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DEGREE SETS FOR HOMOGENEOUSLY TRACEABLE NONHAMILTONIAN GRAPHS

BY

RONALD J. GOULD (ATLANTA, GEORGIA)

A graph G is *homogeneously traceable* if for each vertex v of G there exists a hamiltonian path of G with initial vertex v. While every hamiltonian graph is homogeneously traceable, not every homogeneously traceable graph is hamiltonian. For example, the Petersen graph is a homogeneously traceable nonhamiltonian graph. In [1] it was shown that homogeneously traceable nonhamiltonian graphs exist for all orders p except $3 \leqslant p \leqslant 8$. In the construction presented, every homogeneously traceable nonhamiltonian graph of order 9 and greater contained a vertex of degree 2. R. Frucht (personal communication) asked if there exist homogeneously traceable nonhamiltonian graphs with only large degrees. Of course, the Petersen graph is cubic. It is the object of this paper to give a complete answer to this question.

The following result was established in [3] and will be useful.

LEMMA. *If G is homogeneously traceable of order $p \geqslant 3$, then G is 2-connected.*

It is convenient to construct a class of graphs for use in our main result. Define the graphs H_{2n+1}, $n \geqslant 1$, to consist of 2 disjoint cycles, C: $u_1, u_2, \ldots, u_{2n+1}$ and C': $v_1, v_2, \ldots, v_{2n+1}$, and for each $i = 1, 2, \ldots, 2n+1$ join u_i and v_i by a path P_i of length 2. Denote the vertex of degree 2 on P_i by t_i. These graphs are homogeneously traceable and nonhamiltonian for each $n \geqslant 1$.

The *degree set* of a graph G is the set of degrees of the vertices of G.

THEOREM. *Suppose $S = \{n_0, n_1, \ldots, n_k\}$ is a set of $k+1$ $(\geqslant 1)$ positive integers and $n_i \geqslant 2$ for all i $(0 \leqslant i \leqslant k)$. Then S is the degree set of a homogeneously traceable nonhamiltonian graph unless $S = \{2\}$.*

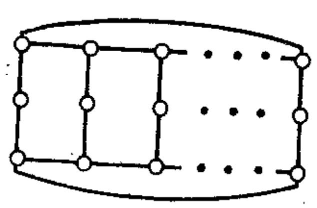

Fig. 1. The graph H_{2n+1}

Proof. Without loss of generality we assume that $n_0 < n_1 < \ldots < n_k$. Suppose $S = \{2\}$. Then, by the Lemma, G is 2-connected. Since G

is 2-regular of order at least 3, then G is a cycle, and hence G is hamiltonian. Thus $\{2\}$ is not the degree set of a homogeneously traceable nonhamiltonian graph. We now consider the converse. Suppose $S \neq \{2\}$. We distinguish three cases.

Case 1. Suppose $3 \in S$. If $S = \{3\}$, the Petersen graph satisfies the Theorem. If $S = \{2, 3\}$, the graph H_3 suffices. Now, if $3 \in S$ but $S \neq \{3\}$ and $S \neq \{2, 3\}$, consider the graph H_l, where l is odd and $l \geqslant \max(3, k)$. We now construct a graph H from H_l.

If $n_0 \neq 2$, then $n_0 = 3$ and each u_i and v_i $(i = 1, 2, \ldots, k)$ has degree 3. For each $i = 1, 2, \ldots, k$ replace the vertex t_i (and its incident edges) by $M_i = K_{n_i+1} - e$, where $e = x_i y_i \in E(K_{n_i+1})$, that is, a copy of the complete graph on $n_i + 1$ vertices minus 1 edge. Then insert the edges $u_i x_i$ and $v_i y_i$. If $l > k$, repeat this argument with M_k replacing each t_j, $k+1 \leqslant j \leqslant l$. Then $\deg_H u_i = \deg_H v_i = 3$ and $\deg_H x = n_i$ for each $x \in V(M_i)$, $i = 1, 2, \ldots, k$; for $l > k$ we have $\deg_H x = n_k$ if $x \in V(M_i)$, $k + 1 \leqslant i \leqslant l$. Thus H has the degree set S.

To see that H is homogeneously traceable note that each M_i is hamiltonian connected as $\deg_{M_i} v \geqslant (|V(M_i)| + 1)/2$ for each $v \in V(M_i)$, $i = 1, 2, \ldots, l$. Thus, by Ore's theorem [2], M_i is hamiltonian connected. To find a hamiltonian path beginning with vertex u_i or v_i $(i = 1, 2, \ldots, l)$ consider the path P in H_l beginning at u_i (or, respectively, at v_i) with vertex t_i replaced by a hamiltonian path through M_i, $i = 1, 2, \ldots, l$. Further, we can find a hamiltonian path with initial vertex $x \in V(M_i)$, $i = 1, 2, \ldots, l$, by beginning with the hamiltonian path P_1 in H_l with initial vertex t_i. If t_i is followed by u_i on P_1, then replace t_i by a hamiltonian $x - x_i$ path in M_i; similarly, if t_i is followed by v_i, then replace t_i by a hamiltonian $x - y_i$ path in M_i. Replace each t_j $(j \neq 1)$ by a hamiltonian $x_j - y_j$ or $y_j - x_j$ path in M_j, the replacement matching the order of u_j and v_j on P_1. Since each M_i $(i = 1, 2, \ldots, l)$ is hamiltonian connected and since there are hamiltonian paths in H_l with initial vertex t_i and second vertex either u_i or v_i, such substitutions yield a hamiltonian path in H with initial vertex t_i. Thus H is homogeneously traceable.

To see that H is not hamiltonian, suppose to the contrary that H is hamiltonian. Then we could start and end some hamiltonian cycle with some vertex $x \in V(M_1)$. Note that the vertices of any M_i must be consecutive (although their particular order may vary) in any hamiltonian cycle of H, as the edges $u_i x_i$ and $y_i v_i$ must be used. However, replacing the subsequence of vertices in M_i with t_i, we produce a hamiltonian cycle in H_l, which is impossible since H_l is nonhamiltonian. Thus H is homogeneously traceable nonhamiltonian with degree set S.

If $n_0 = 2$, we repeat the last argument with vertices t_j $(j = 2, 3, \ldots, l)$, leaving t_1 unchanged. Then $\deg_H t_1 = 2$ and again H has the degree set S.

An analogous argument shows H is homogeneously traceable and nonhamiltonian.

Case 2. Suppose $S = \{n_0, n_1, \ldots, n_k\}$ and $n_i \geq 4$ for $i = 0, 1, \ldots, k$. Again consider the graph H_l, where l is odd and $l \geq \max(3, k)$. We next construct a graph H from H_l.

Remove vertex u_i $(i = 1, 2, \ldots, l)$ and its incident edges and in each case insert a copy of the graph $L_i = K_{n_0+1} - \{x_i f_i, f_i w_i\}$, where $x_i, f_i, w_i \in V(L_i)$. Then remove each vertex v_i $(i = 1, 2, \ldots, l)$ and replace it with a copy of $M_i = K_{n_0+1} - \{r_i s_i, s_i z_i\}$ for $r_i, s_i, z_i \in V(M_i)$. Now insert the edges $x_i w_{i+1}$ and $r_i z_{i+1}$ for $i = 1, 2, \ldots, l-1$ and $x_l w_1$ and $r_l z_1$.

Remove each vertex t_i and its incident edges $(i = 1, 2, \ldots, l)$ and insert a copy of $G_i = K_{n_i+1} - \{a_i b_i, c_i d_i\}$, where $a_i, b_i, c_i, d_i \in V(G_i)$. Then add the edges $f_i a_i, f_i c_i, s_i b_i, s_i d_i$ $(i = 1, 2, \ldots, l)$. If $l > k$, then let $G_i = G_k$ for each $i = k+1, k+2, \ldots, l$.

As before, the graphs G_i, M_i, and L_i $(i = 1, 2, \ldots, l)$ are hamiltonian connected. An argument analogous to that in the last case shows that H is homogeneously traceable. Further, since at most one of the edges $f_i a_i, f_i c_i$ (and similarly $s_i b_i, s_i d_i$) can appear on any hamiltonian path or cycle for each i, an analogous argument shows that H is not hamiltonian. Since H has the degree set S, case 2 is completed.

Case 3. Suppose $S = \{2, n_1, n_2, \ldots, n_k\}$ and $n_i \geq 4$ for $i = 1, 2, \ldots, k$. Here we proceed exactly as in case 2 except vertex t_1 is not changed and the additional edge $f_1 s_1$ is inserted. The graph H (see Fig. 2) so constructed has the degree set S. However, the edge $f_1 s_1$ can appear on no hamiltonian path, so the argument of case 2 shows H to be homogeneously traceable and nonhamiltonian.

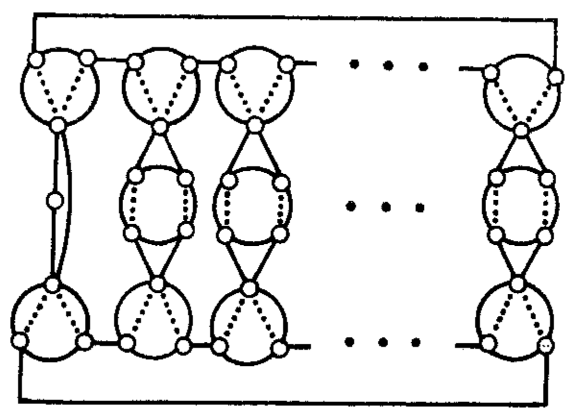

Fig. 2. The graph H (dotted lines represent missing edges in a complete graph)

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DEPARTMENT OF MATHEMATICS
EMORY UNIVERSITY
ATLANTA, GEORGIA

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