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Characterizing forbidden pairs for hamiltonian properties [1]

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Abstract

In this paper we characterize those pairs of forbidden subgraphs sufficient to imply various hamiltonian type properties in graphs. In particular, we find all forbidden pairs sufficient, along with a minor connectivity condition, to imply a graph is traceable, hamiltonian, pancyclic, panconnected or cycle extendable. We also consider the case of hamiltonian-connected graphs and present a result concerning the pairs for such graphs.

1. Introduction

Given a family $\mathscr{F} = \{H_1, H_2, \ldots, H_k\}$ of graphs we say that a graph G is $\mathscr{F}$-free if G contains no induced subgraph isomorphic to any H_i, $i = 1, 2, \ldots, k$. In particular, if $\mathscr{F} = \{H\}$, we simply say G is H-free. We call the graphs in $\mathscr{F}$ *forbidden subgraphs*. The use of forbidden subgraphs to obtain classes of graphs possessing special properties has long been a common graphical technique. In particular, some of the graphs most commonly involved in forbidden families for hamiltonian properties are shown in Fig. 1. It has been pointed out that the star $K_{1,3}$, sometimes called the *claw*, has often been a part of these forbidden families. We shall show the reason for that observation in the course of this paper.

One of the earliest forbidden subgraph results dealing with hamiltonian properties is the following result due to Duffus et al. [3]. The graphs $K_{1,3}$ and N are shown in Fig. 1.

Theorem 1. *Let G be a $\{K_{1,3}, N\}$-free graph. Then*

(1) *if G is connected, then G is traceable and*

(2) *if G is 2-connected, then G is hamiltonian.*

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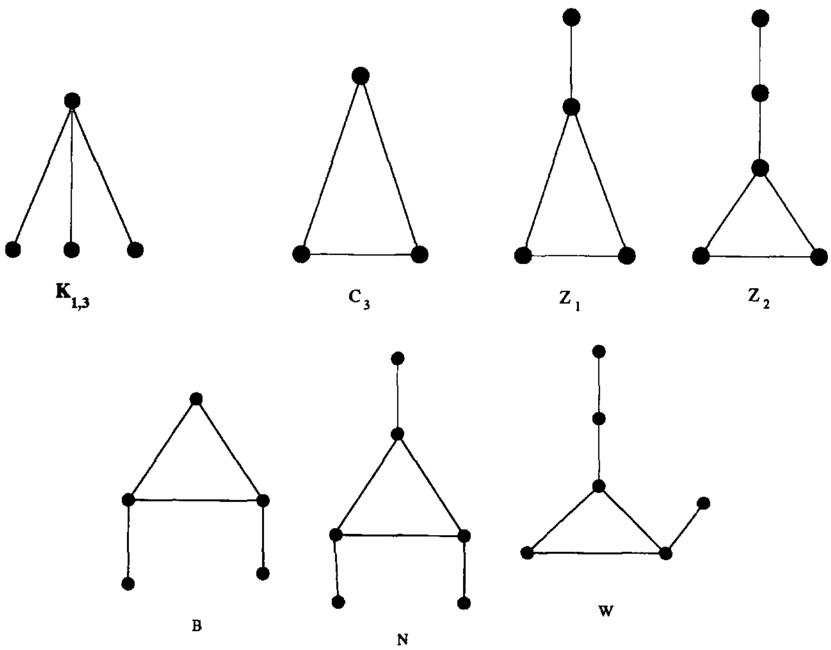

Fig. 1. Common forbidden graphs.

This result is typical of the type we wish to address in this paper. It imposes minor, but necessary, connectivity conditions on the class of graphs defined by a forbidden pair of graphs in order to obtain hamiltonian results. The connectivity conditions used in Theorem 1 are the minimal ones necessary in graphs with the corresponding properties.

If P is a hamiltonian property (like traceable, hamiltonian, pancyclic, etc.), let $k(P)$ denote the least connectivity possible in a graph with property P. Thus, for example if P is traceability, then $k(P)=1$ while if P is hamiltonicity, then $k(P)=2$. In this paper we wish to determine all pairs of connected graphs $\{H_1, H_2\}$ such that any $k(P)$-connected $\{H_1, H_2\}$-free graph will possess hamiltonian property P. In particular, we will consider property P to be each of the following fundamental hamiltonian properties: traceable, hamiltonian, pancyclic, panconnected, and cycle extendable. We shall also consider the problem when P is hamiltonian-connected, however a complete characterization in this case will not be obtained. This idea was introduced by Bedrossian in [1] who considered it for hamiltonian and pancyclic graphs. However, in proving which graphs must be forbidden, he used graphs of small order in his proofs. We shall reexamine his results later and restrict our attention to infinite families of graphs. In doing so, we shall extend Bedrossian's results.

We concentrate on forbidden pairs, however, in the course of our work we will also solve the corresponding problems when only one graph is forbidden. This turns out to be a much more restrictive situation and easier to solve. The question for triples has also been considered and, as you might expect, is considerably more involved. We shall not address triples in this paper.

One case is trivial and we wish to eliminate it from further consideration. Suppose G is connected, has order $n \geqslant 3$ and is P_3-free (here P_k denotes a path on k vertices), then G is easily seen to be a complete graph (which we denote K_n). But if G is complete, then G has every hamiltonian property. Thus, forbidding P_3 alone implies each hamiltonian property P and thus any other graph could be paired with P_3 to obtain the same result. In fact, later we will show that P_3 is the only single graph that solves our problem and thus we will remove it from consideration in forbidden pairs.

We also denote the cycle on n vertices as C_n and the complete bipartite graph with r vertices in one set and m vertices in the other set as $K_{r,m}$. Finally, we define the graphs Z_i, $i = 1, 2, \ldots$ to be a triangle with a path of length i attached to one of its vertices, that is, Z_i is formed by identifying one vertex of a C_3 with an end vertex of a P_{i+1} (see Fig. 1 for Z_1 and Z_2). For convenience we use the notation $A = B$ to denote A is isomorphic to B as well as A is equal to B. This should cause the reader no problems. For other terms not defined here see [6].

2. Traceable graphs

We say a graph G is *traceable* if it contains a spanning path, that is, a path containing all of the vertices of G. In this section we determine which pairs $\{H_1, H_2\}$ ($H_i \neq P_3$, $i = 1, 2$) imply a connected graph G is traceable. We note that Theorem 1 shows the pair $\{K_{1,3}, N\}$ is one such pair. It is also a simple matter to see that if H is any induced subgraph of N, then the pair $\{K_{1,3}, H\}$ will also solve our problem. In particular then, the graphs C_3, P_4, Z_1 and B (see Fig. 1) may each play the role of H. We now show these are the only such pairs of graphs. To do this we will need the example graphs of Fig. 2. Note that each of these graphs represents an infinite family of connected nontraceable graphs.

Theorem 2. *Let R and S be connected graphs ($R, S \neq P_3$) and let G be a connected graph. Then G is $\{R, S\}$-free implies G is traceable if, and only if, $R = K_{1,3}$ and S is one of the following: C_3, P_4, Z_1, B or N.*

Proof. That each of these pairs implies a connected graph is traceable follows from Theorem 1 and our previous comments on induced subgraphs.

Now consider the graph H_0 of Fig. 2, obtained by subdividing the edges of a $K_{1,3}$ an arbitrary number of times. The graph H_0 is clearly connected and nontraceable, so assume without loss of generality that H_0 contains R as an induced subgraph. Further, suppose that R contains an induced P_4. Then note that the graphs H_1 and H_2 (see Fig. 2) are both connected and nontraceable and neither contains an induced P_4. Thus, S must be an induced subgraph of both H_1 and H_2. But then we see that S must be a star, in fact, $S = K_{1,3}$.

Next suppose R does not contain an induced P_4. As R is a subgraph of H_0, then R must contain a vertex of degree 3. But these conditions in H_0 imply $R = K_{1,3}$. Thus, in either case one of our forbidden subgraphs must be $K_{1,3}$.

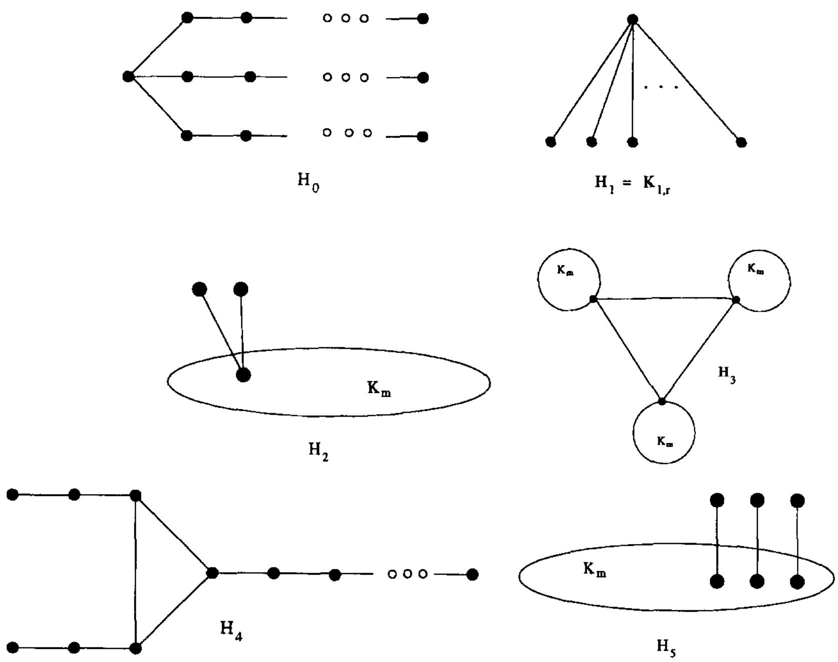

Fig. 2. Connected nontraceable graphs.

For the remainder of this proof we assume without loss of generality that $R = K_{1,3}$. The graph H_3 (see Fig. 2) is connected, nontraceable and contains no induced $K_{1,3}$ and thus, S must be an induced subgraph of H_3. Further, H_3 contains no induced P_5, hence S contains no induced P_5. Similarly, H_5 is claw-free and Z_2-free. Also, H_4 (see Fig. 2) is connected, nontraceable and $K_{1,3}$-free; thus S is an induced subgraph of H_4. Since the largest clique in H_4 is K_3, the same holds for S. But now if S contains no K_3 then S must be P_4, while if S does contain K_3, then S is either $C_3 = K_3, Z_1, B$ or N. This completes the proof. □

We now verify the single forbidden subgraph result for traceable graphs mentioned earlier.

Theorem 3. *Let A and G be connected graphs. Then G is A-free implies G is traceable if, and only if, $A = P_3$.*

Proof. From our earlier remarks we know that if $A = P_3$ then G is traceable. Thus, assume $A \neq P_3$. The graph H_0 of Fig. 2 is not traceable, hence A must be an induced subgraph of H_0. Thus, A is a tree with at most one vertex of degree 3. Similarly, the graphs $K_{1,r}$ ($r \geqslant 3$) imply that A must be a star, in fact, $A = K_{1,3}$. However, the graph H_5 of Fig. 2 is connected, nontraceable and contains no induced $K_{1,3}$. Thus, no other A exists and the result is shown. □

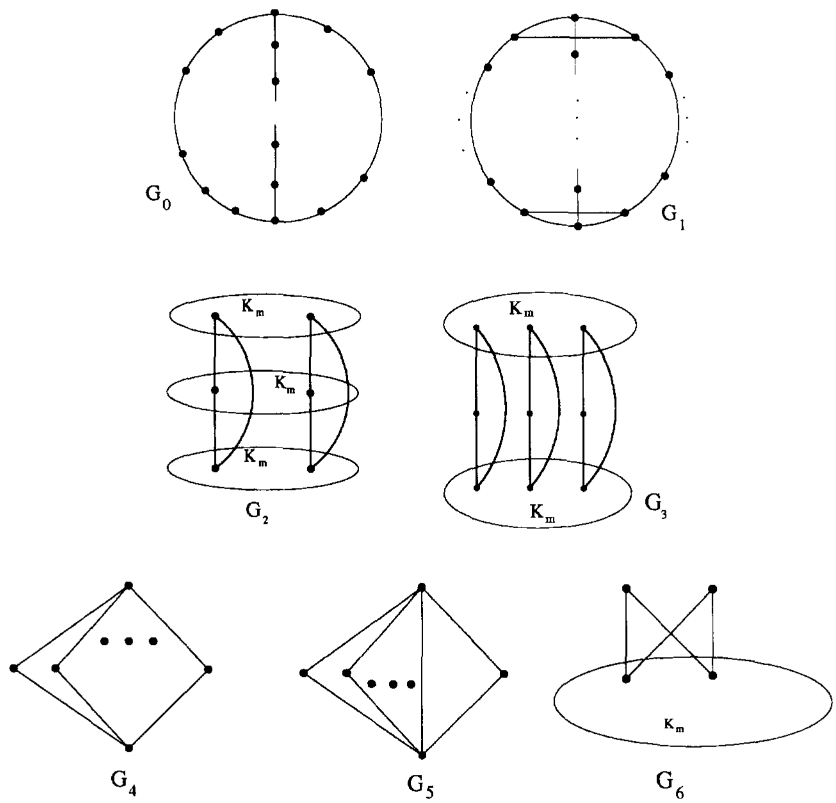

Fig. 3. 2-Connected nonhamiltonian graphs.

3. Hamiltonian graphs

A graph G is *hamiltonian* if G contains a spanning cycle. We now consider the problem of all forbidden pairs that imply a 2-connected graph is hamiltonian. In order to do this we will need several results from the literature as well as the example graphs of Fig. 3, each of which is 2-connected and nonhamiltonian.

Theorem 4 (Broersma and Veldman [2]). *If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph, then G is hamiltonian.*

Theorem 5 (Gould and Jacobson [7]). *If G is a 2-connected $\{K_{1,3}, Z_2\}$-free graph, then G is hamiltonian.*

Theorem 6 (Bedrossian [1]). *If G is a 2-connected $\{K_{1,3}, W\}$-free graph, then G is hamiltonian.*

Theorem 7 (Faudree [4]). *If G is a 2-connected $\{K_{1,3}, Z_3\}$-free graph of order $n \geqslant 10$, then G is hamiltonian.*

A characterization of all pairs that imply a 2-connected graph is hamiltonian was accomplished in [1]. However, as mentioned earlier, graphs of small order were used in the proof to eliminate certain graphs, namely Z_3. However, recently Theorem 7 was verified and this sheds new light on the situation. We now present an extended characterization whose proof is based on infinite families of nonhamiltonian graphs (see Fig. 3).

Theorem 8. *Let R and S be connected graphs $(R, S \neq P_3)$ and G a 2-connected graph of order $n \geqslant 10$. Then G is (R, S)-free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is one of the graphs C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N or W.*

Proof. That each of the pairs implies G is hamiltonian follows from Theorems 1, 4–7 and our remarks about induced subgraphs of forbidden graphs.

Now consider the graphs $G_0, \ldots, G_6$ of Fig. 3. Each is 2-connected and nonhamiltonian. Without loss of generality assume that R is a subgraph of G_1.

Case 1: Suppose that R contains an induced P_4.

Since G_4, G_5, and G_6 are all P_4-free, then S must be an induced subgraph of each of them. But if S is an induced subgraph of G_4, then either S is a star or S contains an induced C_4. However, G_5 is C_4-free, hence S must be a star. Since the only induced star in G_6 is $K_{1,3}$, we have that $S = K_{1,3}$.

Case 2: Suppose that R does not contain an induced P_4.

Then, using G_0 we see immediately that R must be a tree containing at most one vertex of degree 3 and since R contains no induced P_4, we see that $R = K_{1,3}$. Thus, for the remainder of the proof we assume without loss of generality that $R = K_{1,3}$.

Now, S must be an induced subgraph of G_1, G_2, and G_3 (each of which is claw-free). The fact that S is an induced subgraph of G_1 implies that S is a path or S is K_3, possibly with a path off each of its vertices. Suppose that S is a path. Since S is an induced subgraph of G_3 which is P_7-free, we see that if S is a path, it is one of P_4, P_5 or P_6.

Hence, we now assume that S contains a K_3, possibly with a path off each of its vertices. Note that G_3 is Z_4-free. Further, any triangle in G_2 with a path of length 3 off one of its vertices can have no paths off its other vertices (leaving Z_3, Z_2, Z_1, and K_3). Again examining G_2 we see it contains no triangle with a path of length 2 from one of its vertices and a path of length 1 from the other two vertices (leaving B or W). The only remaining possibility is a path of length 1 off each of the vertices of K_3, that is, the graph N. □

Again we turn our attention to the case of only one forbidden subgraph.

Theorem 9. *Suppose A is a connected graph and G is a 2-connected graph. Then G is A-free implies G is hamiltonian if, and only if, $A = P_3$.*

Proof. By our earlier comments we know that if G is P_3-free then, G is complete and hence hamiltonian.

Conversely, the graph G_0 of Fig. 2 is not hamiltonian, hence A must be an induced subgraph of G_0. Thus, A must be a tree with at most one vertex of degree 3. But then G_6 shows that A must be the star $K_{1,3}$ or P_3. However, since G_3 is $K_{1,3}$-free, we see that $A = P_3$. □

4. Pancyclic and panconnected graphs

In this section we characterize those forbidden pairs that imply a 2-connected graph is pancyclic or panconnected. We begin with pancyclic graphs. Recall that G is pancyclic if G contains cycles of all lengths from 3 to $|V(G)|$ and that pancyclic graphs are 2-connected. Once again we must recall earlier works.

Theorem 10 (Faudree [5]). *If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph of order $n \geqslant 10$, then G is pancyclic.*

Theorem 11 (Gould and Jacobson [7]). *If G $(\neq C_n)$ is a 2-connected $\{K_{1,3}, Z_2\}$-free graph of order $n \geqslant 3$, then G is pancyclic.*

With these results in mind we are ready to consider our problem for pancyclic graphs. Once again by considering only infinite families we obtain an extension of Bedrossian's earlier result (which excluded P_6).

Theorem 12. *Let R, S be connected graphs $(R, S \neq P_3)$ and let G $(G \neq C_n)$ be a 2-connected graph of order $n \geqslant 10$. Then G is $\{R, S\}$-free implies G is pancyclic if, and only if, $R = K_{1,3}$ and B is one of P_4, P_5, P_6, Z_1 or Z_2.*

Proof. That each of these pairs implies a 2-connected graph is pancyclic follows from Theorems 10 and 11 and our earlier remarks about induced subgraphs of forbidden graphs.

Conversely, note that G is pancyclic, hence G is hamiltonian. Thus, we may limit our attention to those pairs that imply G is hamiltonian. Hence, $R = K_{1,3}$ and S is one of P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N, or W. However, the graph G_7 of Fig. 4 is a 2-connected, claw-free, nonpancyclic graph which contains no induced B, N or W. Further, G_8 (where the vertices of a K_{2m} are paired and each such pair is connected by a path of length three through two new vertices) is also 2-connected, claw-free and nonpancyclic and is Z_3-free. Thus, our result follows. □

The following result is immediate from Theorem 9.

Theorem 13. *Suppose that A is a connected graph and G is a 2-connected graph. Then G is A-free implies G is pancyclic if, and only if, $A = P_3$.*

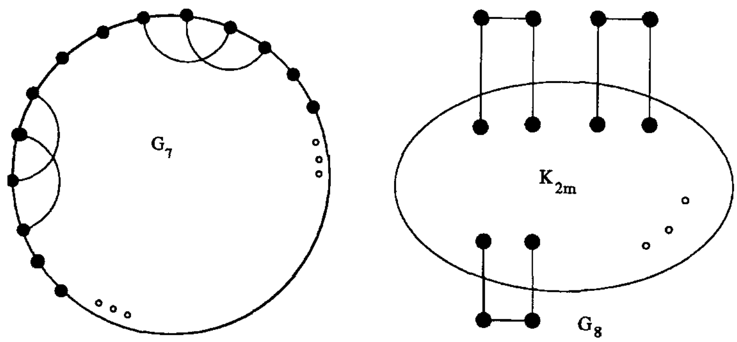

Fig. 4. Two 2-connected nonpancyclic graphs.

We next turn our attention to another strong hamiltonian property. A graph G of order n is said to be *panconnected* if any two vertices of G, say x and y, are joined by paths of all possible lengths l from $dist(x,y)$ to $n-1$. Also recall that panconnected graphs are 3-connected. We begin with the following result.

Theorem 14. *If G is a 3-connected $\{K_{1,3}, Z_1\}$-free graph then G is a complete graph or a complete graph minus a matching. In either case, G is panconnected.*

Proof. A straightforward induction proof can be used to show that any connected $\{K_{1,3}, Z_1\}$-free graph containing a vertex of degree at least 3 is either a complete graph or a complete graph minus a matching. This fact implies G is panconnected. □

For our next result we need several other example families. Let J_1 represent $K_{n,n}$, the family of balanced complete bipartite graphs. Let $J_2 = G_6$, (see Fig. 3). Let J_4 be the point-line incidence graph of a projective plane of order n. It is defined to have a vertex corresponding to each point and to each line of the plane. Two vertices are adjacent provided the point is on the line, that is, we obtain a bipartite graph modeling the incidence of points on lines in the plane. It is well known that such graphs have girth at least 6, are regular, and bipartite. The point-line incidence graph of the Fano plane (the projective plane of order 2) is shown in Fig. 5. The graphs J_3, J_5 and J_6 are also shown in Fig. 5.

Theorem 15. *Let R, S be connected graphs ($R, S \neq P_3$) and let G be a 3-connected graph. Then G is $\{R, S\}$-free implies G is panconnected if, and only if, $R = K_{1,3}$ and $S = Z_1$.*

Proof. The sufficiency follows from Theorem 14.

Conversely, we will first show that one of R and S must be a claw. Thus, suppose that $R, S \neq K_{1,3}$. Without loss of generality assume that R is an induced subgraph of $J_1 = K_{n,n}$. Then $R = K_{1,r}$ where $r \geqslant 4$ or R contains an induced C_4. We now consider two cases.

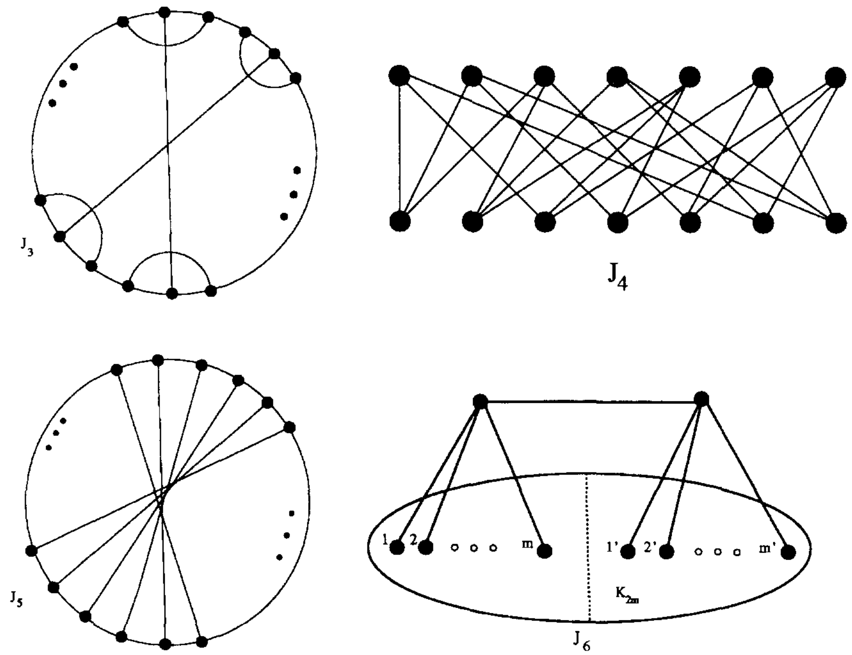

Fig. 5. 3-Connected nonpanconnected graphs.

Case 1: Suppose $R = K_{1,r}$ $(r \geqslant 4)$.

Then R is not an induced subgraph of J_5 (see Fig. 5) as J_5 is regular of degree 3. Thus, S must be an induced subgraph of J_5. Hence we see that S must have girth at least 4. Also note that S must be an induced subgraph of J_2, as R is not an induced subgraph of J_2. But this implies that S must be a star, in fact, $S = K_{1,3}$ contradicting our assumption.

Case 2: Suppose R contains an induced C_4.

Then clearly R is not an induced subgraph of J_4 (the point-line incidence graph of a projective plane which has girth 6). Thus, S must be an induced subgraph of J_4, and so the girth of S must also be at least 6. But S is an induced subgraph of J_2 as well (as J_2 fails to contain an R). Therefore, S must again be a star, contradicting our assumption.

Thus, one of our graphs must be $K_{1,3}$, so without loss of generality suppose that $R = K_{1,3}$. (Note: all graphs used to date in this proof were also not hamiltonian-connected, thus $R = K_{1,3}$ in that problem as well.) Since $R = K_{1,3}$, then S must be an induced subgraph of J_6 and of J_3 as neither contains claws. Note that the longest induced path in J_6 is P_3 which implies that S must contain a cycle. Therefore, S must contain a C_3 with some edges off its vertices. Now since S is an induced subgraph of J_3 we see S contains a triangle and any four vertices containing this triangle will induce at most 4 edges. Similarly, any five vertices containing this triangle will induce at most 5 edges. Finally, we see that S has maximum degree at most 3. Now the only such graphs existing in J_2 are Z_1 and C_3. But then we are left with only Z_1. □

We next state the now obvious result concerning one forbidden subgraph.

Theorem 16. *If A is connected and G is 3-connected then G is A-free implies G is panconnected if, and only if, $A = P_3$.*

We conclude this section with another variation. A graph is said to have a *k-pancyclic ordering* provided the vertices of G can be ordered such that the graph induced by the first j vertices ($j \geqslant k$) is hamiltonian. We now consider such graphs.

Theorem 17. *Let R and S be connected graphs ($R, S \neq P_3$) and let G ($\neq C_n$) be a 2-connected graph of order $n \geqslant 10$. Then G is $\{R, S\}$-free implies G has a 6-pancyclic ordering if, and only if, $R = K_{1,3}$ and $S = P_4$, P_5, P_6, Z_1 or Z_2.*

Proof. If G is $\{R, S\}$-free implies that G has a 6-pancyclic ordering then G is also hamiltonian. Thus, we know that $R = K_{1,3}$ and S is one of C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N, or W. However, consider the graph G_7 as well as G_8 of Fig. 4. Clearly, G_7 has no 6-pancyclic ordering as it has no 6-cycles, while G_8 has no 6-pancyclic ordering as the vertices of degree 2 cannot be incorporated one by one in the ordering. Each graph is claw-free and G_7 is B, N and W-free, while G_8 is Z_3-free. Also, a 2-connected graph being claw-free and C_3-free implies the graph is a cycle. Hence, S is one of P_4, P_5, P_6, Z_1 or Z_2.

Further, Theorem 10 (see [5]) implies that every $\{K_{1,3}, P_6\}$-free graph G has a 6-pancyclic ordering. Thus, we are left with Z_1 and Z_2. However, these follow immediately from Hendry's result (Theorem 18) from the next section. □

5. Cycle extendable graphs

A graph G is said to be *cycle extendable* if any nonhamiltonian cycle can be extended to a cycle containing exactly one more vertex, that is, C is extended to a cycle C' with $V(C') = V(C) \cup \{x\}$ for some vertex x not on C. We say G is *fully cycle extendable* if G is cycle extendable and every vertex of G lies on a triangle. This concept was introduced by Hendry [8]. In that paper he also showed the following:

Theorem 18. *If G is a 2-connected graph of order $n \geqslant 10$ that is $\{K_{1,3}, Z_2\}$-free, then G is cycle extendable.*

With this result in hand we now characterize the forbidden pairs that imply a 2-connected graph is cycle extendable.

Theorem 19. *Let R, S be connected graphs ($R, S \neq P_3$) and G a 2-connected graph of order $n \geqslant 10$. Then G is $\{R, S\}$-free implies G is cycle extendable if, and only if, $R = K_{1,3}$ and S is one of C_3, P_4, Z_1 or Z_2.*

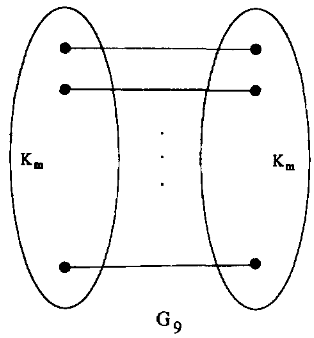

Fig. 6. A non-cycle extendable graph.

Proof. That each of these pairs implies G is cycle extendable follows from Theorem 18 and our comments on induced subgraphs of forbidden graphs.

Conversely, note that if G is cycle extendable then G is hamiltonian and so we may limit our consideration to the pairs listed in Theorem 8. Further, we may assume that $R = K_{1,3}$. The graph G_9 of Fig. 6, formed by taking two copies of K_m and joining corresponding vertices in each copy by an edge, is claw-free and not cycle extendable (in particular, any cycle formed by the vertices of one copy of K_m cannot be extended). Therefore, S must be an induced subgraph of G_9. However, G_9 contains no induced P_5, B, N, W or Z_3. The result now follows. □

The following are corollaries to Hendry's proof of Theorem 18 and the last result. Note that in the next corollary, the cycle extendability requires the use of 3 chords induced by the original cycle. In fact, we can classify types of cycle extendability by the number of cycle chords that must be used in order to extend the cycle. We say a cycle is *t-chord extendable* if it requires exactly t chords to extend the cycle; while a graph G is t-chord extendable if every cycle in G can be extended using at most t chords.

Corollary 20. *Let R,S ($R,S \neq P_3$) be connected graphs and G a 2-connected graph of order $n \geqslant 10$. Then G is $\{R,S\}$-free implies G is 3-chord cycle extendable if, and only if, $R = K_{1,3}$ and S is one of: C_3, P_4, Z_1 or Z_2.*

Corollary 21. *Let R,S ($R,S \neq P_3$) be connected graphs and G a 2-connected graph of order $n \geqslant 10$ with $\delta(G) \geqslant 3$. Then G is $\{R,S\}$-free implies G is 3-chord fully cycle extendable if, and only if, $R = K_{1,3}$ and S is one of: P_4, Z_1 or Z_2.*

Corollary 22. *Let R,S ($R,S \neq P_3$) be connected graphs and G a 2-connected graph of order $n \geqslant 10$. Then G is $\{R,S\}$-free implies G is 0-chord cycle extendable if, and only if, $R = K_{1,3}$ and S is one of C_3, Z_1.*

The graph E_1 in Fig. 7 is claw-free and Z_2-free and is not 0-chord cycle extendable. Any cycle formed from all the vertices except the one of degree 2 cannot be extended

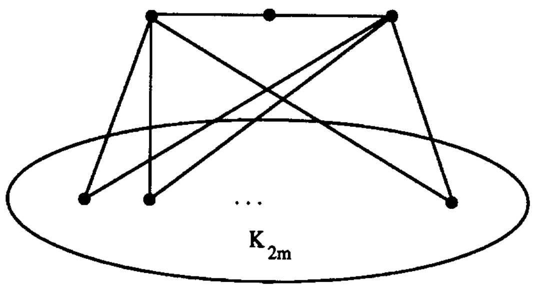

Fig. 7. A graph which is not 0-chord cycle extendable.

without using chords. This is because the neighbors of the vertex of degree 2 are not adjacent on any such cycle. Thus, a natural question is what we can say about such graphs, are they 1 or 2 chord cycle extendable?

We now turn to a situation when 1-chord extendability is obtained.

Theorem 23. *If G is a 2-connected $\{K_{1,3}, Z_2\}$-free graph of order $n \geqslant 10$, then G is 1-chord cycle extendable.*

Proof. Let $C = x_1, x_2, \ldots, x_k, x_1$ be a cycle that is not 1-chord extendable. We can assume that $y_1 \notin V(C)$ and that $x_1 y_1 \in E(G)$. Moreover, since G is 2-connected, there is a path P from y_1 to C that avoids x_1. We will assume that this path is as short as possible over all possible choices of y_1 and the path, which we will denote by $P = y_1, y_2, \ldots, y_t$ with $y_t = x_j$. We can also assume that j is minimal with respect to this property as well. Since G is $K_{1,3}$-free, $x_k x_2 \in E(G)$.

If $t \geqslant 4$, then $\{x_k, x_2, x_1, y_1, y_2\}$ induces a Z_2. Thus, we can assume that $t = 2$ or 3. For $t = 3$, the same set induces a Z_2 unless, without loss of generality, $y_t = x_2$. In this case, $K_{1,3}$-free implies that $x_1 x_3 \in E(G)$ as well. If $x_k x_3 \in E(G)$, then $\{x_k, x_3, x_2, y_2, y_1\}$ induces a Z_2, and if $x_k x_3 \notin E(G)$, then $\{x_2, x_k, x_3, y_2\}$ induces a claw. Therefore we can assume that $t = 2$ and $y_1 x_1$ and $y_1 x_j \in E(G)$ $(2 < j < k)$.

We next investigate the edges between $\{x_k, x_1, x_2\}$ and $\{x_{j-1}, x_j, x_{j+1}\}$, noting that $x_k x_2$ and $x_{j-1} x_{j+1} \in E(G)$. Since C is not 1-chord extendable, $x_2 x_{j+1}$ and $x_k x_{j-1} \notin E(G)$. Since G is $K_{1,3}$-free, $x_1 x_{j-1} \notin E(G)$, as any additional edge on x_1, x_k, y_1 and x_{j-1} allows us to extend C. By similar arguments, $x_1 x_{j+1}$, $x_j x_k$, and $x_j x_2 \notin E(G)$. Also, no Z_2 induced by $\{x_k, x_2, x_1, y_1, x_j\}$ implies that $x_1 x_j \in E(G)$. No Z_2 induced by $\{x_k, x_2, x_1, x_j, x_{j-1}\}$ implies that $x_2 x_{j-1} \in E(G)$, and likewise $x_k x_{j+1} \in E(G)$. Therefore, the structure of edges in the graph induced by $\{x_k, x_1, x_2, x_{j-1}, x_j, x_{j+1}\}$ is completely known.

Now observe that if $y_1 x_i \in E(G)$ for some $i \neq 1, j$, then using the observations of the previous paragraphs we have that $\{x_2, x_1, x_{j-1}, x_{i-1}\}$ induces a claw. Thus, we can assume that y_1 is not adjacent to x_i for any $i \neq 1$ or j.

Let $z = x_{j+2}$. We will now examine the adjacencies of z. If $zx_j \notin E(G)$, then $zx_k \in E(G)$, for otherwise there would be a claw using the vertices $\{x_{j+1}, x_j, z, x_k\}$. However, if $zx_j \in E(G)$, then there is a Z_2 using $\{x_k, z, x_{j+1}, x_j, y_1\}$, a contradiction. Hence we can assume that $zx_j \in E(G)$. Also, $zx_{j-1} \in E(G)$, for otherwise there is a claw centered at x_j using x_{j-1}, z and y_1. The set $\{x_{j-1}, z, x_{j+1}, x_k, x_1\}$ induces a Z_2 unless z is adjacent to a least one of x_k or x_1. However, note that if z is adjacent to x_1, then z must be adjacent to x_k (and also x_2), for otherwise there would be a claw centered at x_1. Thus, we can assume that z is adjacent to x_k. This implies that $zx_1 \in E(G)$, for otherwise $\{x_{j+1}, z, x_k, x_1, y_1\}$ induces a Z_2. Hence z is adjacent to each of x_k, x_1 and x_2. This gives a contradiction, since $\{x_k, x_2, z, x_j, y_1\}$ induces a Z_2. □

We end this section with the expected result on one forbidden graph.

Theorem 24. *If A is connected and G is 2-connected then G is A-free implies G is cycle extendable if, and only if, $A = P_3$.*

6. Hamiltonian-connected graphs

In this section we examine what can be said about graphs in which any two vertices are joined by a spanning path, that is, *hamiltonian-connected* graphs. Unfortunately, we do not have a complete answer in this case. However, recently Shepherd [9] showed that a result similar to Theorem 1 holds.

Theorem 25. *If G is a 3-connected $\{K_{1,3}, N\}$-free graph, then G is hamiltonian-connected.*

We now prove a new result concerning hamiltonian-connected graphs.

Theorem 26. *Let G be a 3-connected graph. If G is $\{K_{1,3}, Z_2\}$-free, then G is hamiltonian-connected.*

Proof. Select vertices u and v and a maximal (hence, nonextendable) $u-v$ path P: $u = v_1, v_2, \ldots, v_m = v$ and assume P is not a hamiltonian path. By an extension of P we shall mean a longer $u - v$ path containing all the vertices of P. Select a vertex w not on P that is adjacent to an interior vertex of P (clearly, this is possible). Since G is 3-connected, there are three vertex disjoint paths from w to P, at least one of which is an edge. Say P_1: $w = x_1, x_2, \ldots, x_{\ell+1} = v_j$ and P_2: $w = y_1, \ldots, y_{b+1} = v_k$ $(j < k)$ are these paths. Without loss of generality we may assume these are shortest paths.

We now consider several cases.

Case 1: Suppose w has disjoint paths to two interior vertices of P, that is, $1 < j < k < m$.

We may assume that no other w to P path occurs in the interval $[v_{j+1}, v_{k-1}]$, that is, P_1 and P_2 are consecutive paths from w to interior vertices of P.

It is now apparent that at least one of $j > 2$ or $k < m - 1$ must hold, as at least one other w to P path exists and it either intersects P prior to v_j or after v_k, and at least one vertex of P must lie between these points of intersection. Thus, we assume without loss of generality that $k < m - 1$.

Since G is claw-free, the edges $v_{j-1}v_{j+1}$ and $v_{k-1}v_{k+1}$ must be in G or we could extend P. Further, all edges from $v_{j-1}, v_{j+1}, v_{k-1}$, and v_{k+1} to vertices of P_1 and P_2 are not in $E(G)$ or again we could easily extend P. Similarly, the edges v_kv_{j+1}, v_kv_{j-1}, v_jv_{k-1} and v_jv_{k+1} all allow us to extend P. If v_jv_{k+2}, $v_{j+1}v_{k+1}$ or $v_{j+1}v_{k+2}$ are in $E(G)$, then P can be extended by

$$v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_{k-1}, v_{k+1}, v_k, \dots, w, \dots, v_j, v_{k+2}, \dots, v_m$$

or

$$v_1, v_2, \dots, v_j, \dots, w, \dots, v_k, v_{k-1}, \dots, v_{j+1}, v_{k+1}, \dots, v_m$$

or

$$v_1, v_2, \dots, v_j, \dots, w, \dots, v_k, v_{k+1}, v_{k-1}, \dots, v_{j+1}, v_{k+2}, \dots, v_m,$$

respectively.

Now $\langle v_{j-1}, v_j, v_{j+1}, x_\ell, x_{\ell-1}\rangle \neq Z_2$, hence $x_{\ell-1}v_j \in E(G)$. But this contradicts the fact P_1 (and P_2) are shortest paths. From this we infer that both P_1 and P_2 are edges, that is, w is the only vertex on either P_1 or P_2 off P.

Next we note that if w is adjacent to any of $v_{j-2}, v_{j+2}, v_{k-2}$ or v_{k+2}, then P can easily be extended. For example, if $wv_{k+2} \in E(G)$, then

$$v_1, v_2, \dots, v_{k-1}, v_{k+1}, v_k, w, v_{k+2}, \dots, v_m$$

extends P.

Since $\langle v_{j-1}, v_j, v_{j+1}, w, v_k\rangle \neq Z_2$, we see that $v_jv_k \in E(G)$. Since $\langle w, v_j, v_k, v_{k+1}, v_{k+2}\rangle \neq Z_2$, we see that $v_kv_{k+2} \in E(G)$. But now, $\langle v_{k+1}, v_{k+2}, v_k, v_j, v_{j+1}\rangle = Z_2$, a contradiction.

Case 2: Suppose the paths P_1 and P_2 from w hit P at v_1, v_m and some interior vertex v_j (clearly, $2 < j < m - 1$).

Subcase 1: Suppose the path P_2 from w to v_m contains at least three vertices.

Let w_1 be the successor of w along P_2 and let v_{j-1} and v_{j+1} be the predecessor and successor of v_j along P. Since G is claw-free and P is of maximal length, we see that v_{j-1} and v_{j+1} must be adjacent. Further, both w and w_1 are nonadjacent to v_{j-1} and v_{j+1}. But then the vertices v_{j-1}, v_j, v_{j+1}, w and w_1 induce a Z_2 unless w_1 is adjacent to v_j. But now the vertices w, w_1, v_j, v_{j+1} and v_{j+2} induce a Z_2. Of the edges that could destroy the Z_2, all but v_jv_{j+2} lead to an easy extension of P. Thus, we suppose that v_jv_{j+2} is an edge of G. If $v_{j+2} \neq v_m$, then we repeat the last argument on $v_{j-1}, v_j, v_{j+1}, w_1$ and w_2 to obtain that v_jw_2 is an edge of G. But then, $\langle v_j, v_{j-1}, w, w_2\rangle$ is isomorphic to $K_{1,3}$. The edges wv_{j-1} and w_2v_{j-1} both allow us to extend P while ww_2 allows us to shorten P_2, a contradiction to our assumptions. Note that $w_2 = v_m$ is possible, but our conclusions still hold in this situation as the induced $K_{1,3}$ on $\{v_m, w_1, v_{m-1}, v_{j-1}\}$ allows us to extend P no matter which of the remaining edges are present in G. In any case, we have a contradiction. Thus, we assume that $v_{j+2} = v_m$. But then, the path

$$v_1, v_2, \ldots, v_{j-1}, v_{j+1}, v_j, w_1, \ldots, v_m$$

extends P, again producing a contradiction, and completing this subcase.

Note that a similar argument applies if the path from w to v_1 contains three or more vertices.

Subcase 2: The vertex w is adjacent to v_1, v_m and v_j.

If the number of components of $G - P$ is two or more, then each of those vertices behaves like w or we would be in a prior case. But this implies that there is a claw centered at v_1 (or v_m), contradicting our conditions.

Thus, the number of components of $G - P$ is exactly one. Call this component C. Suppose that $w' \in V(C)$. If $w'w \in E(G)$, then $w'v_j \in E(G)$ by the Subcase 1 argument of Case 2. Also, w' is adjacent to v_1, v_m and v_j on P. Hence, C must be complete and each vertex of C is adjacent to v_1, v_m and v_j. Further, we see that $|V(C)| = 1$, for otherwise the argument of Subcase 1 implies that Z_2 is an induced subgraph of G.

Hence, in this case we see that any vertex off a maximal length $u-v$ path has degree 3 with adjacencies v_1 and v_m. If the vertex had degree more than 3 it would have two internal adjacencies and we would be back in Case 1. If it was not adjacent to v_1 and v_m and not suitable for Case 1, we would be back in Subcase 1 of this case.

Now consider the paths Q_1: $v_1, w, v_j, v_{j-1}, v_{j+1}, \ldots, v_m$ and Q_2: $v_1, \ldots, v_{j-1}, v_{j+1}, v_j, w, v_m$. There is a maximal path containing Q_i, $i = 1, 2$, missing at most one vertex, which must be v_2 and v_{m-1} respectively (as any one of the interior vertices of P other than v_2 and v_{m-1} will have at least two paths to interior vertices of the maximal paths). Thus, $v_2v_m, v_{m-1}v_1 \in E(G)$. Also, no claw at v_1 implies $v_2v_{m-1} \in E(G)$. Thus, the path Q': $v_1, v_2, v_{m-1}, \ldots, v_{j+1}, v_j, w, v_m$ contains v_1, w, v_2 and v_{m-1}. Hence, the maximal path containing Q' avoids a vertex of degree 3 adjacent to v_1 and v_m. However, there is no such vertex in $G - Q'$, producing the desired contradiction. □

We conclude with a result describing some of the characteristics of the forbidden pairs for hamiltonian-connected graphs.

Theorem 27. *Let R, S be connected graphs ($R, S \neq P_3$) and let G be a 3-connected graph. If G is $\{R, S\}$-free implies G is hamiltonian-connected, then $R = K_{1,3}$ and S satisfies each of the following*:

(a) $\Delta(S) \leqslant 3$,

(b) *The longest induced path in S is at most a P_{12}*,

(c) *S contains no cycles except for C_3*,

(d) *all triangles in S are vertex disjoint*,

(e) *S is claw-free.*

(*Note*: there are only a finite number of possible graphs for S).

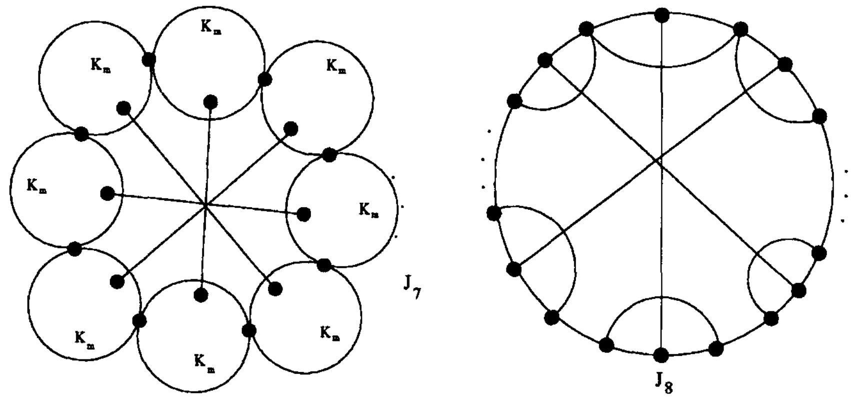

Fig. 8. 3-Connected nonhamiltonian-connected graphs.

Proof. It was shown in Theorem 15 that $R = K_{1,3}$ and we note that all graphs used in that proof are not hamiltonian connected. Hence, the same proof applies here. Consider the claw-free graphs J_7 and J_8 of Fig. 8 and well as J_3 of Fig. 5. The graph S must be an induced subgraph of each of these nonhamiltonian-connected 3-connected graphs.

Now S an induced subgraph of J_3 implies that $\Delta(S) \leqslant 3$; hence (a) follows and (d) follows as well. Then the graph J_7 implies that S contains no P_{13} and so (b) follows. The only induced cycles in J_3 (except for C_3) are C_8, C_{10} etc. On the other hand, J_8 has only C_3, C_7, C_{10}, etc. Thus, (c) follows. Clearly, S is claw-free, hence (e) follows. □

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