On k-Ordered Graphs

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Abstract: Ng and Schultz [J Graph Theory 1 (1997), 45–57] introduced the idea of cycle orderability. For a positive integer k, a graph G is *k-ordered* if for every ordered sequence of k vertices, there is a cycle that encounters the vertices of the sequence in the given order. If the cycle is also a Hamiltonian cycle, then G is said to be *k-ordered Hamiltonian*. We give sum of degree conditions for nonadjacent vertices and neighborhood union conditions that imply a graph is k-ordered Hamiltonian. J Graph Theory 35: 69–82, 2000

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1. INTRODUCTION

Over the years Hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example Hamiltonian connectedness. Recently a new strong Hamiltonian property was introduced in [7].

We say a graph G on n vertices, $n \geq 3$ is *k-ordered* for an integer k, $1 \leq k \leq n$, if for every sequence $S = (x_1, x_2, \ldots, x_k)$ of k distinct vertices in G, there exists a cycle that contains all the vertices of S in the designated order. A graph is *k-ordered Hamiltonian* if for every sequence S of k vertices there exists a Hamiltonian cycle which encounters S in its designated order. We will always let $S = (x_1, x_2, \ldots, x_k)$ denote the ordered k-set. If we say a cycle C contains S, we mean C contains S in the designated order under some orientation.

Ng and Schultz [7] showed the following:

Proposition 1 [7]. *Let G be a Hamiltonian graph on n vertices, $n \geq 3$. If G is k-ordered, $3 \leq k \leq n$, then G is $(k-1)$-connected.*

Theorem 2 [7]. *Let G be a graph of order $n \geq 3$ and let k be an integer with $3 \leq k \leq n$. If*

$$\deg u + \deg v \geq n + 2k - 6$$

for every pair u, v of nonadjacent vertices of G, then G is a k-ordered Hamiltonian graph.

Corollary 3 [7]. *Let G be a graph of order $n \geq 3$ and let k be an integer such that $3 \leq k \leq n$. If*

$$\deg v \geq \frac{n}{2} + k - 3$$

for every vertex v of G, then G is a k-ordered Hamiltonian graph.

The degree condition in the preceding corollary was improved by Kierstead, Sarkozy, and Selkow as follows:

Theorem 4 [6]. *Let $k \geq 2$ be a positive integer and let G be a graph of order n, where $n \geq 11k - 3$. Then G is k-ordered Hamiltonian if $\delta(G) \geq \lceil \frac{k}{2} \rceil + \lfloor \frac{n}{2} \rfloor - 1$.*

One goal of this paper is to improve upon the results obtained by Ng and Schultz in [7]. We obtain a sharp lower bound on the degree sum of nonadjacent vertices that imply a graph is k-ordered Hamiltonian. In particular, we prove the following:

Theorem 5. *Let $k \geq 3$ be a positive integer and let G be a graph of order $n \geq 53k^2$. If for any two nonadjacent vertices x and y, $\deg x + \deg y \geq n + \frac{3k-9}{2}$, then G is k-ordered Hamiltonian.*

We can see that the degree sum condition in Theorem 5 is sharp by considering the following example which was mentioned in [7]. The graph G on n vertices is composed of the three parts: $K_{k-1}, K_k - C_k$, and K_{n-2k+1} containing all the edges between K_{k-1} and $K_k - C_k$ and all edges between K_{k-1} and K_{n-2k+1}. Between K_{n-2k+1} and $K_k - C_k, G$ contains only the edges incident to the even indexed vertices of C_k. This graph is not k-ordered because there is no cycle containing the vertices of C_k in order. For $x \in V(K_{n-2k+1})$ and $y \in V(K_k - C_k), y$ an odd indexed vertex on $C_k, \deg x + \deg y = n + \frac{3k-10}{2}$ for k even. Another interesting characteristic in the graph G above is that for all nonadjacent pairs of vertices $x, y \in V(G), |N(x) \cup N(y)| \geq n-2$. Thus, for a graph to be k-ordered, we need $|N(x) \cup N(y)| > n-2$ which forces the graph to be complete. So there is no nontrivial sufficient condition on unions of neighborhoods of nonadjacent pairs of vertices.

Further, we obtain the following result concerning neighborhood unions of pairs of vertices. The bounds in this theorem are motivated by the following example. Let G be a graph on n vertices with cut set K or order $k-1$ such that $G - K$ has two connected components $C_1 = K_{\lfloor \frac{n-k+1}{2} \rfloor}$, and $C_2 = K_{\lceil \frac{n-k+1}{2} \rceil}$ and G contains all edges between K and C_i for $i = 1$ and 2. The sequence $S = \{x_1, x_2, \ldots, x_{2l}\}$ where $2l = k$ and $x_i \in C_1$ for i odd and $x_i \in C_2$ for i even shows that G is not k-ordered. The neighborhood of pairs of vertices in G is bounded below by $\frac{n+k-2}{2}$.

Theorem 6. *Let k be a positive integer and let G be a k-connected graph of order $n \geq 18k^2$. If $|N(x) \cup N(y)| \geq \frac{n+k}{2}$ for all pairs of distinct vertices $x, y \in V(G)$, then G is k-ordered Hamiltonian.*

Before beginning the proofs of these theorems, we make one general observation. If G is a Hamiltonian graph, then G is k-ordered Hamiltonian for $k = 1, 2$, and 3. By a result in [8], we know the degree sum condition in Theorem 5 implies the graph is Hamiltonian. By results in [5] and [3], we know the neighborhood condition in Theorem 6 implies the graph is Hamiltonian. Thus, in the proofs of these theorems we will immediately assume $k \geq 4$.

2. DEGREE CONDITIONS

In this section we will prove Theorem 5. However, the following result and its corollary, which give sufficient conditions for k-ordered to imply k-ordered Hamiltonian, will make the proof much easier. We say a vertex x is *insertible* in the cycle C if *N(x)* contains consecutive vertices on C. We say a subgraph $H \subseteq G$ is *insertible* in the cycle C if there exists a Hamiltonian path P in H with endpoints x and y such that there exist $a_x \in N(x), a_y \in N(y)$ and a_x and a_y are consecutive vertices on C. If the cycle C is obvious, we say simply the vertex x (or the subgraph H) is *insertible*. Also, we define $N(x, y) = N(x) \cup N(y)$ and $N[x] = N(x) \cup \{x\}$.

Theorem 7. *Let k be a positive integer and let G be a k-connected, k-ordered graph of order* $n \geq 8k^2$. *If for every pair of nonadjacent vertices u and v in V(G)*

$$\deg u + \deg v \geq n,$$

then G is k-ordered Hamiltonian.

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be an ordered subset of the vertices of G. Let C by a cycle of maximum order containing all vertices of S in appropriate order. The k-vertices of S split the cycle C into k intervals: $[x_1, \ldots, x_2], [x_2, \ldots, x_3], \ldots, [x_k, \ldots, x_1]$. Let $L = G - C$. Notice that no vertex of L has more than $|V(C)|/2$ adjacencies to C. This implies that any nonadjacent pair of vertices in L have degree sum at least $|V(L)|$ in L. Thus L is Hamiltonian if it has at least three vertices and complete otherwise. Assume there are vertices $x, y \in L$ with distinct neighbors in one of the intervals of C determined by S, say $[x_i, x_{i+1}]$. Note that we allow $x = y$. Let z_1 and z_2 be the immediate successor and predecessor on C to the neighbors of x and y, respectively, according to the orientation of C. (See Figure 1.) Observe that we can choose x and y and their neighbors in C such that none of the vertices on the interval $[z_1, z_2]$ have neighbors in L. We can also assume that $z_1 \neq z_2$, because $z_1 = z_2$ implies $x = y$ or C is not of maximal order. But neither z_1 nor x can be adjacent to more than half the vertices of C which forces

$$\deg z_1 + \deg x \leq 2\left(\frac{|C|}{2}\right) + |L| - 1 = n - 1,$$

a contradiction.

Let $s = |[z_1, z_2]|$ and $t = |L|$. Because x and y have no neighbors in $[z_1, z_2]$,

$$\deg x + \deg y \leq 2\left((t-1) + \frac{n-s-t+1}{2}\right).$$

Similarly, if z_1 is adjacent to a vertex, say w, on $C - [z_1, z_2], z_2$ cannot be adjacent to the successor, w^+, on C or else the segment $[z_1, z_2]$ could be inserted between w

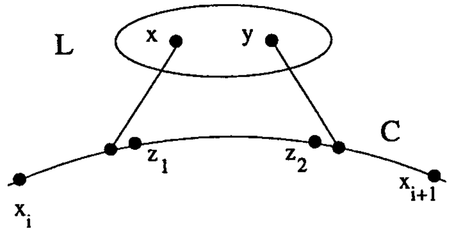

FIGURE 1.

and w^+, while replacing $[z_1, z_2]$ with a path from x to y. Hence,

$$\deg z_1 + \deg z_2 \leq 2(s-1) + n - s - t + 1.$$

Since x and y are both nonadjacent to z_1 and z_2, the initial degree condition forces $\deg x + \deg y + \deg z_1 + \deg z_2 \geq 2n$. But, by the previous two inequalities

$$\deg x + \deg y + \deg z_1 + \deg z_2 \leq 2n - 2,$$

which is a contradiction. Thus on any interval $[x_i, x_{i+1}]$ of C, there exists at most one vertex with neighbors in L. The connectivity, then, requires each segment $[x_i, x_{i+1})$ to have exactly one vertex with a neighbor in L. Also, we know $|C| \geq n/2$ by observing that if y_1 and y_2 are adjacencies of L in consecutive intervals on C, then the successors of these vertices, say w_1 and w_2, are not adjacent (for otherwise C could be extended). Neither w_1 nor w_2 has any adjacencies in L, but one of these vertices, say w_1, has degree at least $n/2$ which forces C to have at least $n/2$ vertices. This forces L to be Hamiltonian connected. Thus, the order of at least one of these k intervals of C must be a function of n, say $f(n) \geq \frac{n}{2k}$. Assume the interval $[x_1, x_2)$ is such a segment. Let z be the unique vertex in this interval with a neighbor in L. Without loss of generality we can assume the interval $[x_1, z)$ contains $f(n)/2$ vertices. Let y by the unique vertex in $[x_2, x_3)$ with a neighbor in L. The connectivity guarantees that if $|L| \geq k$ we can find distinct neighbors of z and y in L. Let y_1 and z_1 be the immediate predecessors of y and z, respectively, on C. Note that if y_1 and z_1 are adjacent, L can be inserted. (See Figure 2).

If $|L| \geq \frac{n}{2} - k$, then y_1 has no neighbors in the interval $[x_1, z_1]$ or C would not be of maximum order. So $\deg y_1 + \deg z_1 \leq 2(n/2 + k - 2) - f(n)/2 < n$, which is a contradiction since $n \geq 8k^2$.

If $|L| < n/2 - k$, then all the vertices in the interval $[x_1, z_1]$ must have degree at least $n - (|L| - 1 + k) = |C| - k + 1$. Thus each is adjacent to all but possibly $k - 1$ of the vertices of C, and, therefore, are insertible. In particular, they are insertible on $C - [x_1, x_2]$. If y_1 and z_1 are not adjacent, we insert z_1 and compare y_1 to the predecessor of z_1, say z_2. If an edge exists here, L can be inserted. If not, we insert z_2, and continue. Thus insertion process must end before reaching x_1 since y_1 also must be adjacent to all but at most $k - 2$ vertices of C. ■

An immediate corollary to Theorem 7 is the following:

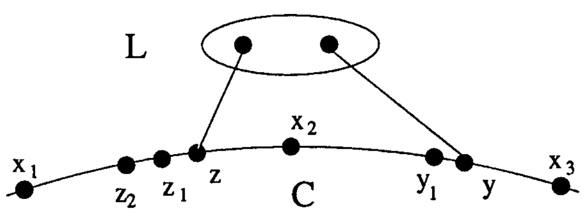

FIGURE 2.

Corollary 8. *Let k be a positive integer and let G be a k-connected, k-ordered graph of order $n \geq 8k^2$. If $\delta(G) \geq \frac{n}{2}$, then G is k-ordered Hamiltonian.*

Before proving theorem 5, we restate it for reference.

Theorem 5. *Let $k \geq 3$ be a positive integer and let G be a graph of order $n \geq 53k^2$. If for any two nonadjacent vertices x and y,* $\deg x + \deg y \geq n + \frac{3k-9}{2}$, *then G is k-ordered Hamiltonian.*

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be an ordered set of vertices of G. Note that by Theorem 7 it is enough to show that G is k-ordered. The proof will be split into cases according to the connectivity of the graph. Observe that the degree condition forces $\mathcal{K}(G) \geq \lceil \frac{3k-5}{2} \rceil$ for $k \geq 4$.

Case 1: $\mathcal{K}(G) \geq 5k$.
Find a cycle in G by finding a shortest x_1–x_2 path P_1, in $G - \{x_3, \ldots, x_k\}$, then the shortest x_2–x_3 path P_2, in $G - \{x_4, \ldots, x_k\} - P_1$, and continue this process to generate $k-1$ internally disjoint paths P_i for $i = 1, \ldots, k-1$. Note that the shortest length path from x_i to x_{i+1} is no more than 5. If not, we can label a path of minimum length as: $x_i = v_1, v_2, \ldots, v_l = x_{i+1}$, where $l \geq 7$. Then v_1, v_4, and v_7 are all mutually nonadjacent and have mutually disjoint neighborhoods in $G - S - \{V(P_1) \cup \cdots \cup V(P_{i-1})\}$. Thus,

$$n \geq \deg(v_1) + \deg(v_4) + \deg(v_7) \geq \frac{3}{2}\left(n + \frac{3k-9}{2}\right) - 5i - (k-i),$$

which is a contradiction for $n \geq \frac{11k}{2}$. But $n \geq \frac{11k}{2}$, since we assume $n \geq 53k^2$. By the connectivity of G, and $x_k x_1$ path must exist.

Case 2: $\frac{3k-1}{2} \leq \mathcal{K}(G) < 5k$.

Subcase A: $\delta < 100k$.
Without loss of generality, we can assume that G is edge-maximal with respect to the property of not being k-ordered (i.e., the addition of any edge makes G k-ordered). Let $\deg v = \delta$ and $F = G - N[v]$. Also, let $L = \{v \in V(G) | \deg v < n/2\}$ (or vertices of low degree) and $H = V(G) - L$ (or vertices of high degree). We claim F is complete and that every $w \in N[v] \cap H$ is adjacent to every vertex in F. Note that for $x \in F, \deg x > n - 100k$. Let x and y be nonadjacent vertices in F. Then the insertion of the edge xy makes G k-ordered. Let C be the smallest cycle in $G + \{xy\}$ that contains S in order. First, we claim that the cycle C contains no more than half the vertices of G. Assume otherwise, and let $|V(C)| = \alpha n$ where $\alpha > 1/2$. Then there exists an interval, say $[x_i, x_{i+1}]$, that contains at least $\alpha n/k$ vertices. Note that no interval can have more than two vertices of L or a smaller cycle is possible. Further, nonadjacent vertices a distance three or more apart on an interval can have no common neighbors off the cycle. Thus, we can find a subset of vertices of $[x_i, x_{i+1}]$, say M, such that $|M| \geq (\alpha n/3k) - 1, M$ is independent, and all vertices of M have degree greater

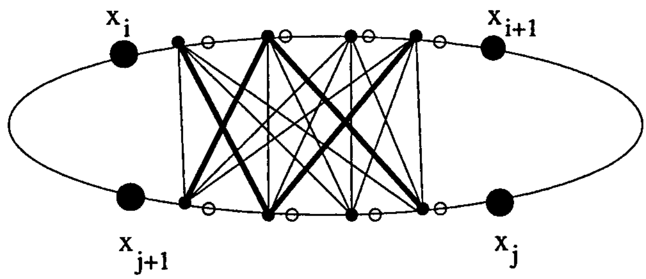

FIGURE 3.

than $n/2$. (One such M would be every third vertex on $[x_i, x_{i+1}]$.) The set M must send at least $\left(\frac{\alpha n}{3k} - 1\right)\binom{n}{2} - (n - \alpha n) = m$ edges to vertices of C. Thus, there exists some interval different from $[x_i, x_{i+1}]$ such that there are at least m/k edges between the two intervals. By the results in ([2] page 311) there exists a $K_{4,4}$ between the two intervals. But such a $K_{4,4}$ can be used to produce a smaller cycle, a contradiction (see Figure 3). Thus, the cycle C can contain at most $n/2$ vertices.

Let $|V(C)| = c$. Becuase $c \leq n/2$, both x and y have distinct neighbors off the cycle. Let $x^* \in N(x) - C, y^* \in N(y) - C$. Then, x^* and y^* are nonadjacent, have no common neighbors off the cycle, and can each have at most three adjacencies in each interval of the cycle for otherwise the length of C will not be maximal. So,

$$n + \frac{3k-9}{2} \leq \deg x^* + \deg y^* \leq (n - c) + 6k.$$

This forces $c < 6k$. But this is impossible since

$$2(n - 100k) \leq \deg x + \deg y \leq n + c. \tag{$*$}$$

(This right hand side of the inequality follows from the fact that x and y can have no common neighbors off the cycle.) We now have shown that F is complete. Note that the same argument applies if we choose $x \in V(F)$ and $y \in N(v) \cap H$. The only difference is the inequality $(*)$ above becomes

$$(n - 100k) + \frac{n}{2} + \frac{3k-9}{4} \leq \deg x + \deg y \leq n + c,$$

which produces the same contradiction.

Partition S into S_L and S_H where $S_H = S \cap H$ and $S_L = S - H$. Note that $S_L \subseteq N[v]$ and $\langle S_L \rangle$ must be complete. Also note that every vertex in S_H is either a vertex in the complete subgraph F or is adjacent to every vertex of F. Assume $|S_L| = l$, so $|S_H| = k - l$. If $l \leq k/2$, then $\mathcal{K}(G - S_H) \geq 2l$. For every $x_i \in S_L$ create a vertex x_i^* such that $N(x_i^*) = N(x_i)$. Let $S_L^* = S_L \cup \{x_i^* : i = 1, \ldots, l\}$. Add a vertex v such that $N(v) = S_L^*$. Let G^* be the graph that results from adding v and $\{x_i^* : x_i \in S_L\}$. Then $G^* - S_H$ is still $2l$-connected. Let M be a set of $2l$ distinct vertices of $V(F - S_H) \subset V(G^*)$. A generalization of Whitney's Theorem [4] implies that there exist $2l$ internally vertex disjoint paths, each starting at the vertex v and ending at a distinct vertex of M. But this implies that, if we return to the graph G, for every vertex x_i in S_L, we can find a pair of internally vertex disjoint paths, starting at x_i and ending at distinct vertices of M, say an x_i–$v_{i,1}$ path and an x_i–$v_{i,2}$ path where $v_{i,j} \in M$ and $v_{i,j} \neq v_{i',j'}$. Now all vertices of S are either in the complete subgraph F of G or have two disjoint paths to F. Thus we can construct a cycle containing S in the appropriate order using these paths to M and edges in F.

If $l = \frac{k}{2} + t$ for $t > 0$, then $\mathcal{K}(G - S_H) \geq \frac{3k-1}{2} - \left(\frac{k}{2} - t\right) = 2l - t - \frac{1}{2}$. Since S_L has over half the vertices of S, there are at least t consecutive pairs (x_i, x_{i+1}). Paths between these are made with edges in S_L, leaving no more than $2l - 2t$ "endpoints" of paths. Construct vertex disjoint paths in $G - S_H$ just as was done in the case $l \leq k/2$.

Subcase B: $\delta \geq 100k$.

Let K be a minimal cut set. Let A and B be the components of $G - K$. Find P_3's from A to B through K with the least number of vertices from S. Because of the minimum degree condition and $|A| \geq |B| \geq 95k$ we know we can avoid a P_3 that is a triple. The number of free P_3's is at least $\frac{3k-1}{2} - (k - d) \geq 2d$ where d is the number of doubles. (Observe that if k is even, $\mathcal{K}(G) \geq 3k/2$ and if k is odd, $d \leq (k-1)/2$.) Finally, for any $x, y \in V(A)$ (or respectively B), if x and y are nonadjacent, $|N_A(x) \cap N_A(y)| \geq \left(n + \frac{3k-9}{2} - 10k\right) - (n - 100k) > 90k$. In particular, there are more than k internally vertex disjoint x–y paths of length 2 in A. Thus, if x_i and x_{i+1} are both in A or both in B we can find a path of length no more than 2 connecting them. Otherwise, we use P_3's ending both in A or both in B to connect x_i and x_{i+1}. Thus, just as in the previous arguments, it is straightforward to construct a cycle containing S.

Case 3: $\frac{3k-5}{2} \leq \mathcal{K}(G) \leq \frac{3k-2}{2}$.

Pick a minimal cut set K. Let A and B be the components of $G - K$. Then a vertex x in A (or respectively in B) is adjacent to every other vertex of $A(B)$ and K except possibly one. To show this consider $x \in V(A)$ and $y \in V(B)$, such that one of them has two nonadjacencies. Then,

$$n + \frac{3k-9}{2} \leq \deg x + \deg y \leq (n - \mathcal{K}(G) - 2) + 2\mathcal{K}(G) - 2$$

which forces $\mathcal{K}(G) \geq (3k-1)/2$. Thus we can find a cycle in G containing S. ■

3. NEIGHBORHOOD CONDITIONS

In this section we will prove the result concerning neighborhood conditions. As before, we will first prove sufficient conditions under which G is k-ordered implies G is k-ordered Hamiltonian.

Theorem 9. *Let k be a positive integer and let G be a graph of order $n \geq 10k$ with $|N(x,y)| \geq \frac{n+k}{2}$ for all pairs of distinct vertices x and y and $\mathcal{K}(G) \geq k+1$. If G is k-ordered, then G is also k-ordered Hamiltonian.*

Proof. Let $S = \{x_1, \ldots, x_k\}$ be an ordered set of vertices of G. Let C be a maximal cycle containing S. (i.e., C cannot be extended by insertions of a vertex or a path.) Let H be the largest component of $G - C, t = |G - C - H|$, and let $|H| = r$. We can think of the elements of S as splitting the cycle C into k intervals: $[x_1, \ldots, x_2], \ldots, [x_k, \ldots, x_1]$.

The proof consists of four cases that depend on the order of H and the magnitude of t.

Case 1: $|H| = r \geq k+1$.

By the connectivity, we know there exist $k+1$ distinct vertices in H with $k+1$ distinct neighbors in C. Thus there exists two vertices in H, say y_1 and y_2, with distinct neighbors, say w_1 and w_2 in a single interval of C. We can pick y_1 and y_2 (and w_1 and w_2) so that w_1 and w_2 are as close as possible on C. Let $z_1, z_2, \ldots, z_s$ be the vertices of C between w_1 and w_2. If z_1 is insertible in C, insert it and proceed to z_2. If z_2 is insertible, do so and proceed to z_3. Do the same starting at z_s, then moving to z_{s-1}, and so forth. If all of the z_i are insertible or all but one are insertible, we close the cycle with a $(y_1$–$y_2)$-path in H producing a larger cycle. If a section remains, say $z_i, z_{i+1}, \ldots, z_j$, then we know $|N(z_i, z_j)| \leq t + s + \frac{n-r-s-t+1}{2}$. Also $|N(y_1, y_2)| \leq r + \frac{n-r-s-t+1}{2}$. Thus, $|N(y_1, y_2)| + |N(z_i, z_j)| \leq n+1$. But this is a contradiction, since by assumption $|N(y_1, y_2)| + |N(z_i, z_j)| \geq n+k$.

Case 2: $2 \leq |H| = r \leq k$.

Suppose H is not complete. Then we can select nonadjacent vertices x and y in H. Thus $\frac{n+k}{2} \leq |N(x,y)| \leq |H| + \frac{n-r-t}{2} = \frac{n+r-t}{2}$ which is a contradiction unless $t = 0, r = k$, and H is complete. Without loss of generality assume the length, s, of the segment of the circle from x_1 to x_2 is maximum. Then from the proof of Case 1 we know at most one vertex along this segment has more than one adjacency to H. Select two vertices in H, say y_1 and y_2, with degree at most one relative to this segment. Then $|N(y_1, y_2)| \leq k + 2 + \frac{n-k-s+1}{2}$ which is a contradiction since $s \geq \frac{n-k}{k}$ and $n \geq 7k$.

Case 3: $|H| = 1, t \geq 1$.

Pair all the vertices in the segment of C from x_i to x_{i+2} starting with x_i. Let w and z be in $G - C$. Certainly, $N(w)$ (and $N(z)$) can have at most one neighbor in each pair. Also, $N(w,z)$ contains at most one vertex from each pair with at most two exceptions, and these two exceptions must have opposite orientation. (For example, if z hit immediate successors of neighbors of w twice, the cycle could be extended.) But now, of the pairs that sit between these two exceptional ones, $N(w,z)$ must miss one altogether or C can be enlarged. Thus, for every segment $P = [x_i, x_{i+1}), |N_P(w,z)| \leq \frac{|P|}{2} + 1$. This forces $|N(w,z)| < (n+k)/2$, which is a contradiction.

Case 4: $|H| = 1, t = 0$.

Let x be the single vertex not in C. We will first show that we can assume deg $x \geq \frac{n+k}{4}$. If this is not the case, then observe that x has at least $k+1$ adjacencies in C. Thus, there is one interval on C determined by S where x has two adjacencies. Let $W = \{w_1, w_2, \ldots w_l\}$ be a section of C between two such consecutive adjacencies of x. Exchange the section W for the vertex x on the cycle C. Note that all the vertices of W, and in fact all the vertices of G except for x, have degree at least $\frac{n+k}{4}$. Now insert as many of the vertices of W as possible. If all the vertices of W can be inserted, we have found a Hamiltonian cycle containing S. If not, we apply the methods of the previous three cases with the one restriction that the unique vertex of small degree should always remain as a vertex on C.

For example, Case 1 applies directly unless there are precisely $k+1$ independent edges from H to C, only one pair of these edges are in a single interval on C, and the vertex x is between the endpoints of the two edges. If $|H| = k+1$, every vertex of H has at least $(n-3k)/4$ neighbors on C. But at most two of the vertices of H can have more than one neighbor in the same interval. Thus $|C| \geq (k-1)(n-3k)/2 > n$ since $k \geq 4$. If $|H| > k+1$, then the degree condition on vertices of H forces $|H| \geq (n-3k)/4$. Thus pick two vertices of H with the fewest neighbors to C. Such a pair forces $|H| \geq \frac{n+k}{2} - k = (n-k)/2$ and $|C| \leq (n+k)/2$. This means every pair of vertices on C without neighbors in H are adjacent. Specifically, if e_1 and e_2 are independent edges from H to C with end vertices a_1 and a_2 on C, then their successors a_1^+ and a_2^+ are adjacent and the cycle can be enlarged.

Because all the vertices of H have high degree, Case 2 occurs only if $|H| = k$. Let $w_i, w_j \in V(H)$. Then $N(w_i, w_j)$ must include half the vertices of C but no consecutive pairs of vertices on C or the cycle could be extended. Consider a section of C labeled a_1, b_1, a_2, b_2, a_3 where $a_1, a_2, a_3 \in N(w_i, w_j)$. If b_1 or b_2 are insertible, the cycle can be enlarged. If neither b_1 or b_2 are insertible, then the path b_1, a_2, b_2 must be insertible and C can be enlarged.

No alteration is necessary to apply Case 3. Thus we can assume that the single vertex x not on the cycle C has degree at least $(n+k)/4$.

We can extend C to include x unless between every pair of consecutive adjacencies of x on C there is at least one uninsertible vertex, say vertex y_i, between the ith consecutive pair of neighbors of x on C. We want to show that there exists some y_i such that deg $y_i > \frac{2n}{7}$. Select l vertices from the set of y_i's, and assume that $d_1 n \leq \deg y_i \leq d_2 n$. Thus,

$$\deg y_i + \deg y_j = |N(y_i, y_j)| + |N(y_i) \cap N(y_j)| \geq n/2 + \alpha_{i,j} n$$

where $\alpha_{i,j} = |N(y_i) \cap N(y_j)|/n$. Let E be the number of edges of G from the set of y_i's to the cycle C. Then,

$$\frac{1}{l-1}\left[\binom{l}{2}\frac{n}{2} + \sum_{\forall i,j} \alpha_{i,j} n\right] \leq |E| \leq l d_2 n.$$

We observe that the average degree of the vertices in $G - \{y_1, \ldots, y_l\}$ to the y_i's is at least $\frac{ld_1 n}{2}$. Thus, $\sum \alpha_{i,j} n \geq n\binom{ld_1}{2}$, so

$$\frac{1}{l-1}\left[\binom{l}{2}\frac{n}{2} + n\binom{ld_1}{2}\right] \geq ld_2 n.$$

We simplify this to get,

$$\frac{1}{4} + \frac{d_1(ld_1 - 1)}{2(l-1)} \leq d_2.$$

Thus,

$$\frac{1}{4} + \frac{1}{32} \leq \frac{1}{4} + \frac{d_1^2}{2} \leq d_2.$$

Now repeat this again by culling the y_i's to a possibly smaller set of vertices whose degree is between $d_2 n \frac{9}{32} n$ and d_3. By the same argument, $d_3 \geq \frac{1}{4} + \frac{(9/32)^2}{32} > \frac{2}{7}$. Note that this implies we can always adjust the vertices of C so that the single vertex not on C has degree $\gamma n > 2n/7$. At this point we need to show that any vertex $x = G - C$ of degree at least γn can be inserted into the cycle C. First we claim that there must be three consecutive adjacencies of x within a single interval of C, say $[x_i, x_{i+1}]$, such that there are at most five vertices of C between the first and third adjacencies of x. If not, we know there must be deg $x - 2k$ triples of neighbors of x within a segment, each of which requires eight vertices. Thus, taking into account vertices that are counted twice we find

$$|C| \geq \frac{7(\gamma n - 2k)}{2} > \frac{7\gamma n}{2} - 7k > n,$$

a contradiction.

So we can assume x has three consecutive adjacencies, a_1, a_2, a_3, in $[x_i, x_{i+1}]$ with at most five vertices between a_1 and a_3. The arguments applied to the case where there are exactly five vertices between a_1 and a_3 are easily applied to the cases where there are fewer than five vertices, so we prove the only case where there are exactly five.

First consider the subcase where there is one vertex between a_1 and a_2 and three between a_2 and a_3. We label this segment: $a_1, w, a_2, y_3, y_2, y_1, a_3$. The single vertex, w, must not be insertible or we could have added x to the cycle. Also w and y_1 are nonadjacent or x would have been inserted. Thus $N(w, y_1)$ includes at most three of the five vertices between a_1 and a_3. So $N(w, y_1)$ must include at least $(n + k - 6)/2$ of the $n - 6$ vertices on the cycle from a_3 to a_1. Since $k \geq 3$, we know $N(w, y_1)$ must include a pair of vertices that are consecutive on C. If $N(w)$ and $N(y_1)$ hit a consecutive pair, then segment w, a_2, y_3, y_2, y_1 can be inserted. If y_1 is insertible, compare w and y_2 as above. Note that inserting y_1 does not effect the insertibility of w. Thus, we either find a pair w and y_i that allows us to enlarge the cycle, or we exhaust the y_i's by inserting them in another place on the cycle.

Second, we consider the case where there are two vertices between each pair of a_i's. We label the segment: $a_1, w_1, w_2, a_2, y_2, y_1, a_3$. One of y_1 and y_2, (and respectively w_1 and w_2) must be insertible and the other not insertible. Assume w_1 is not insertible. Then y_1 is insertible or the degree condition would force y_1 and w_1 to have consecutive neighbors. Insert y_1. Consider w_1 and y_2. If they have consecutive neighbors, the path w_1, w_2, a_2, y_1 is insertible. So it must be the case where w_1 is insertible as a result of the insertion of y_1. Thus w_1, y_1 is an edge. Now we add x to the cycle by inserting w_2 and using the w_1, y_1 edge. By symmetry, y_1 and w_1 must be insertible while y_2 and w_2 both are not. Insert w_1. If y_1 is no longer insertible, it must be that w_1 and y_1 must have consecutive neighbors and the path w_1, w_2, a_2, y_2, y_1 is insertible. So, insert y_1. Now, w_2 and y_2 are unaffected by these insertions (if w_2, y_1 or w_1, y_2 are edges, x can be inserted) and by the degree condition must be insertible as a pair. ■

Before beginning the proof of theorem 6, we restate it for reference.

Theorem 6. *Let k be a positive integer and let G be a k-connected graph of order $n \geq 18k^2$. If $|N(x,y)| \geq \frac{n+k}{2}$ for all pairs of distinct vertices $x, y \in V(G)$, then G is k-ordered Hamiltonian.*

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a set of ordered vertices of G. We first consider the case where the connectivity is exactly k. In this case, let K be a minimal cut set separating G into the two components A and B. By considering any pair of distinct vertices in A (or respectively, B), the neighborhood condition forces $|A| = |B| = \frac{n-k}{2}$, both A and B to be complete, and every vertex in K to be adjacent to all but at most one vertex in A and one vertex in B. Thus it is easy to verify that G is k-ordered Hamiltonian. Thus, we can assume $\mathcal{K} \geq k+1$.

By the previous theorem, for $\mathcal{K}(G) > k$, we need only to show that under the neighborhood condition, G is k-orderable. Again, we split the proof into cases according to the connectivity of G.

Case 1: $\mathcal{K} > 7k$.

We claim there exists a path from x_1 to x_k containing all elements of S in order such that the path from x_i and x_{i+1} contains at most eight vertices. We construct this path by choosing the shortest x_1–x_2 path that avoids S, then the shortest x_2–x_3 path that avoids S and any vertices in the x_1–x_2 path, and so forth. Let $x_i = a_1, a_2, \ldots, a_t = x_{i+1}$ be the shortest $(x_i$–$x_{i+1})$ path in G that avoids S and all vertices used in previous paths. If $t \geq 9$, then we observe that $|N(a_1)| > n/4$, $|N(a_4, a_5)| > n/2$, and $|N(a_8, a_9)| > n/2$ and all these sets can intersect on vertices of S or vertices used on previous paths. Thus,

$$n \geq |N(a_1)| + |N(a_4, a_5)| + |N(a_8, a_9)| - 2[(7(i-1)+1) - (k-1)] > 5n/4 - 14k,$$

which contradicts $n \geq 18k^2$. Thus, we can find an x_1–x_k path that contains S in order and uses at most $7k+1$ vertices. The connectivity guarantees that we can find an x_k–x_1 path to close the cycle.

Case 2: $\mathcal{K}(G) = k + t$ and $t > 0$.

Let K be a minimal cut set, A and B components of $G - K, |A| = n_1$ and $|B| = n_2$. By Hall's Theorem we can form a perfect matching of K into A and K into B, which together form $k + t$ P_3's through K. If both end vertices on such a path are in S, we call the path a triple. If the path has two vertices in S, one of which is the center vertex, it is called a double; one vertex, a single; no vertices, free. Pick matchings such that the paths contain the fewest number of vertices in S. Assume no triples exist. Let $r_1(r_2)$ be the number of doubles with an end vertex in $A(B)$ that is also in S. Let $x, y \in A - S$ and $w, z \in B - S$. Then

$$|N(x,y)| \leq n_1 + k + t - r_1$$

and

$$|N(z,w)| \leq n_2 + k + t - r_2.$$

Thus,

$$n_1 + n_2 + 2k + 2t - r_1 - r_2 \geq n + k,$$

but

$$n_1 + n_2 + 2k + t = n + k.$$

So, $t \geq r_1 + r_2$. Let $r = r_1 + r_2$. Then there exists at least $2r$ free P_3's to provide paths between vertices in A and B. To find paths in A and B we use the fact that each of these graphs is k-linked. A graph is said to be k-linked if for every set of k pairs of vertices of G, say v_i, w_i for $1 \leq i \leq k, G$ contains k internally disjoint v_i–w_i paths for $1 \leq i \leq k$. In [2] it was shown that every $22k$-connected graphs is k-linked. Thus, we want to show that A and B are $22k$-connected. Without loss of generality assume $|A| \leq |B|$. In order to satisfy the neighborhood condition, $|A| \geq (n+k)/2 - (k+t)$ which forces $|B| \leq (n-k)/2$. Let K' be a minimal cut set of B and B' a connected components of $B - K'$ of smallest order. Let $x, y \in V(B')$. Then

$$(n+k)/2 \leq |N(x,y)| \leq |B'| + |K'| + k + t \leq \frac{1}{2}\left(\frac{n-k}{2} - |K'|\right) + |K'| + k + t.$$

Thus, $|K'| \geq (n - k - 4t)/2 \geq 22k$ since $n \geq 18k^2$. The same argument holds for A. Thus we construct the cycle containing S by using the linkage to build the segments between consecutive vertices within A or in B and the P_3's to build the segments between A and B.

If a triple occurs, say the middle vertex is v, we know v must have small degree (a function of k) and consequently we can have at most one triple. Let $x, y \in A - S$ and $w, z \in B - S$. Then

$$|N(x, y)| \geq n_1 + k + t - (r_1 + 1)$$

and

$$|N(w, z)| \geq n_2 + k + t - (r_2 + 1).$$

So,

$$n_1 + n_2 + 2k + 2t - (r_1 + r_2 + 2) \geq n + k.$$

Thus $t \geq r_1 + r_2 + 2$, which provides two additional free P_3's (relative to the previous case of no triples) we can associate with v. If v is not an element of S, we can identify the endvertices of the P_3 containing v with the corresponding endvertices of two free P_3's and proceed as in the previous case. If v is in S, we know v has $t + 1 \geq 2$ neighbors not in S. Let w be such a neighbor. If w is in A or B, it must already be used in a P_3 or the triple would not have occurred. Also, the P_3 using w must not have been able to exchange w for another vertex outside S. But this forces its center to have a large degree to the side opposite w making it a free P_3 or a single P_3. If w is in K it is either on a free P_3 or on a single P_3. If it is a single with vertex x_i, we assign to x_i one of the two extra free P_3's associated with v and use this one for v. The paths from v will use the edges to the two neighbors of v not in S. Thus we can always find two paths from v to either A or B avoiding vertices of S. This completes the proof of Theorem 6. ■

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