Math. Slovaca 30, 1980, No. 2, 175—179

THE MIN-MAX SUPERGRAPH

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Let the degree set (the set of degrees of the vertices) of a graph G be denoted by ϑ_G in which Δ and δ represent the maximum and minimum elements respectively. If S is a finite set of positive integers with $\Delta \in S \subseteq \vartheta_G$, then there exists a graph H with degree set S containing G as an induced subgraph. In the case where $S = \{\delta, \Delta\}$, necessary and sufficient conditions are presented for the order of H to be minimum.

It is well known (see [1], Chap. 1, for example) that for any graph G with maximum degree Δ there exists a Δ-regular graph H containing G as an induced subgraph. (The graph H is called a supergraph of G.) Furthermore, Erdös and Kelly [3] have found a necessary and sufficient set of conditions which determine the minimum order of such a graph H. In this article we generalize the first of these results and extend the second.

The degree set ϑ_G of a graph G is the set of degrees of the vertices of G. If $\vartheta_G = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \ldots < a_n$, then $\delta(G) = \delta = a_1$ is the minimum degree of G and $\Delta(G) = \Delta = a_n$ is the maximum degree of G. As mentioned above, there exists a graph H with degree set $\{\Delta\}$ containing G as an induced subgraph. We first present a generalization of this result.

Theorem 1. *Let G be a graph with degree set ϑ_G and maximum degree Δ and let S be a finite set of positive integers such that $\Delta \in S \subseteq \vartheta_G$. Then there exists a graph H with degree set S such that G is an induced subgraph of H.*

Proof. First, observe that if $S = \vartheta_G$, then we may take $H = G$. We have already noted that the result is true if $S = \{\Delta\}$, so we henceforth assume that $2 \leq |S| < |\vartheta_G|$.

Let $\vartheta_G = \{a_1, a_2, \ldots, a_n\}$ with $\delta = a_1 < a_2 < \ldots < a_n = \Delta$, where $n \geq 3$. Define $G_0 = G$. For $i \geq 1$, define G_i to be that graph consisting of two disjoint copies of G_{i-1} together with those edges joining corresponding vertices, say with the same label v if $\deg_{G_{i-1}} v \notin S$. For each $i = 1, 2, \ldots, n$, define

$$k_i = \min \{m \mid m \geq i, a_m \in S\}$$

and

$$k = \max \{a_{k_i} - a_i \mid i = 1, 2, \ldots, n\}.$$

* Research partially supported by a Faculty Research Fellowship from Western Michigan University.

Then $H = G_k$ has degree set S and contains G as an induced subgraph.

In the case where $S = \{\Delta\}$ Erdös and Kelly described a method for determining the minimum order of the graph H mentioned in the statement of Theorem 1. We do the same thing if $S = \{\delta, \Delta\}$. Prior to presenting a set of conditions which give the minimum order of H in this case, we find it necessary to introduce some terminology.

Let G be a graph of order p and degree set $\vartheta_G = \{a_1, a_2, \ldots, a_n\}$. Let the vertex set $V(G)$ of G be expressed as $V(G) = V_1 \cup V_2 \cup \ldots \cup V_n$, where for $1 \leqslant i \leqslant n$, $|V_i| = m_i \geqslant 1$ such that $v \in V_i$ implies that $\deg_G v = a_i$. Let $V^* = \bigcup_{i=2}^{n-1} V_i$ and let $\sigma = \sum_{v \in V^*} (\Delta - \deg_G v)$ denote the regular deficiency of G. Further, let H be a graph having degree set $\{\delta(G), \Delta(G)\}$ and containing G as an induced subgraph, and let $I = V(H) - V(G)$ be the set of vertices that need to be added to G in order to obtain H. From the $s = |I|$ vertices in H which are not in G, let s_1 have degree δ and $s_2 = s - s_1$ have degree Δ in H. Let j represent the number of vertices in V_1 that have degree Δ in H. Then $0 \leqslant j \leqslant m_1$, and $j = m_1$ forces s_1 to be at least one. Let $F = \langle I \rangle$ denote the subgraph of H induced by the set I. With $k = |E(F)|$ denoting the size of F, we observe that $0 \leqslant k \leqslant s(s-1)/2$.

The graph H contains $(m_1 - j + s_1)$ vertices of degree δ and $(p - m_1 + j + s_2)$ vertices of degree Δ. Hence

$$\delta(m_1 - j + s_1) + \Delta(p - m_1 + j + s_2) \quad \text{is even.} \tag{1}$$

We may observe that $H = G$ in case $V^* = \emptyset$. Otherwise, if $u \in V_1$ and $\deg_H u = \Delta$, then $s \geqslant \Delta - \delta$; and, if $v \in V_2$ so that $\deg_G v = a_2$ and $\deg_H v = \Delta$, then $s \geqslant \Delta - a_2$. Thus,

$$s = s_1 + s_2 \geqslant \begin{cases} \Delta - a_2 & \text{if } j = 0 \\ \Delta - \delta & \text{if } j \geqslant 1. \end{cases} \tag{2}$$

We may count the number e of edges in H between the sets $V(G)$ and I in two ways. The set V_1 has j vertices of degree Δ in H and the vertices in V^* result in the regular deficiency σ. Then $e = \sigma + j(\Delta - \delta)$. Moreover the graph F has size k, and the set I contains s_1 vertices of degree δ and s_2 vertices of degree Δ in H. So $e = \delta s_1 + \Delta s_2 - 2k$. Thus

$$\delta s_1 + \Delta s_2 - 2k = \sigma + j(\Delta - \delta). \tag{3}$$

We also observe that these e edges induce a bipartite graph on the set $V(G) \cup I$. It is possible to describe this more precisely, which we now do. Let $\mathcal{F}$: $f_1, f_2, \ldots, f_s$ denote the sequence of degrees of the vertices of F, where $\sum_{i=1}^{s} f_i = 2k$. For each permutation π on $\{1, 2, \ldots, s\}$, consider the sequence $\mathcal{S}_1$: $b_1, b_2, \ldots, b_s$, where

$$b_i = \begin{cases} \Delta - f_{\pi(i)} & \text{if} \quad 1 \leq i \leq s_2 \\ \delta - f_{\pi(i)} & \text{if} \quad s_2 < i \leq s_1 + s_2 \end{cases}$$

is nonnegative. Also consider the sequence $\mathscr{S}_2$ whose terms are $(\Delta - \deg_G v)$, where $v \in V^*$ if $j=0$, and $1 \leq j \leq m_1$ implies that $v \in V^* \cup V_1(j)$, where $V_1(j)$ denotes a j-element subset of V_1. Then $\mathscr{S}_2$ has $n = p - (m_1 + m_n) + j$ terms. Let us write this sequence as $\mathscr{S}_2$: $c_1, c_2, \ldots, c_n$.

The pair of sequences $\mathscr{S}_1 : b_1, b_2, \ldots, b_s$ and $\mathscr{S}_2 : c_1, c_2, \ldots, c_n$ is called bigraphical (see [2]) if there exists a bipartite graph B with partite sets $U_1 = \{u_1, u_2, \ldots, u_s\}$ and $U_2 = \{w_1, w_2, \ldots, w_n\}$ such that $\deg_B u_i = b_i$, $1 \leq i \leq s$, and $\deg_B w_j = c_j$, $1 \leq j \leq n$. Necessary and sufficient conditions were obtained in [2] for a pair of sequences of nonnegative integers to be bigraphical. We state one such condition for later use.

Theorem 2. *Let $\mathscr{S}_1 : b_1, b_2, \ldots, b_s$ and $\mathscr{S}_2 : c_1, c_2, \ldots, c_n$ be a pair of sequences of nonnegative integers with*

$$b_1 \geq b_2 \geq \ldots \geq b_s,$$
$$c_1 \geq c_2 \geq \ldots \geq c_n,$$

and

$$\sum_{i=1}^{s} b_i = \sum_{j=1}^{n} c_j.$$

Then the pair of sequences $(\mathscr{S}_1; \mathscr{S}_2)$ is bigraphical if and only if the pair of sequences $(\mathscr{S}'_1; \mathscr{S}'_2)$ is bigraphical where

$$\mathscr{S}'_1 : b_1 - 1, b_2 - 1, \ldots, b_{c_1} - 1, b_{c_1+1}, \ldots, b_s$$

and

$$\mathscr{S}'_2 : c_2, c_3, \ldots, c_n.$$

We can now state the following condition:

there exists a graphical sequence $\mathscr{F}$ for which some pair of sequences $(\mathscr{S}_1; \mathscr{S}_2)$ is bigraphical. (4)

We have now shown that the conditions (1)—(4) are necessary for a graph H of minimum order $p + s$ (where $s = s_1 + s_2$) to exist. These conditions also prove to be sufficient. In order to see this let G be a given graph with degree set $\vartheta_G = \{a_1, a_2, \ldots, a_n\}$, where $\delta = a_1 < a_2 < \ldots < a_n = \Delta$ and $n \geq 2$, and let $s = s_1 + s_2$ (where s_1, s_2 are nonnegative integers) be the least positive integer for which there exist integers j and k, $0 \leq j \leq m_1$ and $0 \leq k \leq s(s-1)/2$ such that (1)—(4) are satisfied. By (4) there exists a graphical sequence $\mathscr{F}$: $f_1, f_2, \ldots, f_s$. Let F be a graph having degree sequence $\mathscr{F}$ where, then, the size of F is k. Also by (4) some pair of sequences $(\mathscr{S}_1; \mathscr{S}_2)$ is bigraphical, so there exists a bipartite graph B with partite sets $U_1 = \{u_1, u_2, \ldots, u_s\}$ and $U_2 = \{w_1, w_2, \ldots, w_n\}$ such that $\deg_B u_i = b_i$, $1 \leq i \leq s$, and

$\deg_B w_j = c_j$, $1 \leqslant j \leqslant n$. We now define a graph H by $V(H) = U_1 \cup V(G)$, where $U_1 = V(F)$, $U_2 = V^* \cup V_1(j)$ and $E(H) = E(B) \cup E(F) \cup E(G)$. Clearly G is an induced subgraph of H and $\vartheta_H = \{\delta, \Delta\}$. Thus, the following result has been verified.

Theorem 3. *Let G be a graph with degree set $\vartheta_G = \{a_1, a_2, \ldots, a_n\}$, where $\delta = a_1 < a_2 < \ldots < a_n = \Delta$ and $n \geqslant 2$. Let H be a graph with degree set $\vartheta_H = \{\delta, \Delta\}$ containing G as an induced subgraph. A necessary and sufficient condition that $p+s$ be the least possible order for H is that $s = s_1 + s_2$ is the least integer satisfying:*

(1) $\delta(m_1 - j + s_1) + \Delta(p - m_1 + j + s_2)$ *is even,*

(2) $s = s_1 + s_2 \geqslant \begin{cases} \Delta - a_2 & \text{if } j = 0 \\ \Delta - \delta & \text{if } j \geqslant 1, \end{cases}$

(3) $\delta s_1 + \Delta s_2 - 2k = \sigma + j(\Delta - \delta)$, *and*

(4) *there exists a graphical sequence $\mathcal{J}$ for which some pair of sequences $(\mathcal{S}_1; \mathcal{S}_2)$ is bigraphical.*

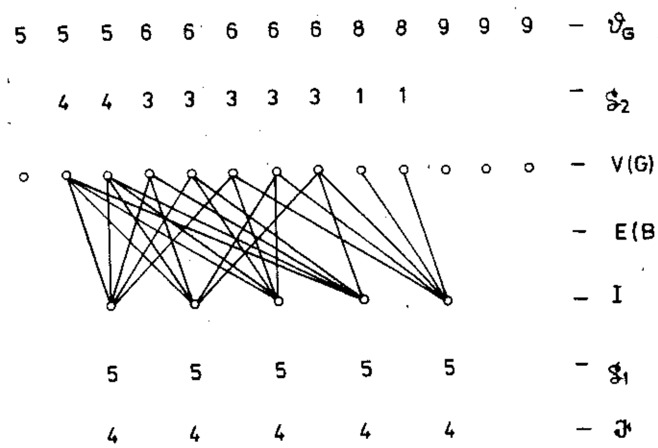
We illustrate the procedure by an example. Let G be a graph with degree sequence

$$9, 9, 9, 8, 8, 6, 6, 6, 6, 6, 5, 5, 5.$$

Here $\delta = 5$, $\Delta = 9$, $\sigma = 17$, $m_1 = 3$, $p = 13$ and $a_2 = 6$. By (1), $5(3 - j + s_1) + 9(13 - 3 + j + s_2)$ is even, and this implies that s_1 and s_2 have opposite parity and s is odd. Condition (2) implies that $s_1 + s_2 \geqslant 3$ if $j = 0$; and $j = 1, 2$ or 3 implies that $s_1 + s_2 \geqslant 5$, since $s = s_1 + s_2$ is odd and at least 4. Also, (3) states that $5s_1 + 9s_2 - 2k = 17 + 4j$.

Consider $s = s_1 + s_2 = 3$. Then j must be zero, and $0 \leqslant k \leqslant \binom{3}{2} = 3$. (i) If $s_1 = 0$ and $s_2 = 3$, then $k = 5$. (ii) If $s_1 = 1$ and $s_2 = 2$, then $k = 3$ and $F \cong K_3$. This implies that $\mathcal{J}$: 2, 2, 2; $\mathcal{S}_1$: 7, 7, 3; and $\mathcal{S}_2$: 3, 3, 3, 3, 3, 1, 1. Here the conditions (1), (2) and (3) hold. Moreover the sequence $\mathcal{J}$ is graphical. But a repeated application of Theorem 2 shows that the pair of sequences $(\mathcal{S}_1; \mathcal{S}_2)$ is not bigraphical. So (4) fails to hold. (iii) If $s_1 = 2$ and $s_2 = 1$, then $k = 1$ and $F \cong K_1 \cup K_2$. Hence $\mathcal{J}$: 1, 1, 0; $\mathcal{S}_1$: 9, 4, 4 or $\mathcal{S}_1$: 8, 5, 4; and $\mathcal{S}_2$: 3, 3, 3, 3, 3, 1, 1. Once again we use Theorem 2 to observe that $(\mathcal{S}_1; \mathcal{S}_2)$ is not bigraphical. (iv) If $s_1 = 3$ and $s_2 = 0$, then $k < 0$. Thus, $s \geqslant 5$.

We consider $s_1 = 0$, $s_2 = 5$, $j = 2$. Then $k = 10$ and $F \cong K_5$. Now $\mathcal{J}$: 4, 4, 4, 4, 4; $\mathcal{S}_1$: 5, 5, 5, 5, 5; and $\mathcal{S}_2$: 4, 4, 3, 3, 3, 3, 3, 1, 1. The pair $(\mathcal{S}_1; \mathcal{S}_2)$ is easily seen to be bigraphical (by Theorem 2). In the figure below we have shown the essential sequences and the graph B. ($E(G)$ and $E(F)$ are not shown.) Clearly $\vartheta_H = \{5, 9\}$.



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Received June 14, 1978

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Резюме

Пусть ϑ_G обозначает множество всех степеней вершин графа G, max $\vartheta_G = \Delta$, min $\vartheta_G = \delta$. Если S – множество такое, что $\Delta \in S \subseteq \vartheta_G$, то существует граф H с множеством $\vartheta_H = S$, для которого G является порожденным подграфом. В случае $S = \{\delta, \Delta\}$ находится необходимое и достаточное условие для того, чтобы число вершин графа H было минимальным.