73

A Note on Cycles in 2-Factors of Line Graphs

Ronald J. Gould*
Emory University
Atlanta GA 30322

Emily A. Hynds
Emory University
Atlanta GA 30322

February 11, 1998

Abstract

We provide a generalization to the well-known result of Harary and Nash-Williams characterizing graphs with Hamiltonian line graphs. Our generalization allows us to characterize those graphs whose line graphs contain a 2-factor with exactly k ($k \geq 1$) cycles.

1 Introduction

Line Graphs and 2-factors have both been popular areas of study for many years. In this paper, we combine these two topics in a generalization of a classic result by Harary and Nash-Williams [2]. Their result characterizes the graphs G whose line graphs $L(G)$ are Hamiltonian. Our generalization allows us to consider arbitrary 2-factors with k cycles in line graphs.

We define a *dominating circuit* of a graph G to be a circuit of G with the property that every edge of G either belongs to the circuit or is adjacent to an edge of the circuit. The result we wish to generalize is the following result by Harary and Nash-Williams [2].

Theorem 1 *Let G be a graph without isolated vertices. Then $L(G)$ is hamiltonian if and only if $G \simeq K(1,n)$, for some $n \geq 3$, or G contains a dominating circuit.*

All graphs in this paper will be simple finite graphs. Given a graph G, we say that G contains a *k-system that dominates* if G contains a collection of k edge disjoint circuits and stars, $(K(1,n_i), n_i \geq 3)$, such that each edge of G is either contained in one of the circuits or stars, or is adjacent to one of the circuits. Terms that are not defined here can be found in [1].

*Supported by O.N.R. Grant N00014-97-1-0499.

2 The Generalization

We now present a generalization of Theorem 1.

Theorem 2 *Let G be a graph with no isolated vertices. The graph $L(G)$ contains a 2-factor with k ($k \geq 1$) cycles if, and only if, G contains a k-system that dominates.*

Proof: The case when $k = 1$ is Theorem 1. We will now proceed by induction on k. Assume the theorem holds for all $k \leq n$, where $n > 1$. For one direction we will assume that we have a graph G such that $L(G)$ contains a 2-factor with $n + 1$ cycles; $C_1, C_2, ..., C_{n+1}$. Now, consider the subgraph of $L(G)$ induced by the vertices of $C_1, ..., C_n$. This subgraph is the line graph of a subgraph G^* of G containing only the edges corresponding to the vertices of $C_1, C_2, ..., C_n$.

Every vertex v_i of $V(C_{n+1})$ corresponds to exactly one edge e_i in $E(G)$. Hence, we can also form $L(G^*)$ by removing all of the vertices of C_{n+1} from $L(G)$. Consequently, we can form the subgraph G^* of G by removing the edges e_i from $E(G)$ that correspond to the vertices of C_{n+1} in $L(G)$. By induction, G^* contains an n-system that dominates. Let S be such an n-system. Now we will look at the subgraph H of G that is induced by the edges that we removed to get G^*. In other words, H uses all of the removed edges and any vertices that they are adjacent to in G. Then $L(H)$ is the subgraph of $L(G)$ induced by the vertices of C_{n+1}. Clearly $L(H)$ is hamiltonian, and so by induction, H is a star or a circuit that dominates the edges of H. Call this 1-system S_1. Therefore, $S \cup S_1$ is an $n+1$-system that dominates in G.

Conversely, suppose G is a graph that contains an $n + 1$-system that dominates with stars and circuits $S_1, S_2, ..., S_{n+1}$. Let G^* be the subgraph of G that is induced by the vertices of the stars and circuits $S_1, ..., S_n$ and any vertices corresponding to edges that are dominated by the circuits in this collection but are not a part of S_{n+1}. Then by induction, $L(G^*)$ contains a 2-factor with n cycles. Now let H be the subgraph of G that is induced by the edges that are not in G^*. The graph H must be a star or a circuit that dominates the edges of G not included in G^*. So, by induction, $L(H)$ is hamiltonian. Because G^* and H are edge disjoint, $L(G^*)$ and $L(H)$ are vertex disjoint subgraphs of $L(G)$ that include all of the vertices of $L(G)$. Consequently, $L(G)$ contains a 2-factor with $n + 1$ cycles. □

We now show a special application of Theorem 2.

Corollary 3 *The graph $L(K_n)$ ($n \geq 3$) contains a 2-factor with k cycles for $k = 1, 2, ..., \lfloor \frac{n(n-1)}{6} \rfloor$.*

Sketch of Proof: We proceed by induction on n, the order of the complete graph. The result is clear for $n = 3$, as $L(K_3)$ is a triangle. For

$n > 3$, we separate the proof of the induction step into 2 cases for $n \equiv 1$ or $2 (\bmod 3)$ and $n \equiv 0 (\bmod 3)$. The idea is to show that K_n contains a k-system that dominates for $k = 1, 2, ..., \lfloor \frac{n(n-1)}{6} \rfloor$.

Case 1 *Suppose* $n \equiv 1$ *or* $2 (\bmod 3)$.

We begin by breaking K_n into an edge disjoint K_{n-1} and star $K(1, n-1)$. We then use induction on the K_{n-1} to get an l-system that dominate its edges for $l = 1, 2, ..., \frac{(n-1)(n-2)}{6}$. To these systems we add $1, ..., \lfloor \frac{n-1}{3} \rfloor$ stars from the $K(1, n-1)$. It is easy to see that these, along with the fact that K_n is Hamiltonian, give us a k-system that dominates for $k = 1, 2, ..., \lfloor \frac{n(n-1)}{6} \rfloor$.

Case 2 *Suppose* $n \equiv 0 (\bmod 3)$.

Using the technique from Case 1 we can produce a k-system that dominates K_n for $k = 1, 2, ..., \lfloor \frac{n(n-1)}{6} - 1 \rfloor$. It remains only to find a k-system that dominates K_n for $k = \frac{n(n-1)}{6}$. We separate K_n into an edge disjoint K_{n-2} and H, where H is the subgraph of K_n obtained by removing the edges of the K_{n-2}. Let v and w be the vertices of K_n that are in H but not in the K_{n-2}. By induction, we get an $\frac{(n-2)(n-3)}{6}$-system that dominates the K_{n-2}. To these we add a cycle C from K_n with vertices v, w, and a vertex x from the K_{n-2}. Finally, we add $\frac{n-3}{3}$ stars isomorphic to $K(1, 3)$ from H that are edge disjoint from C and have center v, and another $\frac{n-3}{3}$ stars isomorphic to $K(1, 3)$ from H that are edge disjoint from C and have center w. This gives us a total of $\frac{n(n-1)}{6}$ edge disjoint stars and circuits that dominate K_n, which proves the corollary for this case. □

Remark: It is now clear that Theorem 1 can be restated (without concern for isolated vertices) as: Given a graph G, then $L(G)$ is hamiltonian if and only if G contains a 1-system that dominates. Thus, Theorem 2 holds without the condition that G contains no isolated vertices.

References

[1] M. Behzad, G. Chartrand, and L. Lesniak, *Graphs and Digraphs*, Prindle, Webber, and Schmidt, Massachusetts, 1979.

[2] F. Harary and C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs. *Canad. Math. Bull.* (1965) 701-710.