On k-Linked Graphs

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Abstract

Abstract: A graph G is said to be *k-linked* if G has order at least $2k$ and for any ordered set $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ of $2k$ vertices, G contains vertex disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i connects v_i and w_i for $i = 1, 2, \ldots, k$. Many have studied the question of the minimum connectivity necessary to imply a graph is k-linked. Here we consider adding additional conditions, such as forbidden subgraphs, which reduce the connectivity level necessary for the graph to be k-linked. We also consider powers of graphs, as any such edge density condition is also natural to consider when dealing with k-linked graphs.

1 Introduction

One generalization of the idea of connectivity in graphs is the following: A graph G is said to be *k-linked* if G has order at least $2k$ and for any

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ordered set of $2k$ vertices $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$, G contains vertex disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i connects v_i and w_i for $i = 1, 2, \ldots, k$. For $k = 1$ this reduces to the standard definition of a connected graph. Given vertices v_i, w_i $(i = 1, 2, \ldots, k)$, the collection of vertex disjoint paths $P_1, P_2, \ldots, P_k$ is called a *k-linkage*. Further, given two vertices u, v and paths $P_1, P_2, \ldots, P_t$ joining u and v, we say these paths are *internally disjoint* provided $V(P_i) \cap V(P_j) = \{u, v\}$, for $i \neq j$.

Larman and Mani [6] as well as Jung [5] considered the problem of the existence of a smallest integer $f(k)$ such that every $f(k)$-connected graph is k-linked. Clearly, $f(1) = 1$ while Jung [5] proved that $f(2) = 6$. There are 5−connected planar graphs which are not 2−linked. For example, in the graph of Figure 1, the pairs x, y and a, b have no 2-linkage. It is easy to see that in this graph any $x - y$ path must intersect any $a - b$ path. Thomassen [8] characterized graphs that are not 2-linked. However, the problem of determining $f(k)$ for $k \geq 3$ appears to be difficult and remains open. The graph $K_{3k-1} - kK_2$ shows that for $k \geq 3$, $f(k) \geq 3k - 2$. Recently, Bollobás and Thomason [1] have shown that if $\kappa(G) \geq 22k$, then G is k−linked, hence $f(k) \leq 22k$.

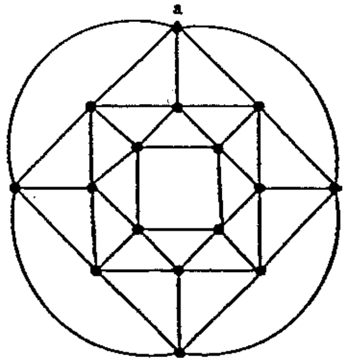

Figure 1: Planar, 5-connected, but not 2-linked graph.

It is reasonable to expect that under certain conditions one could determine the connectivity necessary for a graph to be k−linked. One approach which has been useful in other path or cycle problems is restricting attention to the class of graphs free of a particular family of subgraphs (see [3]). We say a graph G is $\{H_1, H_2, \ldots, H_t\}-$*free* if G contains no induced subgraph isomorphic to H_i for any $i = 1, 2, \ldots, t$. If $t = 1$ we simply say G is H_1-*free*, or that H_1 is a *forbidden subgraph* of G.

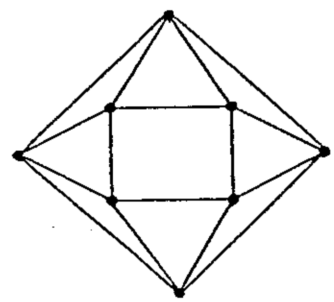

Figure 2: A 4-connected $K_{1,3}$-free but not 2-linked graph, X.

The following notation will be useful. Given a path P containing vertices x and y, we denote by $P[x, y]$ the subpath of P from x to y, including both x and y. We term $P[x, y]$ the $x - y$ segment of the path P. Similarily, $P(x, y)$ is the subpath of P from x to y which does not include the end vertices x or y. When paths P and Q intersect, they may have one vertex in common or many consecutive vertices in common. We will consider the entire subpath of consecutive vertices in common to both paths as one *intersection of the paths*. Note that distinct intersections may share vertices.

All graphs in this paper are simple, without loops or multiple edges. For terms not defined here, see [2].

2 Forbidden Subgraphs

With the use of forbidden subgraphs, we are sometimes able to reduce the connectivity needed for a graph to be 2$-$linked and at times establish a level of connectivity sufficient to show a graph is k-linked when $k \geq 3$. We begin with a result on the 2$-$linked case.

Theorem 1 *If G is a 5-connected $K_{1,3}$-free graph of order $n \geq 5$, then G is 2-linked.*

Proof: Suppose G is not 2-linked and consider two pairs of vertices, x, y and u, v where linkage fails. Since there exists at least five internally disjoint $x - y$ paths in G, we begin with such a path system, say Ψ.

Since 2-linkage fails for the pairs x, y and u, v, it must be the case that every path from u to v intersects every path from x to y. In particular then, the following lemma must hold.

Lemma 1 *Suppose Ψ is an internally disjoint $x - y$ path system and Q is a $u - v$ path which does not contain x or y. Also suppose that over all such $u - v$ paths, Q has the fewest path intersections with Ψ, then*

(i) Q will intersect each path in Ψ only at internal vertices,

(ii) Q determines an ordering, say $P_1, P_2, \ldots, P_5$ of these paths, and

(iii) given this ordering, there will be a segment of Q from P_i to P_{i+1} for $i = 1, 2, 3, 4$. Thus, Q will never reintersect a path of Ψ after leaving that path.

Proof. Given the $x - y$ path system Ψ, clearly Q must intersect each of the five paths or a 2-linkage would result, a contradiction to our assumption. By our conditions, this intersection is only possible at internal vertices of the path. Hence, (i) holds.

Clearly, upon following Q from u to v there is a first $x - y$ path encountered, call it P_1. On continuing to follow Q eventually a new path (call it P_2) in Ψ is intersected. We continue in this manner until the ordering of the five paths of Ψ is determined. Hence, (ii) holds.

Clearly there is a segment of Q from u to P_1 (maybe only u itself) and a segment of Q from P_1 to P_2. Now suppose there is no segment of Q from P_i to P_{i+1} for some $i \geq 3$. Then upon leaving the path P_i, the path Q must either proceed to P_{i+2} (or if P_{i+2} does not exist, to v) or the path Q must reintersect an earlier path. In the first case if Q proceeds to P_{i+2} without intersecting P_{i+1}, a contradiction to our order of intersection results; while if Q proceeds on to v, a 2-linkage would result using P_{i+1}, again a contradiction. In the second case, if Q reintersects an earlier path, say P_j with segment $[z^1, z^2]$ and if $P_j[x^1, x^2]$ is the segment of P_j corresponding to the first intersection of Q with P_j, then replace Q with

$$Q^* = Q[u, x^1], P_j[x^1, z^2], Q(z^2, v].$$

The path Q^* has fewer intersections in Ψ than Q, no matter where z^1 or z^2 lie on P_j, a contradiction to our choice of Q. Thus, (iii) holds and hence, Lemma 1 is proved. □

Lemma 2 *If G is not 2-linked, then all $u - v$ paths must intersect the paths of Ψ (in this case $P_1, \ldots, P_5$) in this order (ignoring repeated intersections of the same path).*

Proof. If this were not the case, then some $u - v$ path, say Q^* would, without loss of generality, intersect P_i prior to intersecting P_j, for $j < i$. Recall, Q is a $u - v$ path with fewest intersections. Suppose $[u^1, u^2]$ is the first intersection of Q and P_i and that $[x^1, x^2]$ is the first intersection of Q^* and P_i. But then, by Lemma 1 (iii), following the segments $Q^*[u, x^1], P_i(x^1, u^2), Q[u^2, v]$ we produce a $u - v$ path that avoids P_j, creating a 2-linkage, a contradiction. Hence, Lemma 2 is proved. □

Finally, given this ordering of paths $P_1, \ldots, P_5$, if path P_i has initial vertex u_{i1}, we note that $< x, u_{11}, u_{31}, u_{51} > \cong K_{1,3}$ unless one of the edges $u_{11}u_{31}, u_{11}u_{51}$, or $u_{31}u_{51}$ is present in G. Suppose that $u_{11}u_{31}$ is in G. Further suppose that $[x^1, x^2]$ is the intersection of Q with P_1 and $[z^1, z^2]$ is the intersection of Q with P_3. Then

$$Q[u, x^1], P_1[x^1, u_{11}), u_{11}, u_{31}, P_3(u_{31}, z^2), Q(z^2, v]$$

produces a $u - v$ path that avoids P_2 since by Lemma 1 (iii) Q never reintersects P_2. A similar argument applies if the edges $u_{11}u_{51}$ or $u_{31}u_{51}$ exist in G. Thus, in all cases we are able to complete the linkage, proving the theorem. □

Example 1: The graph of Figure 3 is $K_{1,3}$-free, 4-connected, but not two linked. By repeating the basic interior pattern we can construct an infinite family with these properties. Thus, Theorem 1 is best possible in terms of the connectivity condition.

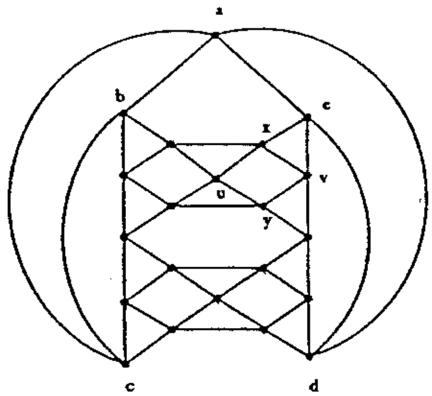

Figure 3: $K_{1,3}$-free, 4-connected, but not 2-linked graph.

In order to see that this graph is $K_{1,3}$-free, simply note that the neighborhood of each vertex is $2K_2$. In order to see that this graph is not 2-linked, note that it is planar and if we consider the pairs x, y and u, v, then any $x - y$ path would completely surround either u or v. Finally, to see that this graph is 4-connected, note that the vertices a, b, c, d and e play a fundamental role in preventing small cut sets. □

We now extend Theorem 1 for values of k greater than two.

Theorem 2 *If G is $4(k-1)+1$-connected and $K_{1,3}$-free ($k \geq 2$), then G is k-linked.*

Proof: We proceed by induction on k. For $k = 2$ Theorem 1 provides the base result. Now assume a $(k-1)$-linkage exists in G for any $k-1$ pairs of vertices and suppose that for the pairs $(v_1, u_1), \ldots, (v_{k-1}, u_{k-1})$, (x, y) no k-linkage exists.

Now by the induction hypothesis, the pairs $(v_1, u_1), \ldots, (v_{k-1}, u_{k-1})$ can be linked, so suppose $P_1, \ldots, P_{k-1}$ is such a linkage. Then it must be the case that any $x - y$ path somehow intersects this system. However, we know that there exist $4k-3$ internally disjoint paths from x to y. Now among all $(k-1)$-linkages joining v_i with u_i, $i = 1, 2, \ldots, k-1$, select one with the smallest number of intersections with the $x - y$ path system.

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Since G is $4(k-1)+1$ connected, there are at least $4k-3$ $(k \geq 2)$ paths from x to y, there must exist some $v_i - u_i$ path that intersects five of these $x-y$ paths first (in the sense of shortest path length). If this were not the case there would exist an $x-y$ path that misses all $(k-1)$ paths in the linkage, and we could extend to a k-linkage for these k pairs, contradicting our assumption.

Now order these five paths as was done in Lemma 1. Then again using the $K_{1,3}$ centered at x and the resultant edge that must be in G, we can build a path system with a smaller number of intersections with the $x-y$ paths, contradicting our assumption on the system. Thus, G is k-linked and the result is proved. □

An argument similar to that of the last theorem provides the following easy generalization.

Theorem 3 *If G is $K_{1,t}$-free $(t \geq 3)$ and has connectivity at least $(2t-2)(k-1)+1$, then G is k-linked.*

The next lemma will be useful in several results.

Lemma 3 *If G is a t-connected graph that is not 2-linked and x, y and u, v are pairs with no linkage, then for any $x-y$ path system Ψ of t internally disjoint paths, each such path that does not contain u or v must have length at least $t-1$.*

Proof: Suppose not, say the path P_i in Ψ does not contain u or v but has at most $t-1$ vertices (and hence length less than $t-1$). Then, removing all the vertices of P_i from G leaves a connected graph. Hence, there is still a $u-v$ path in $G-V(P_i)$, which must miss the $x-y$ path P_i. Thus, a 2-linkage would exist, a contradiction to our assumptions, proving the lemma. □

We next turn our attention to forbidden paths. The following result is straightforward, so we omit it's proof.

Theorem 4 *A 5-connected P_7-free graph of order $n \geq 8$ is 2-linked. Further, a 4-connected P_5-free graph of order $n \geq 9$ is also 2-linked.*

Example: Note that Theorem 4 says a 4-connected P_5-free graph is 2-linked, again an improvement on the general value of f(2). The graph of Figure 4 is clearly 3-connected and P_5-free. This graph is not

2-linked as the pairs a, b and x, y can not be linked. Clearly, any path joining either pair and missing the other pair requires use of the edge uv.

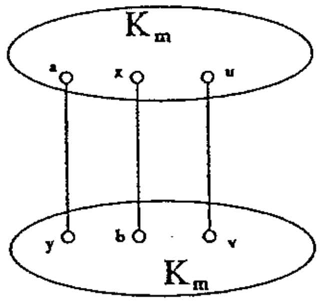

Figure 4: A 3-connected P_5-free, but not 2-linked graph.

We now turn to a result whose proof is similar in technique to those we have seen earlier, but uses a larger set of forbidden subgraphs. Let the graph Z_i be defined as a triangle with a path of length i attached to one of its vertices.

Theorem 5 *If G is a 5-connected $\{K_{1,4}, Z_3\}$-free graph, then G is 2-linked.*

Proof. Suppose that G is not 2-linked and that x, y and u, v are two pairs of vertices with no 2-linkage. Let Ψ be an $x - y$ path system with each path as short as possible, say $P_i : x, u_{i,1}, \ldots, u_{i,j_i}, y$ where $1 \leq i \leq 5$ and $j_i \geq 1$. Also assume the order of the paths is determined by the order of intersection with a shortest $u - v$ path Q.

If u and v are not vertices in Ψ, then by Lemma 3 each path in Ψ contains at least 3 internal vertices. Since G is $K_{1,4}$-free, then some edge of the form $u_{1,1}u_{2,1}$ or $u_{4,1}u_{5,1}$ is in G or a 2-linkage can be found (this may take repeated applications of the $K_{1,4}$−free property). Without loss of generality suppose that $u_{1,1}u_{2,1}$ is an edge of G. But then $< x, u_{1,1}, u_{2,1}, u_{4,1}, u_{4,2}, u_{4,3} > \cong Z_3$. However, the addition of any edge to this graph either produces a 2-linkage or shortens the overall sum of the path lengths in Ψ, a contradiction in either case. Thus, G must be 2-linked.

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Note that the graph X of Figure 2 satisfies the hypothesis of the last result except with connectivity 4, however it is not 2-linked. The graph B (commonly called the bull) is a triangle with one edge off two distinct vertices of the triangle.

Theorem 6 *If* G *is a 4-connected* $\mathcal{F} = \{K_{1,3}, B\}$*-free graph, then* G *is 2-linked or* G *contains an induced* X.

Proof. Suppose that G does not contain X as an induced subgraph. Also suppose G is 4-connected, $\mathcal{F}$-free and that G is not 2-linked. Say that x, y and u, v are two pairs of vertices in G with no 2-linkage. Since G is 4-connected, we know that there exists a system of at least four internally disjoint $x - y$ paths. Over all such systems, suppose that Ψ is one with the least total path length sum, that is, Ψ has the smallest possible sum of the path lengths in the system.

Note that x and y are not adjacent, for if they were then the graph $G - \{x, y\}$ would still be 2-connected and a 2-linkage would exist, contradicting our assumptions. Similarly, u and v are not adjacent. Thus, each path in Ψ has at least one internal vertex. If u and v are not in Ψ, then by Lemma 3 each path of Ψ contains at least two internal vertices.

Let Ψ consist of the paths $P_i : x, u_{i,1}, u_{i,2}, \ldots, u_{i,j_i}, y$ where $1 \leq i \leq 4$ and $j_i \geq 2$ for each i. Further, assume that the ordering P_1, P_2, P_3, P_4 is determined by the order of intersection with the shortest $u - v$ path Q (as was done earlier).

By repeatedly considering potential induced $K_{1,3}$'s centered at x and using the vertices $u_{i,1}$ for $i = 1, 2, 3, 4$ and the fact that any edge of the form $u_{i,1}u_{t,1}$ where $t > i+1$ would allow a 2-linkage to exist, we see that $e_1 = u_{1,1}u_{2,1}$, $e_2 = u_{2,1}u_{3,1}$ and $e_3 = u_{3,1}u_{4,1}$ are all in $E(G)$.

We now consider two cases based upon whether u and v are in Ψ or not.

Case 1. Suppose that at least one of u and v are not in Ψ.

Without loss of generality we suppose that u is not in Ψ. Then by Lemma 3 all paths of Ψ (except possibly P_4) contain at least two internal vertices. First note that $< x, u_{2,1}, u_{1,1}, u_{2,2}, u_{1,2} > \cong B$ unless one of the edges $u_{2,1}u_{1,2}$ or $u_{1,1}u_{2,2}$ is in G. Without loss of generality suppose that $u_{2,1}u_{1,2}$ is in $E(G)$. Then $< u_{4,1}, x, u_{2,1}, u_{1,1}, u_{2,2} > \cong B$

unless $u_{1,1}u_{2,2}$ is in $E(G)$. (Both the above arguments use the fact that all other potential edges allow a 2-linkage to occur or shorten the sum of the path lengths in Ψ, each a contradiction to our assumptions.) Finally, $< u_{2,1}, x, u_{2,2}, u_{1,2} > \cong K_{1,3}$ unless $u_{1,2}u_{2,2}$ is an edge of G. Again all other potential edges shorten the path sum of Ψ.

Now by repeating the above arguments we can force the edges $u_{2,t}u_{1,t+1}$, $u_{1,t}u_{2,t+1}$ and $u_{2,t+1}u_{1,t+1}$ for as many values of $t \geq 1$ as exist on both P_1 and P_2.

If the first intersection of the $u - v$ path Q with P_1 is at $u_{1,m}$ and Q leaves P_2 at $u_{2,s}$, we consider two cases.

If $s < m$, then the two paths $Q[u, u_{1,m}], u_{1,m}u_{2,m-1}, P_1(u_{2,m-1}, u_{2,s}]$, $Q(u_{2,s}, v]$ and $P_2[x, u_{2,s-1}]$, $u_{2,s-1}u_{1,s}$, $P_1(u_{1,s}u_{1,m-1}]$, $u_{1,m-1}$, $u_{2,m}$, $P_2(u_{2,m}, y]$ form a 2-linkage, contradicting our assumptions.

If $s \geq m$ a similar construction again shows a 2-linkage is present in G. Thus in either situation a contradiction is reached, ending this case.

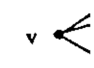
Case 2. Suppose that both u and v are in Ψ.

Then we know by the ordering of the paths in Ψ that u is on P_1 and v is on P_4. If P_1 and P_2 each have two or more internal vertices an argument similar to the last case will lead to a contradiction. Since each path must have at least one internal vertex, from Lemma 3 we may assume that the only internal vertex in P_1 is u and that the only internal vertex in P_4 is v. Further, u and v are each adjacent to x and y as they are the only internal vertices of the paths. Now, $u = u_{1,1}$ is adjacent to $u_{2,1}$ and $v = u_{4,1}$ is adjacent to $u_{4,1}$. Thus, we know we have the structure of Figure 5.

Consider $< x, u, u_{2,1}, u_{2,2}, v > \cong B$. The only edge that does not lead to an immediate contradiction is $uu_{2,2}$, thus it must be in G. By symmetry, $vu_{3,2}$ is an edge of G. Now $< u, x, y, u_{2,2} > \cong K_{1,3}$ unless $yu_{2,2}$ is an edge of G (again all other possibilities lead to a contradiction). A similar argument shows $yu_{3,2}$ is also an edge. Finally, $< x, u, u_{2,2}, y, u_{3,2} > \cong B$ unless $u_{2,2}u_{3,2}$ is an edge of G.

The only other possible edges are $u_{31}u_{22}$ and $u_{21}u_{32}$. If both these edges are present, u, u_{21}, u_{32}, v and x, u_{31}, u_{22}, y form a 2-linkage. Without loss of generality then suppose $u_{21}u_{32}$ is an edge of G.

Since G is 4-connected and $n \geq 9$, there exists a vertex w not in $V(\Psi)$. Further, if there exist internally disjoint paths from w to x and



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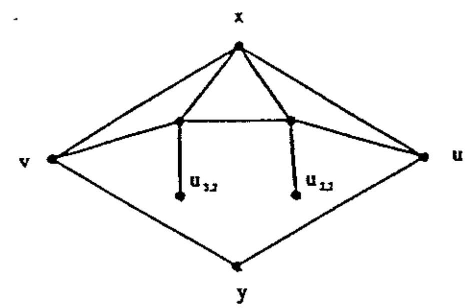

Figure 5: The situation in G.

y or to u and v that do not intersect $u_{21}, u_{22}, u31$ or u_{32}, a 2-linkage clearly exists.

Thus, we may assume that some w not in $V(\Psi)$ is adjacent to one of the internal vertices of P_2 or P_3.

Subcase 1: Suppose w is adjacent to u_{21}.

Now $< u_{21}, u_{32}, u_{22}, v, w >\cong B$. The edges vu_{21}, vu_{22}, and wv each create a 2-linkage. Thus, either wu_{22} or wu_{32} must be in G.

Subcase 1a. Suppose wu_{22} is an edge of G.

Now $< w, u_{21}, u_{22}, x, y >\cong B$. The edges wx, wy and xy are easily seen to create 2-linkages. The edges xu_{22} and yu_{21} each shorten paths in Ψ, contradicting our assumptions. Thus, a contradiction is reached in this case.

Subcase 1b. Suppose wu_{32} is an edge of G.

Now $< w, u_{21}, u_{32}, x, y >\cong B$. Again wx, wy or xy create 2-linkages, while xu_{32} or yu_{21} shorten paths in Ψ. Thus, this case also leads to a contradiction.

Since neither wu_{22} nor wu_{32} can be edges of G, a contradiction is reached and Case 1 is complete. By symmetry, wu_{32} also is not an edge of G.

Subcase 2: Suppose wu_{31} is an edge of G.

Then $< u_{31}, u_{32}, u_{21}, w, y >\cong B$. The edge wy creates a 2-linkage, while if we consider wu_{32}, an argument similar to that given in Subcase 1a leads to a contradiction. The edges yu_{31} or yu_{21} each shorten paths in Ψ Thus, wu_{21} must be an edge of G and we are back in Case 1.

By symmetry, wu_{22} is also not an edge of G. Hence, w has no adjacencies to internal vertices of P_2 or P_3 which means that our assumption that $u_{21}u_{32}$ was an edge of G cannot hold. By symmetry, $u_{31}u_{22}$ is also not an edge of G. But now, $< V(\Psi) > \cong X$, and our result is complete. That is, in all cases, either G is 2-linked or X is an induced subgraph of G. □

3 Powers of Graphs

We now turn our attention to another natural approach to the k-linked problem. Since dense graphs would seem more likely to be k-linked, it is reasonable to consider powers of graphs. The kth power G^k of a connected graph G is that graph with $V(G^k) = V(G)$ and $uv \in E(G^k)$ if, and only if, $1 \leq dist_G(u, v) \leq k$.

Theorem 7 *If G is a connected graph of order at least $2k$, then G^{2k-1} is k-linked.*

Proof. We proceed by induction on the order of G. If $|V(G)| = 2k$, then as G is connected, G^{2k-1} is complete and hence is easily seen to be k-linked. We now assume that for all connected graphs of order $n-1 \geq 2k$ that G^{2k-1} is k-linked. Let G be a connected graph of order $n > 2k$ and let $S = \{v_1, \ldots, v_{2k}\}$ be an ordered collection of $2k$ distinct vertices in $V(G)$. Select a vertex v that is not a cut vertex of G and consider $H = G - \{v\}$.

If $v \notin S$, then by the induction hypothesis, H^{2k-1} is k-linked, hence a k-linkage exists in G^{2k-1} as well.

If $v \in S$, then without loss of generality suppose that $v = v_{2k}$, where v_{2k} is to be linked to v_{2k-1}. Now select a vertex $u \notin S - \{v_{2k-1}, v_{2k}\}$ such that $dist_G(u, v) \leq 2k-1$. Since $|S - \{v_{2k-1}, v_{2k}\}| = 2k-2$, $n > 2k$ and G is connected, such a vertex must exist. Again let $H = G - \{v\}$. Now by the induction hypothesis, the graph H^{2k-1} is k-linked. Thus, the set $S' = S - \{v_{2k-1}\} + \{u\}$ has a linkage. But then the path joining v_{2k-1} and u in H^{2k-1} can be modified to a v_{2k-1} to v_{2k} path by using the edge from u to v_{2k} that must exist in G^{2k-1}. This produces a linkage for S in G^{2k-1} and completes the proof. □

Example. The last result is sharp since the graph P_m^{2k-2} is not k-linked when $m \geq 2k-1$. This is easy to see since an end vertex of P_m only has degree $2k-2$ in P_m^{2k-2}. □

Lemma 4 is a direct conseq
Lemma 6 is due to Hobbs [4].

Lemma 4 *If G is a k-connec subsets of $V(G)$ with $|S_1| = |S_2|$ paths $P_1, \ldots, P_k$ such that P_i is $i = 1, 2, \ldots, k$.*

Lemma 5 *If G is a graph with a complete subgraph on $2k$ vert*

Proof. Consider an ordered se in G. Denote the vertices of son
Suppose that $|A \cap B| = t$,
Since the graph $G' = G - T$ is
exist $2k - t$ vertex disjoint patl
If $t = 2k$, then $A \subset V(K_2$
Thus, we suppose $0 \leq t < 2k$.
$v_{t+1}, \ldots, v_{2k}$ are in $A - B$. F
$v_i = b_i$, $i = 1, \ldots, t$. By Lemi
paths in $G - T$ from the set A
We now show that any pai
vertices in T are to be linked,
vertex v_j in $A - T$ is to be linke
from v_j to b_j in $B - T$ follow
if two vertices of $A - T$ are t
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these paths in $B - T$. Thus, in
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Lemma 6 *[4]. If G is m-con is km-connected.*

Theorem 8 *If G is an s-conn $s \geq 3$ and $st \geq 4k$, then G^t is*

Proof. Assume first that
vertices $S = \{v_1, \ldots, v_k, w_1, \ldots$
to be linked. If $dist_G(v_i, w_i) \leq$
the vertices v_i and w_i are adj

Lemma 4 is a direct consequence of Menger's Theorem [7], while Lemma 6 is due to Hobbs [4].

Lemma 4 *If G is a k-connected graph and S_1 and S_2 are disjoint subsets of $V(G)$ with $|S_1| = |S_2| = k$, then there exist k vertex disjoint paths $P_1, \ldots, P_k$ such that P_i is a $u_i - v_i$ path with $u_i \in S_1$ and $v_i \in S_2$, $i = 1, 2, \ldots, k$.*

Lemma 5 *If G is a graph with connectivity at least $2k$ and G contains a complete subgraph on $2k$ vertices, then G is k-linked.*

Proof. Consider an ordered set $A = \{v_1, \ldots, v_{2k}\}$ of distinct vertices in G. Denote the vertices of some complete subgraph of order $2k$ as B.

Suppose that $|A \cap B| = t$, where $0 \leq t \leq 2k$, and let $T = A \cap B$. Since the graph $G' = G - T$ is $2k - t$ connected, by Lemma 4 there exist $2k - t$ vertex disjoint paths in G' from the set $A - T$ to $B - T$.

If $t = 2k$, then $A \subset V(K_{2k})$ and the linkages are trivial to find. Thus, we suppose $0 \leq t < 2k$. Say vertices $v_1, \ldots, v_t$ are in $A \cap B$ and $v_{t+1}, \ldots, v_{2k}$ are in $A - B$. Further, let $V(B) = \{b_1, \ldots, b_{2k}\}$ where $v_i = b_i$, $i = 1, \ldots, t$. By Lemma 4, there exist $2k - t$ vertex disjoint paths in $G - T$ from the set $A - T$ to $B - T$.

We now show that any pair of vertices in A can be linked. If two vertices in T are to be linked, simply use the edge between them. If a vertex v_j in $A - T$ is to be linked with a vertex v_m in T, follow the path from v_j to b_j in $B - T$ followed by the edge from b_j to v_m. Finally, if two vertices of $A - T$ are to be linked, follow their corresponding paths to $B - T$ and the appropriate edge joining the endvertices of these paths in $B - T$. Thus, in all cases the linkages can be formed, G is k-linked and the lemma proved. □

Lemma 6 *[4]. If G is m-connected with order at least km, then G^k is km-connected.*

Theorem 8 *If G is an s-connected graph of order at least $st+2$, where $s \geq 3$ and $st \geq 4k$, then G^t is k-linked.*

Proof. Assume first that $s \geq 4$. Consider an ordered set of distinct vertices $S = \{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ in $V(G)$, where the pair (v_i, w_i) is to be linked. If $dist_G(v_i, w_i) \leq t$, for each i with $1 \leq i \leq k$, then in G^t the vertices v_i and w_i are adjacent, and we clearly have a k-linkage.

Hence, at least one of the pairs must be at distance more than t. Say the pair (v_j, w_j) is at distance greater than t in G. Since G is s-connected there exist at least s internally disjoint $v_j - w_j$ paths and each of these paths must have length at least $t-1$.

Now consider the vertices on these s paths which are at distance at most $\lfloor 2k/s \rfloor = \lfloor (4k/s)/2 \rfloor$ from v_j. Then in G^t, these vertices, along with v are all adjacent. Since $t \geq 4k/s$, we see that $s\lfloor \frac{2k}{s} \rfloor + 1 \geq 2k$, hence G^t contains a K_{2k}. Thus, by Lemma 5, G^t is k-linked. □

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