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SHORT CYCLES IN HAMILTONIAN GRAPHS

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Abstract

The girth of a graph with a hamiltonian cycle and t chords will be investigated. In particular, for any integer $t > 0$ let $g(t)$ denote the smallest number such that any hamiltonian graph G with n vertices and $n+t$ edges will have girth at most $g(t)n+c$, where c is a constant independent of n. It will be shown that there exist constants c_1 and c_2 such that $(c_1(\log t))/t \leq g(t) \leq (c_2(\log t)/t)$. For small values of t, $(1 \leq t \leq 8)$, $g(t)$ will be determined precisely.

*Research is partially supported by ONR Grant No. N00014-94-J-1085
†Research is partially supported by ONR Grant No. N00014-94-J-1085
‡Research is partially supported by ONR Grant No. N00014-94-J-1098
§Research is partially supported by ONR Grant No. N00014-94-J-1050

1 INTRODUCTION

The girth $g(G)$ of a graph G is the order of the smallest cycle of the graph. In general, the girth of a graph will decrease as edges are added. More specifically, as chords are added to a hamiltonian cycle in a graph, the girth of the graph will be forced to decrease. We will investigate the relationship between the girth of a hamiltonian graph and the number of chords of the hamiltonian cycle. More specifically, the following function g will be investigated.

Definition 1 *For any positive integer t, let $g(t)$ denote the smallest number such that any hamiltonian graph G with n vertices and $n+t$ edges will have girth $g(G) \leq g(t)n + c$, where c is a constant (independent of n).*

In section 2 the following theorem, which gives asymptotically sharp upper and lower bounds for the function g, will be proved.

Theorem 1 *For any positive integer $t > 1$, there exist positive constants c_1 and c_2 such that*

$$(c_1(\log t))/t \leq g(t) \leq (c_2(\log t))/t.$$

A hamiltonian graph G with n vertices and $n+t$ edges will have $g(G) \leq g(t)n + c$ for some constant c. If t is a constant then, it is possible that the girth $g(G)$ is still a positive fraction of n ($g(G) \geq c_1((\log t)n)/t$ for some constant c_1), and it is always true that $g(G) \leq (c_2(\log t)n)/t$ for some constant c_2. However, if $t \geq \log n$, then $g(G) = o(n)$, as the following corollary indicates.

Corollary 1 *If G is a hamiltonian graph with n vertices and $n + \log n$ edges, then $g(G) \leq c((\log\log n)n)/\log n$ for some constant c. In general, this bound cannot be improved except for the constant c.*

There are several other interesting special cases of the upper bound in Theorem 1; the cases when $t = n^\alpha$ for $(0 < \alpha < 1)$, $t = \epsilon n$ for some $(0 < \epsilon < 1)$, and when $t = \epsilon(\log n)n$ are examples of this. The following is a corollary of the proof of Theorem 1.

Corollary 2 *(i) If G is a hamiltonian graph with n vertices and $n + n^\alpha$ edges for some $0 < \alpha < 1$, then $g(G) \leq c(\log n)n^{1-\alpha}$ for some constant c.*

(ii) If G is a hamiltonian graph with n vertices and $n + \epsilon n$ edges for $0 < \epsilon < 1$, then $g(G) \leq c(\log n)$ for some constant c.

(iii) If G is a hamiltonian graph with n vertices and $n + \epsilon(\log n)n$ edges for $0 < \epsilon < 1$, then $g(G) \leq c$ for some constant c.

In section 3 precise values of $g(t)$ will be determined for $1 \leq t \leq 8$. Specifically, the following theorem will be proved.

Theorem 2 *For $1 \leq t \leq 8$, the values of $g(t)$ are $g(1) = g(2) = 1/2$, $g(3) = g(4) = 1/3$, $g(5) = 3/10$, and $g(6) = g(7) = g(8) = 1/4$.*

2 GENERAL BOUNDS

A result of Erdős and Sachs on the existence of regular graphs of given minimum girth will be needed in the proof of Theorem 1. In [2] the following was proved, but the form stated here comes from [1], page 57.

Theorem 3 *[2] Given a positive integer $r \geq 3$, there exists an r-regular graph of girth at least r with at most*

$$\left(\frac{r-1}{r-2}\right) \lfloor (r-1)^{r-1} + (r-1)^{r-2} + (r-4) \rfloor \leq r^r$$

vertices.

Let $H_{r,m}$ denote an r-regular graph of order m and girth at least r assured by Theorem 3. Thus $m \leq r^r$. Before we begin the proof of Theorem 1, some additional notation will be introduced. By $C = (x_0, x_1, \cdots, x_{n-1}, x_0)$ we will mean a cycle of length n containing the vertices $\{x_0, x_1, \cdots, x_{n-1}\}$ in the order indicated. For $i < j$, the interval of vertices on C strictly between x_i and x_j will be denoted by $(x_i, x_{i+1}, \cdots, x_j)$. When the endvertices x_i and x_j are included in the interval, it will be denoted by $[x_i, x_{i+1}, \cdots, x_j]$. A *chord* R of C is just an edge $x_i x_j$ between 2 non-consecutive vertices x_i and x_j in C. Any such chord R determines 2 cycles related to C, namely $C_1 = (x_i, x_j, x_{j+1}, \cdots, x_i)$ and $C_2 = (x_i, x_j, x_{j-1}, \cdots, x_i)$. Collectively these two cycles use each of the edges of C precisely once, so the sum of their lengths is precisely $n+2$. Two chords $R = x_i x_j$ and $S = x_k x_\ell$ are *intersecting chords* if $i < k < j < \ell$, and if not, then they are *parallel chords.* If R and S are parallel chords of C, then there are 3 cycles associated with these chords. One cycle contains the chord R, one contains both R and S, and the other contains the chord S, and collectively the chords use each edge of the cycle precisely once. Thus, the sum of the lengths of the 3 cycles is $n+4$. There is a corresponding collection of cycles for larger numbers of parallel chords. Also, there is an analogous collection of cycles for intersecting chords. For example, if R, S, and T are three pair-wise intersecting chords of C, then there are 6 cycles determined by these chords and 3 of these cycles share no edges on the cycle C. Each of the 3 cycles contain precisely 2 of the chords and collectively they use each edge on the cycle precisely once, and so the sum of the lengths of these 3 cycles is $n+6$.

Proof: (of Theorem 1) We verify the lower bound for $g(t)$ by describing an example of a graph G of appropriate order, size, and girth. In fact, it is sufficient to show for t large enough and n sufficiently large, there is a hamiltonian graph G with n vertices and $n+t$ edges that has $g(G) \geq c((\log t)n)/t$ for some constant c.

Denote the vertices of $H_{r,m}$ by $\{y_0, y_1, \cdots, y_{m-1}\}$. Consider a cycle $C = (x_0, x_1, \cdots, x_{n-1}, x_0)$ of order n that is divisible by $2mr$ (i.e. $n = 2mrk$ for some integer k). Partition the vertices of C into $2m$ intervals of consecutive vertices of C, each of length rk. For each integer i, $(0 \leq i \leq 2m-1)$, let $I_i = [x_{irk}, x_{(i+1)rk})$. For each of the intervals $I_{2i} = [x_{2irk}, x_{(2i+1)rk})$, mark the r vertices $x_{2irk}, x_{(2ir+1)k}, \cdots, x_{((2i+1)r-1)k}$ in this interval. Therefore, rm of the $2mrk$ vertices are marked. For each of the r edges incident to the vertex $y_\ell \in H_{r,m}$, identify one of the r marked vertices in the interval $I_{2\ell} = [x_{2\ell rk}, x_{(2\ell+1)rk})$. Now, if $y_i y_j$ is an edge in $H_{r,m}$, then place an edge between the two marked vertices (one from I_{2i} and one from I_{2j}) of C identified with the edge $y_i y_j$. Denote by G_n the graph obtained from C by adding the $rm/2$ edges. Let $t = rm/2$. Note that $\Delta(G_n) = 3$.

The graph G_n has n vertices and $n+t$ edges. Any cycle in G_n that uses any of the vertices of one of the odd intervals I_{2i+1} will use all of the vertices of I_{2i+1} and thus will contain at least rk vertices. If a cycle of G_n contains no vertices from any of the odd intervals I_{2i+1}, then the cycle must contain at least r chords of C_n, since $H_{r,m}$ has girth at least r. Therefore, the cycle must contain at least rk vertices from the intervals of length k associated with the marked vertices. This implies that $g(G_n) \geq rk = n/2m$. Since $2t/r = m \leq r^r$, we have $t \leq r^{r+1}/2$, and so $r \geq c' \log t$ for some constant c'. Hence, $g(G_n) \geq n/2m = rn/4t \geq c(\log t)n/t$ for some constant c. As a consequence $g(t) \geq c(\log t)/t$ for some constant c.

Next, the upper bound for $g(t)$ will be verified. Let G be a hamiltonian graph with n vertices and $n+t$ edges. We will show that there is a constant c such that $g(G) \leq c(\log t)n/t$. Assume that this is not true.

Let C be a hamiltonian cycle of G. Select an appropriate and small p $(3 \leq p < \log t)$ and partition the vertices of C into $\lfloor t/p \rfloor$ intervals, each of length approximately pn/t. By assumption, there are no chords in any of these intervals, for otherwise there would be too small a cycle. Collapse each of the intervals to one vertex to obtain the multigraph H induced by the chords of C. Therefore, H has $\lfloor t/p \rfloor$ vertices and t edges, and so the average degree in H is at least $2p$. Thus, there is a subgraph L of H that has minimum degree at least $\delta \geq p$.

The minimum degree condition on L will place an upper bound on the girth of L. Given a vertex $v \in V(L)$, the number of vertices a distance precisely k from v will be a least $(\delta - 1)^k$ if $k \leq g(L)/2 - 1$. This implies that $(\delta - 1)^{g(L)/2-1} < |V(L)|$. In our case this gives the inequality

$$(p-1)^{g(L)/2-1} < t/p,$$

from which it follows that $g(L) \leq c' \log t$ for some constant c'. As a consequence of this we have that $g(H) \leq c' \log t$. Since $g(G) \leq (pn/t)g(H)$, it follows that $g(G) \leq c(\log t)n/t$ for some constant c. This completes the proof of the upper bound on $g(t)$ and of Theorem 1. □

3 SMALL ORDER CASES

The structure of the examples and the techniques of the proofs to determine the values of $g(t)$ for small values of t are similar to those used in the general case. Before giving the proof of Theorem 2, we will describe a set of 8 families of examples $\{H_1, H_2, \cdots H_8\}$ that will be used to verify the upper bounds for $g(t)$. Each example H_i will be obtained by adding i chords to a cycle $C = (x_0, x_1, \cdots, x_{n-1}, x_0)$. These graphs are pictured in Figure 1.

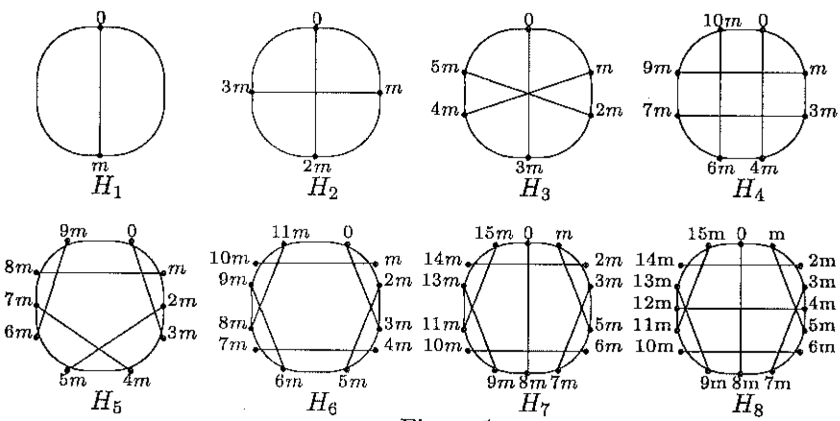

Figure 1

For H_2, assume that $n = 4m$ and add the 2 chords x_0x_{2m} and x_mx_{3m}. It is straightforward in this case to see that $g(H_2) = 2m + 1 = n/2 + 1$. The graph H_1 is obtained from H_2 by deleting the chord x_mx_{3m}, and also $g(H_1) = n/2 + 1$.

For H_3, assume that $n = 6m$ and add the 3 chords x_0x_{3m}, x_mx_{4m}, and $x_{2m}x_{5m}$. The endvertices of these cycles partition the vertices of C_n into 6 intervals each with $m = n/6$ vertices. Any cycle of H_3 must use all of the vertices in at least 2 of these intervals, and so it follows that $g(H_3) = 2m + 2 = n/3 + 2$.

For H_4, assume that $n = 12m$ and add the 4 chords x_0x_{4m}, x_mx_{9m}, $x_{3m}x_{7m}$ and $x_{6m}x_{10m}$. The endvertices of these cycles partition the vertices of C_n into 8 intervals with 4 of the intervals having $m = n/12$ vertices and the other 4 having $m = n/6$ vertices. Any cycle of H_4 must use all of the vertices in at least 2 of the intervals with $n/6$ vertices, one of the intervals with $n/6$ vertices and 2 with $n/12$ vertices, or 4 of the intervals with $n/12$ vertices. Therefore, $g(H_4) = 4m + 2 = n/3 + 2$.

For H_5, assume that $n = 10m$ and add the 5 chords x_0x_{3m}, $x_{2m}x_{5m}$, $x_{4m}x_{7m}$, $x_{6m}x_{9m}$, and $x_{8m}x_m$. The endvertices of these cycles partition the vertices of C_n into 10 intervals each with $m = n/10$ vertices. It is not difficult to check that any cycle of H_5 must use all of the vertices in at least 3 of these intervals, and so it follows that $g(H_5) = 3m + 1 = 3n/10 + 1$.

In the graph H_6, which has a construction that is similar to the construction for H_5, assume that $n = 12m$ and add the 6 chords x_0x_{3m}, $x_{2m}x_{5m}$, $x_{4m}x_{7m}$, $x_{6m}x_{9m}$, $x_{8m}x_{11m}$, and $x_{10m}x_m$. The endvertices of these cycles partition the vertices of C_n into 12 intervals each with $m = n/12$ vertices. It is not difficult to check that any cycle of H_6 must use all of the vertices in at least 3 of these intervals, and so it follows that $g(H_6) = 3m+1 = n/4+1$.

The graph H_7 is a subgraph of H_8, so we will first describe the graph H_8. Assume that $n = 16m$ and add the 8 chords $x_{1m}x_{5m}$, $x_{3m}x_{7m}$, $x_{6m}x_{10m}$, $x_{9m}x_{13m}$, $x_{11m}x_{15m}$, $x_{14m}x_{2m}$, x_0x_{8m},and $x_{4m}x_{12m}$. The endvertices of these cycles partition the vertices of C_n into 16 intervals each with $m = n/16$ vertices. It can be checked that any cycle of H_8 must use all of the vertices in at least 4 of these intervals, and so it follows that $g(H_8) = 4m + 1 = n/4 + 1$. The graph H_7 is obtained from H_8 by deleting any one of the 8 chords, say $x_{6m}x_{14m}$, and $g(H_7) = n/4 + 1$.

Proof: (of Theorem 2) The lower bounds for $g(1), g(2), \cdots, g(8)$ follow directly from the examples $H_1, H_2, \cdots, H_8$. To complete the proof it is sufficient to verify the appropriate upper bounds for $g(1)$, $g(3), g(5)$ and $g(6)$. Let $C = (x_0, x_1, \cdots, x_{n-1}, x_0)$ be a cycle of length n, and for $i = 1, 3, 5$ and 6, let G_i be a graph obtained from C by adding i chords. Except for a few special situations, the nature of the proofs for each of the special cases will be the same. A set of cycles will be described that will collectively use each edge of the cycle C precisely once. This permits the sum of the lengths of the set of cycles to be calculated, and the average length of the cycles gives an upper bound on the girth of the graph.

Claim: $g(1) = 1/2$

Clearly $g(G_1) \leq n/2+1$, since any chord of C determines 2 cycles that share precisely 2 vertices and collectively each of the edges of C once, and so one of the cycles has at most $(n+2)/2 = n/2+1$ vertices. This implies $g(G_1) \leq n/2+1$ and $g(2) \leq g(1) \leq 1/2$.

Claim: $g(3) = 1/3$

If 2 of the chords in G_3 are parallel, then these 2 chords determine three cycles such that the sum of the number of vertices in the 3 cycles is $n+4$. Therefore, one of the cycles contains at most $(n+4)/3$ vertices, which implies $g(G_3) \leq (n+4)/3$. If no pair of the chords of G_3 are parallel, then they are intersecting. These 3 chords imply the existence of 3 cycles such that the sum of the number of vertices in the 3 cycles is $n+6$. Hence, one of the cycles contains at most $(n+6)/3$ vertices, and so $g(G_3) \leq n/3+2$ and $g(4) \leq g(3) \leq 1/3$.

Claim: $g(5) = 3/10$

We will assume there is no constant c such that $g(G_5) \leq 3n/10 + c$, and show that this leads to a contradiction. By the same arguments used in the previous case, if there are 3 parallel chords in G_5, then there are 4 cycles such that the sum of their lengths is $n+4$ and $g(G_5) \leq (n+4)/4$. Also, if there are 4 intersecting chords, then $g(G_5) \leq (n+8)/4$. Therefore, we will assume that G_5 does not contain either 3 parallel or 4 intersecting chords.

Each chord R of G_5 determines 2 cycles, which we will denote by C_1 and C_2. The chord also partitions the remaining 4 chords into 3 categories: those that intersect the chord R, those associated with the cycle C_1, and those associated with C_2. If there is a chord associated with each of C_1 and C_2, then there are 3 parallel chords, a contradiction. Hence, we can assume that the other 4 chords of C either intersect R or are associated with one of the cycles, say C_1. If $g(C_2) \leq n/4 + c$ for some constant c, then we are done. Thus, we can assume $g(C_1) < 3n/4$. If there are 3 chords associated with C_1, then since $g(3) = 1/3$, there is a cycle in G_5 with at most $(3n/4)/3 + c < n/4 + c < 3n/10 + c$ for some constant c, a contradiction. Hence there are at most 2 chords in C_1, and so there must be at least 2 chords intersecting R. From this point on, we will assume that each chord of G_4 has at least 2 intersecting chords.

We will next show that G_5 does not contain 2 pairs of intersecting chords that are parallel. Assume that R_1, R_2, R_3, R_4 are 4 such chords. Thus, there exists 8 vertices $v_1, v_2, \cdots, v_8$ on C, appearing in the order on the cycle indicated by the their index, such that $R_1 = v_1v_3, R_2 = v_2v_4, R_3 = v_5v_7$, and $R_4 = v_6v_8$. There is another chord $R = u_1u_2$ of G_5. Since each chord of G_5 must intersect at least 2 intersecting chords, the chord R must intersect each of the chords R_1, R_2, R_3, R_4, and so we can assume that $u_1 \in (v_2, v_3)$, and that $u_2 \in (v_6, v_7)$. There are 4 cycles in G_5 that share no edges on the cycle C, namely the cycles $C_1 = (v_1, v_3 \cdots, v_4, v_2, \cdots, v_1)$,

$C_2 = (v_5, v_7 \cdots, v_8, v_6, \cdots, v_5)$, $C_3 = (u_1, u_2, \cdots, v_7, v_5, \cdots, v_4, v_2, \cdots, u_1)$ and $C_4 = (u_1, u_2, \cdots, v_6, v_8, \cdots, v_1, v_3, \cdots, u_1)$. The sum of the number of vertices in these 4 cycles is $(n+10)$, so there is a cycle of length at most $(n+10)/4$. This implies that $g(G_5) \leq (n+10)/4$, a contradiction.

We will also show that G_5 does not contain 3 intersecting chords. Assume that $R_1 = u_1u_2$, $R_2 = v_1v_2$, and $R_3 = w_1w_2$ are 3 such chords with the order of these vertices on the cycle being $u_1, v_1, w_1, u_2, v_2, w_2$. There is a fourth chord, which we will denote by $R = x_1x_2$. With no loss of generality, we can assume that $x_1 \in (u_1, v_1)$. Since G_5 does not contain 4 intersecting chords, we can assume with no loss of generality that either $x_2 \in (v_1, w_1)$ or $x_2 \in (w_1, u_2)$. We will consider these two situations in Case 1 and Case 2 that follow.

Case 1: Suppose $x_2 \in (v_1, w_1)$.

We will denote the fifth chord by $S = y_1y_2$. Since each chord, in particular R, must intersect at least 2 other chords, we have that $y_1 \in (x_1, x_2)$. By symmetry, we can assume $y_1 \in (x_1, v_1)$. Using the fact that each chord must intersect at least 2 chords, symmetry, and the forbidden structures such as 4 intersecting chords, there are 3 possibilities for y_2; either $y_2 \in (v_2, w_2)$, $y_2 \in (w_2, u_1)$, or $y_2 \in (w_1, u_2)$. We will consider these subcases independently.

Subcase 1.1: Suppose $y_2 \in (v_2, w_2)$.

The sum of the number of vertices in the 4 cycles $C_1 = (u_1, u_2, \cdots, w_1, w_2, \cdots, u_1)$, $C_2 = (y_1, y_2, \cdots, v_2, v_1, \cdots, y_1)$, $C_3 = (u_1, u_2, \cdots, v_2, v_1, \cdots, x_2, x_1, \cdots, u_1)$, and $C_4 = (y_1, y_2, \cdots, w_2, w_1, \cdots, x_2, x_1, \cdots, y_1)$ is $n+10$. Therefore, one of the cycles C_1, C_2, C_3 or C_4 has at most $(n+10)/4$ vertices.

Subcase 1.2: Suppose $y_2 \in (w_2, u_1)$.

For future reference denote this graph by H_1^*. Each of the following 10 cycles uses 3 of the 10 intervals of the cycle C: $C_1 = (y_1, y_2, \cdots, u_1, \cdots, x_1, \cdots, y_1)$, $C_2 = (x_1, x_2, \cdots, v_1, \cdots, y_1, \cdots, x_1)$, $C_3 = (v_1, v_2, \cdots, w_2, w_1, \cdots, x_2, \cdots, v_1)$, $C_4 = (w_1, w_2, \cdots, y_2, \cdots, u_1, u_2, \cdots, w_1)$, $C_5 = (x_1, x_2, \cdots, w_1, w_2, \cdots, y_2, y_1, \cdots, x_1)$, $C_6 = (v_1, v_2, \cdots, w_2, \cdots, y_2, y_1, \cdots, v_1)$, $C_7 = (u_1, u_2, \cdots, w_1, \cdots, x_2, x_1, \cdots, u_1)$, $C_8 = (w_1, w_2, \cdots, v_2, \cdots, u_2, \cdots, w_1)$, $C_9 = (u_1, u_2, \cdots, v_2, v_1 \cdots, y_1, y_2, \cdots, u_1)$, and $C_{10} = (u_1, u_2, \cdots, v_2, v_1 \cdots, x_2, x_1, \cdots, u_1)$. The sum of the lengths of these cycles is $3n+20$, and so one of the cycles has at most $(3n+20)/10 = 3n/10 + 2$ vertices.

Subcase 1.3: Suppose $y_2 \in (w_1, u_2)$.

For future reference denote this graph by H_2^*. Each of the following 10 cycles uses 3 of the 10 intervals of the cycle C: $C_1 = (x_1, x_2, \cdots, v_1, \cdots, y_1, \cdots, x_1)$, $C_2 = (y_1, y_2, \cdots, u_2, \cdots, v_2, v_1 \cdots, y_1)$, $C_3 = (u_1, u_2, \cdots, v_2, \cdots, w_2, \cdots, u_1)$, $C_4 = (y_1, y_2, \cdots, u_2, u_1, \cdots, x_1, \cdots, y_1)$, $C_5 = (x_1, x_2, \cdots, w_1, w_2, \cdots, u_1, \cdots, x_1)$, $C_6 = (v_1, v_2, \cdots, w_2, w_1, \cdots, x_2, \cdots, v_1)$, $C_7 = (u_1, u_2, \cdots, y_2, \cdots, w_1, w_2 \cdots, u_1)$, $C_8 = (x_1, x_2, \cdots, w_1, \cdots, y_2, y_1 \cdots, x_1)$, $C_9 =$

$(u_1, u_2, \cdots, v_2, v_1 \cdots, x_2, x_1, \cdots, u_1)$, and $C_{10} = (v_1, v_2, \cdots, w_2, w_1 \cdots, y_2, y_1, \cdots, v_1)$. The sum of the lengths of these cycles is $3n + 20$, and so one of the cycles has at most $(3n + 20)/10 = 3n/10 + 2$ vertices.

Case 2: Suppose $x_2 \in (w_1, u_2)$.

We will again denote the fifth chord by $S = y_1y_2$. Using the symmetry of the graph, the fact that each chord must intersect at least two other chords, and to avoid Case 1 for any triple of intersecting chords, we can assume without loss of generality that $y_1 \in (v_1, w_1)$ and $y_2 \in (w_2, u_1)$. In this case consider the 4 cycles $C_1 = (x_1, x_2, \cdots, u_2, u_1, \cdots, x_1)$, $C_2 = (y_1, y_2, \cdots, w_2, w_1, \cdots, y_1)$, $C_3 = (v_1, v_2, \cdots, w_2, w_1, \cdots, x_2, x_1 \cdots, v_1)$, and $C_4 = (u_1, u_2, \cdots, v_2, v_1, \cdots, y_1, y_2, \cdots, u_1)$. The sum of the lengths of these 4 cycles is $(n + 10)/4$, and so $g(G_5) \leq (n + 10)/4$ in this case. This gives a contradiction that verifies that G_5 does not contain 3 intersecting chords.

Select 2 parallel chords, which we will denote by $R_1 = u_1u_2$ and $R_2 = v_1v_2$ with u_1, u_2, v_2, v_1 being the order of these vertices on the cycle C. There must be 2 chords that intersect R_1, which we will denote by $S_1 = x_1x_2$ and $S_2 = y_1y_2$. If the two chords S_1 and S_2 intersect, then there will be 3 intersecting chords, a contradiction. Hence, we can assume that S_1 and S_2 are also parallel. At least one of S_1 or S_2 must intersect R_2. With no loss of generality we can assume that $x_1 \in (u_1, y_1)$, $y_1 \in (x_1, u_2)$, $x_2 \in (v_2, v_1)$, and that either $y_2 \in (v_2, x_2)$ or $y_2 \in (u_2, v_2)$. There is an additional chord in G_5, which we will denote by $T = z_1z_2$.

We will first consider the case when $y_2 \in (v_2, x_2)$. However to avoid the existence of 3 parallel chords, T must intersect at least one of R_1 and R_2 and at least one of S_1 and S_2. This will imply the existence of 3 intersecting chords, a contradiction. In other words, it is straightforward to check that the chord T implies the existence of either 3 parallel chords or 3 intersecting chords.

In the case when $y_2 \in (u_2, v_2)$, observe that the chord T must intersect both the chord R_2 and the chord S_2, since each chord must intersect at least 2 chords. Thus, with no loss of generality, we can assume that $z_1 \in (u_2, y_2)$ and $z_2 \in (v_2, x_2)$. Now, the 10 cycles $C_1 = (u_1, u_2, \cdots, y_1, \cdots, x_1, \cdots, u_1)$, $C_2 = (y_1, y_2, \cdots, z_1, \cdots, u_2, \cdots, y_1)$, $C_3 = (z_1, z_2, \cdots, v_2, \cdots, y_2, \cdots, z_1)$, $C_4 = (v_1, v_2, \cdots, z_2, \cdots, x_2, \cdots, v_1)$, $C_5 = (x_1, x_2, \cdots, v_1, \cdots, u_1, \cdots, x_1)$, $C_6 = (u_1, u_2, \cdots, z_1, z_2, \cdots, x_2, x_1, \cdots, u_1)$, $C_7 = (y_1, y_2 \cdots, v_2, v_1, \cdots, u_1, u_2, \cdots, y_1)$, $C_8 = (z_1, z_2 \cdots, x_2, x_1, \cdots, y_1, y_2, \cdots, z_1)$, $C_9 = (v_2, v_1 \cdots, u_1, u_2, \cdots, z_1, z_2, \cdots, v_2)$, $C_{10} = (x_2, x_1 \cdots, y_1, y_2, \cdots, v_2, v_1, \cdots, x_2)$ collectively use each of the edges of C precisely 3 times and each of the chords 4 times. Hence, thus sum of their lengths is $3n + 20$, and one of the 10 cycles has length at most $(3n + 20)/10 = 3n/10 + 2$. This gives a contradiction which completes the verification that $g(5) = 3/10$. For future reference denote this graph by H_3^*.

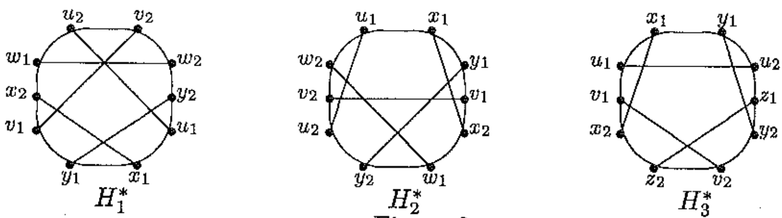

Figure 2

Claim:$g(6) = 1/4$

We will assume there is no constant c such that $g(G_6) \leq n/4 + c$, and show that this leads to a contradiction. The arguments for G_6 are identical to those for G_5 and will borrow heavily from the observations of that case. In particular, the arguments used in proving $g(5) = 3/10$ also imply that G_6 does not contain 3 parallel chords or 4 intersecting chords and that each chord of G_6 must intersect at least 3 chords. The deletion of a chord from G_6 gives a graph G_5. In all of the cases considered in proving that $g(5) = 3/10$, it was shown that $g(G_5) \leq n/4 + c$ for some constant c, except for the three graphs H_1^*, H_2^*, and H_3^* (see Figure 2). Hence we can assume that G_6 is one of the graphs H_i^* $(1 \leq i \leq 3)$ with an additional chord R. The chord R must be positioned in each of the graphs H_i^* such that there does not exists 3 parallel chords, 4 intersecting chords, and such that each chord intersects at least 3 chords. In a straightforward way it can be checked that this is impossible. This contradiction completes the proof that $g(6) = 1/4$ and also the proof of Theorem 2. □

Actually more was proved than was indicated in the statement of Theorem 2. The range of values for the constant c used in the definition of $g(t)$ is very restricted. If we define $g^*(t)$ to be the least upper bound of the girth $g(G)$ of any hamiltonian graph G with n vertices and $n + t$ edges, then the proof of Theorem 2 implies that $g^*(1) = g^*(2) = n/2 + 1$, $g^*(3) = g^*(4) = n/3 + 2$, $3n/10 + 1 \leq g^*(5) \leq 3n/10 + 2$, and $n/4 + 1 \leq g^*(i) \leq n/4 + 5/2$ for $i = 6, 7$ and 8.

It is very likely with some tedious case analysis that $g(9)$ and $g(10)$ can be determined and probably $g(9) = g(10) = 1/5$. The following example implies that $g(9) \geq g(10) \geq 1/5$. Let H_{10} be the graph obtained from the cycle $C = (x_0, x_1, \cdots, x_{n-1}, x_0)$ with $n = 20m$ by adding the 10 chords $x_0x_{10m}, x_{2m}x_{6m}, x_{4m}x_{8m}, x_{5m}x_{15m}, x_{7m}x_{11m}, x_{9m}x_{13m}, x_{12m}x_{16m}, x_{14m}x_{18m}, x_{17m}x_m, x_{19m}x_{3m}$. Any cycle of G must contain all of the vertices in at least 4 of the intervals of the C determined by the 10 chords, and so $g(H_{10}) = 4m + 1 = n/5 + 1$.

References

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