$ such that $az, bz \in E(G)$ then the path $z^-P[a,b]z$ contains a and b. For this reason, let $Z[z^-,z]$ be a path such that $V(Z(z^-,z)) \subseteq V(P(x,y))$ and $V(Z(z^-,z)) \supseteq N(z)\cap V(P(x,y))$. Let $x^* \in N_H(x)$ and $y^* \in N_H(y)$ and

$$Q[x,z] = xx^*Hy^*yy^-P[y^+,z^-]Z[z^-,z].$$

Let $Q^*[x,z]$ be a maximal path contaiting all vertices of $Q[x,z]$. Since $P[x,z]$ is a maximal $x-z$ path, $W = V(P(x,y)) - V(Q^*(x,z)) \neq \emptyset$. By our hypothesis, $z \in N(W)\cap V(Q^*[x,z])$ which contradicts the choice of the path $Z[z^-,z]$. □

3 Proof of Theorem 8

Suppose that to the contrary there exists a 3-connected $\{K_{1,3}, P_6\}$ -free graph such that there are two vertices x, z so that G does not contain an $x-z$ hamiltonian path. By Lemma 2.1, we only need to show that $N(H)\cap V(P[x,z]) = \{x,y,z\}$ for each maximal $x-z$ path $P[x,z]$ and each component H of $G-$ $V(P[x,z])$, where $y \in V(P(x,z))$. Suppose, to the contrary and without loss of generality, there exists a maximal $x-z$ path $P[x,z]$ and a component H such that there are three vertices $u,v,w \in N(H)\cap V(P[x,z])$ with u, $v \in V(P(x,z))$ and u,v,w are in that order along $P[x,z]$.

Since G is $K_{1,3}$-free, u^-u^+, $v^-v^+ \in E(G)$. Since $P[x,z]$ is a maximal $x-z$ path, there exists a vertex in $P(u,v)$ which is not adjacent to u. Let u_2 be the first such vertex along $P(u,v)$ and let $u_1 = u_2^-$. We define v_1, v_2 in the same way except in $P(v,w)$ not in $P(u,v)$. Since P cannot be extended by H, the following holds.

- For each $j = 1,2$, $uv_j \notin E(G)$. Otherwise, if $v_j \neq v^+$, the path

  $$P[x,u^-]P[u^+,v^-]P[v^+,v_j^-]vHuv_jP[v_j,z]$$

  gives a contradiction to the maximality of $P[x,z]$; if $v_j = v^+$, the path

  $$P[x,u^-]P[u^+,v]HuP[v_j,z]$$

is longer than P, a contradiction to the maximality of $P[x,z]$. Similarly, we have

- For each $j = 1,2$, $u_jv \notin E(G)$. And,
- For each $i = 1,2$ and $j = 1,2$, $u_iv_j \notin E(G)$.

Summarizing the above results, we obtain that

$$E(\{u,u_1,u_2\},\{v,v_1,v_2\}) = \emptyset.$$

Thus, $u_2u_1uHvv_1v_2$ is an induced path of at least 6 vertices, a contradiction. □

4 Proof of Theorem 10

Suppose that to the contrary there exists a 3-connected $\{K_{1,3}, Z_3\}$ -free graph such that there are two vertices x, z so that G does not contain an $x-z$ hamiltonian path. By Lemma 2.1, we only need to show that $N(H) \cap V(P[x,z]) = \{x, y, z\}$ for each maximal $x-z$ path $P[x,z]$ and each component H of $G-$ $V(P[x,z])$, where $y \in V(P(x,z))$. Suppose, to the contrary and without loss of generality, there exist a maximal $x-z$ path $P[x,z]$ and a component H such that there are three vertices $u, v, w \in N(H) \cap V(P[x,z])$ with $u, v \in V(P(x,z))$ and u, v, w are in the order along $P[x,z]$. Furthermore, we assume that $|V(P[x,u])|$ is maximum under the above conditions.

Since G is claw-free, then u^-u^+, $v^-v^+ \in E(G)$ and $w^-w^+ \in E(G)$ if $w \neq z$. We will devote the remainder of the proof into two cases. Since $P[x,z]$ is a maximal $x-z$ path, the following hold.

$$
\begin{aligned}
E(\{u, u^+, u^{+2}\}, \{v, v^+, v^{+2}\}) &\subseteq \{uv\}, \text{ and} \\
E(\{u, u^{-1}, u^{-2}\}, \{v, v^{-1}, v^{-2}\}) &\subseteq \{uv\}.
\end{aligned}
$$

Case 4.1 *The edge uv is not present.*

Since $u^{+2}u^+uHvv^+$ does not induce a Z_3, $uu^{+2} \notin E(G)$. Similarly, uu^{-2}(if $u \neq x^+$), vv^{+2}, $vv^{-2} \notin E(G)$. By the maximality of $P[x,z]$, $|V(P(u^{+1}, v^{-1}))| \geq 1$. Since $u^-u^+uHvv^-$ does not induce a Z_3, then $u^+v^- \in E(G)$. Since $u, u^{+2}, v^- \in N(u^+)$ and $uu^{+2}, uv^- \notin E(G), u^{+2}v^- \in E(G)$. If $|V(P(u^+, v^-))| \geq 2$, then $u^{+2}v^{-2} \in E(G)$ since $u^{+2}, v^{-2}, v \in N(v^+)$ and $u^{+2}v, v^{-2}v \notin E(G)$. Then, $u^{+2}v^{-2}v^{-1}vHu$ induces a Z_3, a contradiction. Thus, $u^{+2} = v^{-2}$.

Since $P[x,z]$ is a maximal $x-z$ path, $uw^-, vw^- \notin E(G)$ and G is $K_{1,3}$-free, $N(w) \not\supseteq \{u, v\}$ holds.

If $w \neq z$, then $N(w) \cap \{u^-, u^+, u^{+2}, v^-, v^+\} = \emptyset$ since $w^-w^+ \in E(G)$ and $P[x,z]$ is a maximal $x-z$ path. Since $u^+u^{+2}v^-vHw$ does not induce a Z_3, $vw \in E(G)$. For the same reason, $uw \in E(G)$, which contradicts $N(w) \not\supseteq \{u, v\}$. Therefore, $w = z$.

Since $w = z$, $N(z^-) \cap \{u^-, u^{+2} = v^{-2}, v^{-1}\} = \emptyset$. Since G is $K_{1,3}$-free and $N_H(z) \neq \emptyset$, $N(y) \cap \{u^-, u^{+2}, v^-\} = \emptyset$. If $uz \notin E(G)$, that the subgraph induced by $v^-u^{+2}u^+uHz$ is not Z_3 implies that $u^+z \in E(G)$. If $vz \notin E(G)$, that the subgraph induced by $u^+u^{+2}v^-vHz$ is not Z_3 implies that $u^+z \in E(G)$. Thus, $u^+z \in E(G)$ holds. Then, $\{u^+, u^{+2}, u^-, z\}$ induces an $K_{1,3}$ centered at u^+, a contradiction.

Case 4.2 *The edge uv is present.*

Since G is $K_{1,3}$-free, $N_H(u) = N_H(v)$. Let $u^* \in N_H(u)$.

Since $u^{+2}u^+uvv^+v^{+2}$ does not induce a Z_3, $uu^{+2} \notin E(G)$. Similarly, uu^{-2}(if $u \neq x^+$), vv^{+2}, $vv^{-2} \notin E(G)$. By the maximality of $P[x,z]$, $|V(P(u^{+1}, v^{-1})| \geq 1$.

Since $u^-u^+uvv^+v^{+2}$ does not induce a Z_3, either $u^-v^+ \in E(G)$ or $u^-v^{+2} \in E(G)$. If $u^-v^+ \in E(G)$, then $u^-v^{+2} \in E(G)$ since $vv^{+2}, u^-v \notin E(G)$ and G is $K_{1,3}$ -free. Thus, $u^-v^{+2} \in E(G)$ holds.

We claim that $u^+u^{+3} \in E(G)$. Suppose, to the contrary, that $u^+u^{+3} \notin E(G)$. Considering the subgraph induced by $u^*vuu^+u^{+2}u^{+3}$, we have either $uu^{+3} \in E(G)$ or $u^{+3}v \in E(G)$ since G is Z_3-free. If $uu^{+3} \in E(G)$, then $u^+u^{+3} \in E(G)$ since $u^*, u^+, u^{+3} \in N(u)$ and $u^+, u^{+3} \notin N(u^*)$, a contradiction. Thus, $u^{+3}v \in E(G)$. Since G is $K_{1,3}$-free, Note that $u^{+4} \in V(P[u,v])$. If $u^{+2}u^{+4} \in E(G)$, the path $P[x,u^-]u^+uu^*P[v,u^{+4}]u^{+2}u^{+3}P[v^+,z]$ is an extension of P by H, a contradiction. Thus, $u^{+2}u^{+4} \notin E(G)$. Similarly, $u^+u^{+4} \notin E(G)$. Considering the subgraph induced by $u^-uu^+u^{+2}u^{+3}u^{+4}$, we have either $u^-u^{+4} \in E(G)$ or $uu^{+4} \in E(G)$. Clearly, that $uu^{+4} \in E(G)$ implies that $u^-u^{+4} \in E(G)$. Thus, $u^-u^{+4} \in E(G)$ holds. The path $P[x,u^-]P[u^{+4},v]u^*P[u,u^{+3}]P[u^+,z]$ shows that P is not a maximal $x-z$ path, a contradiction.

We claim that $|V(P(u^{+1}, v^{-1}))| = 1$. Suppose, to the contrary, $|V(P(u^{+1}, v^{-1}))| \geq 2$. Considering the Z_3-free subgraph induced by $u^{+3}u^{+2}u^{+}uvv^{+}$, we have either $u^{+3}v \in E(G)$ or $u^{+3}v^{+} \in E(G)$. Since $u^{+3}v \in E(G)$ implies that $u^{+3}v^{+} \in E(G)$, $u^{+3}v^{+} \in E(G)$ holds. Considering the subgraph induced by $u^{-}u^{+}uvv^{-}v^{-2}$, we have either $u^{-}v^{-} \in E(G)$ or $u^{-}v^{-2} \in E(G)$. Clearly, that $u^{-}v^{-} \in E(G)$ implies that $u^{-}v^{-2} \in E(G)$. Thus, $u^{-}v^{-2} \in E(G)$ holds. Similarly, we can show that $u^{+2}v^{-} \in E(G)$. Then, the path, $P[x, u]u^{*}vv^{-}u^{+2}u^{+}P^{-}[v^{-2}, u^{+3}]P[v^{+}, z]$, contains all vertices of $P[x, z]$ and vertex u^{*}, a contradiction to the maximality of $P[x, z]$. Hence, $|V(P(u^{+}, v^{-})| = 1$.

If $w \neq z$, in the same manner, we can show that $|V(P(v^{+1}, w^{-1}))| = 1$. Since $u^{-}v^{+2} \in E(G)$ and $v^{+2} = w^{-2}$, the path, $P[x, u^{-}]P^{-}[v^{+2}, u]u^{*}P[w, z]$, shows that $P[x, z]$ is not a maximal path, a contradiction. Thus, $w = z$.

Consider the path $P[x, u^{-}]u^{+}uu^{*}vv^{-}P[v^{+}, z]$ and extend it to a maximal path $Q[x, z]$. Clearly, u^{+2} is not on $Q[x, z]$ otherwise $P[x, z]$ is not a maximal path and u^{+2} forms a component of $G - V(Q[x, z])$. In the same manner as the above, we see that $u^{+2}z \in E(G)$. Since $P[x, z]$ is a maximal path, $u^{+2}z^{-} \notin E(G)$ and $z^{*}z^{-} \notin E(G)$, where $z^{*} \in N_H(z)$. Then, $\{z, z^{-}, u^{+2}, z^{*}\}$ induces a $K_{1,3}$, a contradiction. □

5 Proof of Theorem 9

Suppose that to the contrary there exists a 3-connected $\{K_{1,3}, W\}$ -free graph such that there are two vertices x, z so that G does not contain an $x - z$ hamiltonian path. By Lemma 2.1, we only need to show that $N(H) \cap V(P[x, z]) = \{x, y, z\}$ for each maximal $x - z$ path $P[x, z]$ and each component H of $G-$ $V(P[x, z])$, where $y \in V(P(x, z))$. Suppose, to the contrary and without loss of generality, there exists a path $P[x, z]$ and a component of $G - V(P[x, z])$ such that there are three vertices $u, v, w \in N(H) \cap V(P[x, z])$ with $u, v \in V(P(x, z))$ and u, v, w are in that order along $P[x, z]$.

Since G is $K_{1,3}$-free, $u^{-}u^{+}, v^{-}v^{+} \in E(G)$ and $w^{-}w^{+} \in E(G)$ if $w \neq z$. Since $P[x, z]$ is a maximal path,

$$\begin{aligned} E(\{u^{-2}, u^{-}, u\}, \{v^{-2}, v^{-}, v\}) &\subseteq \{uv\}, \text{ (if } u^{-2} \text{ exists) and} \\ E(\{u, u^{+}, u^{+2}\}, \{v, v^{+}, v^{+2}\}) &\subseteq \{uv\}, \text{ and} \\ E\{u^{-1}, u, v^{-2}, v^{-1}, v\}, \{w^{-1}\}) &= \emptyset. \end{aligned}$$

If $uv \in E(G)$, then $N_H(u) = N_H(v)$ since G is $K_{1,3}$-free. Let $u^{*} \in N_H(u)$. From the above inequalities, we see that uv^{-}, $u^{-}v^{-}$, $uv^{-}2$, $u^{-}v^{-}2 \notin E(G)$. Therefore, $u^{-}uu^{*}vv^{-}v^{-2}$ induces a W, a contradiction. Thus, $uv \notin E(G)$. Similarly, we can show that $vw \notin E(G)$. If $uw \in E(G)$, then $N_H(u) = N_H(w)$ since G is $K_{1,3}$-free. Let $u^{*} \in N_H(u) = N_H(w)$ such that $dist(u^{*}, v)$ is minimum. Let $R[v, u^{*}]$ be the shortest path from v to u^{*} in the subgraph induced by $V(H) \cup \{v\}$, then $N(V(R[v, u^{*})) \cap \{u, w\} = \emptyset$. The subgraph induced by $v^{+}R[v, u^{*}]wuu^{+}$ contains an induced W, a contradiction. Therefore, $uw \notin E(G)$.

Let $Q[u, w]$ be a shortest path from u to w in the subgraph induced by $V(H) \cup \{u, w\}$. Let $R[v, v^{*}]$ be a path from v to $Q[u, w]$ such that

1. $|R[v, v^{*}]|$ is minimum;

2. $|Q[u, v^{*}]|$ is minimum under the above condition.

By the minimalities above, $Q[u, v^{*}]R^{-}[v^{*}, v]$ is an induced path. Let a be the predecessor of v^{*} and b be the successor of v^{*} on the path $Q[u, w]$ and let c be the predecessor of v^{*} on the path of

$P[v, v^*]$. Since G is $K_{1,3}$-free, $bc \in E(G)$. Then, the subgraph induced by $Q[u, v^*]bcR^-[c, v]v^-v^{-2}$ contains an induced W, a contradiction. $\square$

References

[1] Bedrossian, P., *Forbidden subgraphs and minimum degree conditions for hamiltonicity,* Ph.D. Thesis, Memphis State University, 1991

[2] Bondy, J.A., Murty, U.S.R., *Graph theory with applications,* Macmillan, London and Elsevier, New York (1976).

[3] Broersma, H.J., Veldman, H.J., *Restrictions on induced subgraphs ensuring hamiltonicity or pancyclcity of $K_{1,3}$ -free graphs,* Contemporary Methods in Graph Theory (R. Bodendiek), BI-Wiss.-Verl.., Mannheim-Wien-Zurich, (1990) 181-194

[4] Duffus, D., Gould, R.J., Jacobson, M.S., *Forbidden subgraphs and the hamiltonian Theme,* The Theory and Applications of Graphs, ed. by Chartrand, Alavi, Goldsmith, Lesniak, and Lick, (1981) 297-316.

[5] Faudree, R.J., Gould, R.J., *Characterizing forbidden pairs for hamiltonian properties,* Discrete Math, 173(1997) 45-60.

[6] Faudree, R.J., Ryjacek, Z,. Schiermeyer, I., *Forbidden subgraphs and pancyclicity,* Congesssuss Numeratum 109 (1995), 13-32.

[7] Faudree, R.J., Ryjacek, Z., Schiermeyer, I., *Forbidden subgraphs and cycle extendablitity,* J. Combin. Math. Combin. Comput., to appear.

[8] Gould, R.J., Jacobson, M.S., *Forbidden subgraphs and hamiltonian properties of graphs,* Discrete Mathematics, 42(1982) 189-196.

[9] Shepherd, F.B., *Hamiltonicity in claw-free graphs,* J. Combin. Theory Ser. B 53(1991) 173-194.