Degree Conditions for 2-Factors

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ABSTRACT

For any positive integer k, we investigate degree conditions implying that a graph G of order n contains a 2-factor with exactly k components (vertex disjoint cycles). In particular, we prove that for $k \leq (n/4)$, Ore's classical condition for a graph to be hamiltonian $(k = 1)$ implies that the graph contains a 2-factor with exactly k components. We also obtain a sufficient degree condition for a graph to have k vertex disjoint cycles, at least s of which are 3-cycles and the remaining are 4-cycles for any $s \leq k$. © 1997 John Wiley & Sons, Inc.

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1. INTRODUCTION

All graphs considered in this paper are simple finite graphs. Let G be a graph. The minimum degree of G will be denoted by $\delta(G)$ and we define

$$\sigma_2(G) = \min\{d(u) + d(v)\colon\ uv \notin E(G), u, v \in V(G)\},$$

that is, $\sigma_2(G)$ is the minimum value of the sum of degrees of pairs of nonadjacent vertices. A hamiltonian cycle of G is a cycle of G which contains every vertex of G. The following two results are classic theorems about hamiltonian graphs.

Theorem 1 (Dirac [5]). *Let G be a graph of order $n \geq 3$. If the minimum degree $\delta(G) \geq n/2$, then G is hamiltonian.*

Theorem 2 (Ore [9]). *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is hamiltonian.*

For any graph G, F is a 2-factor of G if and only if F is a union of vertex disjoint cycles that span $V(G)$. Both Dirac and Ore's Theorems imply that G has a 2-factor consisting of exactly one cycle. The main purpose of this paper is to investigate degree conditions sufficient to imply a graph G has a 2-factor which is a union of exactly k vertex disjoint cycles for any fixed k. The following result is obtained.

Theorem 3. *Let k be a positive integer and let G be a graph of order $n \geq 4k$. If $\sigma_2(G) \geq n$, then G has a 2-factor with exactly k vertex disjoint cycles.*

Since the complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ (n odd) does not contain a 2-factor, the degree sum condition n in the above theorem is in some sense the best possible. Also note that the complete bipartite graph $K_{m,m}$ does not contain k vertex disjoint cycles if $m < 2k$. Thus, the condition $n \geq 4k$ in the above theorem is in some sense the best possible too.

In Theorem 4 we obtain for free a natural corollary to Theorem 3. This result corresponds to a similar generalization of Dirac's Theorem.

Theorem 4. *Let k be a positive integer and let G be a graph of order $n \geq 4k$. If $\delta(G) \geq n/2$, then G has a 2-factor with exactly k vertex disjoint cycles.*

For the case $k = 2$, the following much stronger result was obtained by El-Zahar.

Theorem 5 (El-Zahar [6]). *Let G be a graph of order n and let $n_1 \geq 3$ and $n_2 \geq 3$ be two integers such that $n_1 + n_2 = n$. If the minimum degree $\delta(G) \geq \lceil (n_1/2) \rceil + \lceil (n_2/2) \rceil$, then G has two vertex disjoint cycles C_1 and C_2 of length n_1 and n_2, respectively.*

In fact, in the same paper, El-Zahar conjectured that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k (n_i \geq 3)$ with minimum degree

$$\delta(G) \geq \left\lceil \frac{n_1}{2} \right\rceil + \left\lceil \frac{n_2}{2} \right\rceil + \cdots + \left\lceil \frac{n_k}{2} \right\rceil,$$

then contains k vertex disjoint cycles of length $n_1, n_2, \ldots, n_k$, respectively.

If El-Zahar's conjecture is true, then it follows that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k (n_i \geq 3)$ with $\delta(G) \geq (2n/3)$, then G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ of lengths $n_1, n_2, \ldots, n_k$, respectively. Sauer and Spencer [10] and independently Catlin [3] proved the more general result that every graph of order n with minimum degree exceeding $(1-(1/2\Delta))n-1$ contains every given subgraph of order n and maximum degree Δ.

This implies that El-Zahar's conjecture holds with $\delta(G) \geq \frac{3n}{4} - 1$. For $\Delta = 2$, Catlin [3] established a stronger result.

Theorem 6. *If G is a graph of order $n = n_1 + n_2 + \cdots + n_k (n_i \geq 3)$ with $\delta(G) \geq \frac{2n}{3} + O(n^{1/3})$, then G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ of lengths $n_1, n_2, \ldots, n_k$, respectively.*

Recently Aigner and Brandt [1] obtained the following improvement which gives the best possible degree bound.

Theorem 7. *If G is a graph of order $n \geq n_1 + n_2 + \cdots + n_k$ $(n_i \geq 3)$ with $\delta(G) \geq (2n-1)/3$ then G contains k vertex disjoint cycles of lengths $n_1, n_2, \ldots, n_k$, respectively.*

The following result gives a sufficient condition for a graph to have k disjoint cycles which are either triangles or 4-cycles.

Theorem 8. *Let $s \leq k$ be two nonnegative integers and let G be a graph of order $n \geq 3s + 4(k-s)$. If $\sigma_2(G) \geq (n + s/2)$, then G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that*

$$\begin{aligned} |V(C_i)| &= 3 \text{ for } 1 \leq i \leq s \\ |V(C_i)| &\leq 4 \text{ for } s+1 \leq i \leq k \end{aligned}$$

that is, the first s cycles are triangles and the others are either triangles or 4-cycles.

The proof of Theorem 3 will be immediate from two lemmas. Theorem 4 follows directly from Theorem 3. There are also several lemmas needed for the proof of Theorem 8. For this reason, we will place the proofs of Theorems 3 and Theorem 8 in separate sections.

Generally, we will follow [8] for notation and terminology. Let G be a graph and let V_1 and V_2 be two subsets of vertices of G. The subgraph of G induced by the vertex set V_1 will be denoted by $\langle V_1 \rangle$. Further, $E(V_1, V_2)$ will denote the set of edges in G with one endvertex in V_1 and the other in V_2. Let H and K be two subgraphs of G. We use $V(H)$ and $E(H)$ denote the vertex set and the edge set of H, respectively, while $e(H)$ denotes the number of edges in H. For simplicity, let $E(H, K)$ stand for $E(V(H), V(K))$ and $e(H, K)$ for $|E(H, G)|$.

The following two results will be heavily used in our proofs.

Theorem 9 (Corradi-Hajnal [4]). *Let k be a positive integer and G a graph of order n. If $n \geq 3k$ and the minimum degree $\delta(G) \geq 2k$, then G contains k vertex disjoint cycles.*

Theorem 10 (Justesen [7]). *If G is a graph of order $n \geq 3k$ such that*

$$d(u) + d(v) \geq 4k$$

for all pairs u and v of nonadjacent vertices, then G contains k vertex disjoint cycles.

2. THEOREM 3

As stated earlier, the proof of Theorem 3 will follow immediately from the next two Lemmas.

Lemma 1. *Let G be a graph of order $n \geq 4k$. If $\sigma_2(G) \geq n$ then G contains a 2-factor with at least k cycles.*

Proof. Let L be the set of vertices of G with degree less than $n/2$. Observe that the vertices of L form a complete subgraph of G.

By Justesen's Theorem G contains k vertex disjoint cycles. Choose a subgraph H of at least k disjoint cycles which cover the largest number of vertices of L, and subject to this requirement, the largest number of vertices altogether. Suppose H is not a spanning subgraph. Certainly $G - V(H)$ is a forest, otherwise we could have added a cycle to H, contradicting our choice of H. Let v be a vertex of degree at most one in $G - V(H)$. Further, we assume v has the smallest degree in G subject to the condition of degree at most one in $G - V(H)$. Clearly, $N(v)$ does not contain two consecutive vertices of any cycle in H. If $d(v) \geq n/2$, it is readily seen that $G - V(H) = K_2, d(v) = n/2$ and v is adjacent to every other vertex of each cycles. Let w be the other vertex in $G - V(H)$. $d(w) \geq n/2$ by the choice of v, hence $d(w) = n/2$ and w is adjacent to every other vertex of each cycle in H. Then, it is not difficult to see that we can add vertices v and w to a cycle of H to form a new cycle, a contradiction. Thus, suppose $v \in L$. If v has two neighbors in a cycle C of H, then we can add v to C by possibly removing a segment of C, no vertex of which is in L. In this way we obtain a subgraph H' with the same number of cycles as H and which covers more vertices of L. This contradicts our choice of H. Hence, v has at most one neighbor in every cycle of H.

Since $\sigma_2(G) \geq n$ implies $d(v) \geq \delta(G) \geq 2$, there must be a cycle C in which v has a neighbor w. Let xy be an edge of C which is not incident with w. As neither x nor y are adjacent to v each of them can have at most $d(v) - 1$ non-neighbors or our degree sum condition would fail to hold. One non-neighbor of x (and y) is v, and if v has a neighbor in $G - V(H)$, this is also a non-neighbor of x and y because of our choice of H. If v has a neighbor in $r \geq 1$ cycles of H, then both x and y have at most $r - 2$ non-neighbors in $H - V(C)$, or again our condition on σ_2 would be violated. In particular then there must be a cycle C' in which v has a neighbor and all but at most one of the possible edges are joining xy to C'. It is easy to check that this yields two cycles which contain the following: v, the neighbors of v on C and C', x and y, again contradicting the choice of H. ∎

Lemma 2. *Let G be a graph of order n containing a 2-factor with $k \geq 2$ cycles. If $\sigma_2(G) \geq n$ then G contains a 2-factor with $k - 1$ cycles.*

Proof. Among the 2-factors of G with exactly k cycles let H be one for which a smallest component is as small as possible. Let C be such a shortest cycle of H. Again let L be the set of vertices of G with degree smaller than $n/2$. It is easily seen that the degree sum condition implies that G is connected. Thus, there must be an edge vw joining C to another component C'. Since $k \geq 2$ we have $|V(C)| \leq n/2$ and therefore any vertex of $V(C) \setminus L$ must have a neighbor in $G - V(C)$. Choose $v \in V(C) \setminus L$ unless all vertices of C are in L and choose $w \in V(C')L$ if possible. Fix an orientation of C and C' respectively. Now consider the two predecessors v^-, w^- and the two successors v^+, w^+ of v and w on C and C', respectively. Since both pairs are ends of a hamiltonian path of the subgraph G' induced by $V(C) \cup V(C')$, either this subgraph contains a hamiltonian cycle, in which case we have found the desired 2-factor with exactly $k - 1$ components, or

$$d_{G'}(v^+) + d_{G'}(v^-) + d_{G'}(w^+) + d_{G'}(w^-) \leq 2(|V(G')| - 1).$$

Since

$$\begin{aligned} d_{G-G'}(v^+) + d_{G-G'}(v^-) + d_{G-G'}(w^+) + d_{G-G'}(w^-) &\geq 2\alpha_2(G) - 2(|V(G')| - 1) \\ &\geq 2(|V(G) - V(G')| + 1) \end{aligned}$$

we obtain that one of the following statements must hold:

(i) $d_{G-G'}(v^+) + d_{G-G'}(v^-) > |V(G) - V(G')|$, or
(ii) $d_{G-G'}(w^+) + d_{G-G'}(w^-) > |V(G) - V(G')|$.

If (i) holds then we obtain edges v^+x, v^-y incident to an edge xy of another cycle $C'' \neq C, C'$. Hence, we obtain $k-1$ disjoint cycles covering all vertices of G except v. If v is adjacent to both ends of an edge in one of the $k-1$ cycles we are done. If $v \in L$ this follows from the fact that the edge v^+v^{++} belongs to one of the cycles and $v^+, v^{++} \in L$ (or our choice of v is violated), and if $v \notin L$ this follows from $d(v) \geq n/2$.

Thus, we may assume that (i) does not hold. Hence (ii) holds and by a similar reasoning we are done if $w \notin L$ or if w is adjacent to two consecutive vertices of C or C'. Hence, we assume that $w \in L$ and w is not adjacent to two consecutive vertices of C or C'. In this case neither v^+ nor v^- are adjacent to w. In particular, $v^+ \notin L$ and $v^- \notin L$. Since all neighbors of v^+ and v^- on C' are in L (by the initial choice of the edge vw) and are all neighbors of w, both v^+ and v^- have at most $|C'|/2 - 1$ neighbors on C'. Since $v^+w, v^-w \notin E(G)$ we obtain

$$\begin{aligned} d_C(v^+) + d_C(v^-) &\geq 2(\sigma_2(G) - d(w)) - |V(C')| + 2 - |V(G) - V(G')| \\ &\geq n + 1 - |V(C')| + 2 - |V(G) - V(G')| \\ &= |V(C)| + 3. \end{aligned}$$

As $|V(C)| \geq 3$, one of the vertices, say v^+, has at least three neighbors in the subgraph induced by C, in particular, a neighbor u different from v, v^{++}. We obtain a new cycle $v^+, v^{++}, \ldots, u, v^+$ and a hamiltonian path P from u^+ to w^+ in the remaining part of G'. If the remaining part of G' is hamiltonian this contradicts the choice of H, because the resulting 2-factor has a shorter cycle than C. Otherwise u^+ and w^+ are not adjacent and $d_{G-V(P)}(u^+) + d_{G-V(P)}(w^+) > n - |V(P)|$, so that we obtain the desired 2-factor with $k-1$ cycles, which completes the proof. ∎

The proof of Theorem 3 is now an immediate consequence of Lemmas 1 and 2.

3. THEOREM 8

3.1. Lemmas

Lemma 3. *Let C_1 and C_2 be two vertex disjoint cycles of G and let $m_1 = |V(C_1)|$ and $m_2 = |V(C_2)|$. If $m_1 \geq m_2 \geq 4$ and $e(C_1, C_2) \geq (m_1m_2 + 1)/2$, then the subgraph induced by $V(C_1) \cup V(C_2)$ contains two vertex disjoint cycles C_1^* and C_2^* such that C_1^* is a triangle.*

Proof. Suppose, to the contrary, there do not exist two such cycles. Let $C_1 = x_1x_2 \cdots x_{m_1}x_1$ and $C_2 = y_1y_2 \cdots y_{m_2}y_1$. Since $e(C_1, C_2) \geq (m_1m_2 + 1)/2$, there is a vertex in C_2, say y_1, such that $|N(y_1) \cap V(C_1)| \geq (m_1 + 1)/2$. This implies that $N(y_1)$ contains two consecutive vertices of C_1, say x_2 and x_3.

Suppose $e(\{x_2, x_3\}, C_2) \geq 2m_2 - 1$, that is, G contains every possible edge, with at most one exception, from $\{x_2, x_3\}$ to C_2. Without loss of generality, we assume that the possible edge $x_3y_{m_2}$ is missed. Then, triangles $x_2y_1y_{m_2}x_2$ and $x_3y_2y_3x_3$ give two desired cycles, a contradiction. Thus $e(\{x_2, x_3\}, C_2) \leq 2m_2 - 2$.

Suppose there is another vertex y_i $(i \neq 1)$ that is adjacent to both x_2 and x_3. Since the desired cycles C_1^* and C_2^* do not exist, we have either

(1) $E(C_2, C_1 - \{x_2, x_3\}) \subseteq \{y_kx_j\}$ for some k and j or;
(2) Each of y_1 and y_i is adjacent to exact one vertex in $C_1 - \{x_2, x_3\}$.

In either case, we have $e(C_2, C_1 - \{x_2, x_3\}) \leq 2$. Hence $e(C_1, C_2) \leq (2m_2 - 2) + 2 = 2m_2$. Thus, $(m_1m_2 + 1)/2 \leq 2m_2$, which implies that $m_1 \leq 3$, a contradiction.

Thus, $|N(y_i) \cap \{x_2, x_3\}| \leq 1$ for each $2 \leq i \leq m_2$. Since the vertices y_1, x_2, x_3 form a triangle and the desired cycles C_1^* and C_2^* do not exist, it follows that $e(C_1 - \{x_2, x_3\}, C_2 - \{y_1\}) \leq 1$. Thus,

$$\begin{aligned} e(C_1, C_2) &= e(\{x_2, x_3\}, C_2 - y_1) + e(C_1 - \{x_2, x_3\}, C_2 - y_1) + e(C_1, y_1) \\ &\leq (m_2 - 1) + 1 + m_1 = m_1 + m_2. \end{aligned}$$

Hence we have $(m_1m_2 + 1)/2 \leq m_1 + m_2$, which contradicts the fact $m_1 \geq m_2 \geq 4$. ∎

Lemma 4. *Let C_1 and C_2 be two vertex disjoint cycles with lengths $m_1 = |V(C_1)| = 3$ and $m_2 = |V(C_2)| \geq 5$. If $e(C_1, C_2) \geq (3m_2 + 1/2)$, then $\langle V(C_1 \cup C_2)\rangle$ contains a triangle C_1^* and a cycle C_2^* such that C_1^* and C_2^* are vertex disjoint and $|V(C_2^*)| < m_2$.*

Proof. Let $C_1 = x_1x_2x_3x_1$ and $C_2 = y_1y_2 \cdots y_{m_2}y_1$. Suppose, to the contrary, there are no such two cycles C_1^* and C_2^*. Since $e(C_1, C_2) \geq (3m_2 + 1)/2$, there is a vertex, say x_1, such that

$$|N(x_1) \cap V(C_2)| \geq \frac{m_2 + 1}{2} \geq 3.$$

If $N(x_2) \cap N(x_3) \cap V(C_2) \neq \emptyset$, without loss of generality, let $y_2 \in N(x_2) \cap N(x_3)$. Since G contains no such C_1^* and C_2^* and $|N(x_1) \cap V(C_2)| \geq 3$, we have $N(x_1) \cap V(C_2) = \{y_1, y_2, y_3\}$. From the fact $|N(x_1) \cap V(C_2)| \geq (m_2 + 1)/2$, we obtain $m_2 = 5$. Since $e(C_1, C_2) \geq (3 \times 5 + 1)/2 = 8$, we have either $|N(x_2) \cap V(C_2)| \geq 3$ or $|N(x_3) \cap V(C_2)| \geq 3$. Without loss of generality, we assume that $|N(x_2) \cap V(C_2)| \geq 3$. Since $y_2 \in N(x_1) \cap N(x_3)$, in the same manner as the above, we see that $N(x_2) \cap V(C_2) = \{y_1, y_2, y_3\}$.

If $x_3y_1 \in E(G)$, then $x_3y_1y_2x_3$ and $y_3x_1x_2y_3$ are desired triangles, which is a contradiction. Thus, $x_3y_1 \notin E(G)$ and similarly $x_3y_3 \notin E(G)$. Since $e(C_1, C_2) \geq 8, N(x_3) \cap \{y_4, y_5\} \neq \emptyset$. Without loss of generality, assume that $x_3y_4 \in E(G)$. Then there are two cycles $C_1^* = y_1x_1x_2y_1$ and $C_2^* = x_3y_2y_3y_4x_3$, a contradiction.

Therefore, $N(x_2) \cap N(x_3) \cap V(C_2) = \emptyset$. Suppose that $|N(x_1) \cap V(C_2)| \geq m_2 - 1$, where $N(x_1) \supseteq \{y_1, y_2, \ldots, y_{m_2-1}\}$. In this case, we claim $e(\{x_2, x_3\}, C_2) \leq 2$. Otherwise, assume $e(\{x_2, x_3\}, C_2) \geq 3$. Without loss of generality, we assume that $|N(x_2) \cap V(C_2)| \geq 2$. Since there does not exist a pair of cycles with the desired properties, we have $|N(x_2) \cap V(C_2)| = 2$ and $m_2 = 5$ and we have either

$$\begin{aligned} N(x_2) \cap V(C_2) &= \{y_2, y_4\}, \\ N(x_2) \cap V(C_2) &= \{y_1, y_3\}. \end{aligned}$$

Without loss of generality, we assume that $N(x_2) \cap V(C_2) = \{y_2, y_4\}$. Then, it is readily seen that $N(x_3) \cap V(C_2) = \emptyset$, which gives $e(\{x_2, x_3\}, V(C_2)) \leq 2$, a contradiction. Thus, we have $|N(x_1) \cap V(C_2)| \leq m_2 - 2$.

Since $|N(x_1) \cap V(C_2)| \geq (m_2+1)/2$, $N(x_1)$ contains two consecutive vertices of C_2, say y_2 and y_3. Since $N(x_1) \supseteq \{y_2, y_3\}$, we have $e(\{x_2, x_3\}, C_2 - \{y_2, y_3\}) \leq 2$. Further, equality holds only if either

$$E(\{x_2, x_3\}, C_2 - \{y_2, y_3\}) = \{x_2y_1, x_3y_4\},$$

or

$$E(\{x_2, x_3\}, C_2 - \{y_2, y_3\}) = \{x_3y_1, x_2y_4\}.$$

Since $N(x_2) \cap N(x_3) \cap V(C_2) = \emptyset$, it follows that $e(\{x_2, x_3\}, C_2) \leq 4$. Thus

$$\frac{3m_2+1}{2} \leq e(C_1, C_2) \leq (m_2 - 2) + 4 \leq m_2 + 2,$$

which implies that $m_2 \leq 3$, a contradiction. ∎

Lemma 5. *Let $C_1 = x_1x_2x_3x_1$ be a triangle and $C_2 = y_1y_2y_3y_4y_1$ be a 4-cycle in a graph G such that C_1 and C_2 are vertex disjoint and $e(C_1, C_2) \geq 9$. Then the induced subgraph $\langle V(C_1 \cup C_2)\rangle$ contains two vertex disjoint triangles C_1^* and C_2^*.*

Proof. To the contrary, we assume that $\langle V(C_1 \cup C_2)\rangle$ contains no two vertex disjoint triangles. Without loss of generality, we also assume that

$$|N(x_1) \cap V(C_2)| \geq |N(x_2) \cap V(C_2)| \geq |N(x_3) \cap V(C_2)|.$$

Clearly, $|N(x_1) \cap V(C_2)| \geq 3$. Since $e(C_1, C_2) \geq 9$ and $|N(x_1) \cap V(C_2)| \leq |V(C_2)| = 4$, we have $e(\{x_2, x_3\}, C_2) \geq 5$. By the Pigeonhole Principle, $N(x_2) \cap N(x_3) \cap V(C_2) \neq \emptyset$. Assume that $y_2 \in N(x_2) \cap N(x_3)$. If $x_1y_4 \in E(G)$, then $x_1y_3 \notin E(G)$ and $x_1y_1 \notin E(G)$, which implies that $|N(x_1) \cap V(C_2)| \leq 2$, a contradiction. Thus $x_1y_4 \notin E(G)$. Since $|N(x_1) \cap V(C_2)| \geq 3$, we have $N(x_1) \cap V(C_2) = \{y_1, y_2, y_3\}$. Since $e(C_1, C_2) \geq 9$, we have

$$|N(x_1) \cap V(C_2)| = |N(x_2) \cap V(C_2)| = |N(x_3) \cap V(C_2)| = 3.$$

Further, in the same manner as we did for x_1, we can show that,

$$N(x_i) \cap V(C_2) = \{y_1, y_2, y_3\} \quad \text{for } i = 1, 2, 3.$$

Clearly, there are two triangles $y_1x_1x_3y_1$ and $x_2y_2y_3x_2$, a contradiction. ∎

3.2. Proof of Theorem 8

Note that $n \geq 3s + 4(k-s)$ implies $n + s \geq 4k$. By Theorem 10, we know that G contains k vertex disjoint cycles. Let $C_1, C_2, \ldots, C_k$ be k vertex disjoint cycles such that

(1) the number of triangles in $\{C_1, C_2, \ldots, C_k\}$ is as large as possible;
(2) subject to condition 1, $\sum_{1\leq i\leq k} |V(C_i)|$ is minimum.

Let $m_i = |V(C_i)|$. Without loss of generality, assume that

$$m_1 \leq m_2 \leq \cdots \leq m_k.$$

Claim 1. *If $m_k \geq 4$, then*

$$e\left(C_k, \bigcup_{1\leq i\leq k-1} C_i\right) > \frac{m_k \times (\sum_{1\leq i\leq k-1} m_i + s)}{2}.$$

Proof. By the minimality of $\sum_{1\leq i\leq k} m_i$, we know that C_k is an induced cycle. Then $e(\langle C_k\rangle) = m_k$ and $\sum_{x\in V(C_k)} d(x) \geq (m_k \times (n+s)/2)$.

For each vertex $v \notin \cup_{1\leq i\leq k} V(C_i)$, we have $|N(v) \cap V(C_k)| \leq 2 < (m_k/2)$. Thus,

$$\begin{aligned} e\left(C_k, \bigcup_{1\leq i\leq k-1} C_i\right) &> \frac{m_k(n+s)}{2} - \frac{m_k}{2}\left(n - \sum_{1\leq i\leq k-1} m_i\right) \\ &= \frac{m_k \times (\sum_{1\leq i\leq k-1} m_i + s)}{2}. \end{aligned}$$

∎

Claim 2. *The inequality $m_k \leq 4$ holds.*

Proof. Suppose, to the contrary, that $m_k \geq 5$. By Claim 1, we see there is a cycle C_i such that $e(C_i, C_k) \geq (m_i m_k + 1)/2$.

If $m_i \geq 4$, by Lemma 3, $\langle V(C_i \cup C_k)\rangle$ contains two vertex disjoint cycles C_i^* and C_k^* such that $V(C_i^*)$ is a triangle, which contradicts condition (1) of the choice of $C_1, C_2, \ldots, C_k$.

Thus $m_i = 3$. By Lemma 4, $\langle V(C_i \cup C_k)\rangle$ contains two vertex disjoint cycles C_i^* and C_k^* such that C_i^* is a triangle and $|V(C_k^*)| < |V(C_k)|$, a contradiction. ∎

Assume that $m_1 = m_2 = \cdots = m_t = 3$ and $m_{t+1} = \cdots = m_k = 4$. To finish the proof, we only need to show that $t \geq s$. Suppose, to the contrary, $t < s$. If there is a cycle C_i with $t+1 \leq i \leq k-1$ such that $e(C_i, C_k) \geq \lceil (4 \times 4 + 1)/2 \rceil = 9$. Then by Lemma 3, $G(V(C_i \cup C_k))$ contains a triangle and a cycle, which contradicts to our choice of $C_1, \ldots, C_k$. Thus,

$$e\left(C_k, \bigcup_{t+1\leq i\leq k-1} C_i\right) \leq 8(k-t-1).$$

From Claim 1, we have

$$\begin{aligned} e\left(C_k, \bigcup_{1\leq i\leq k-1} C_i\right) &\geq \frac{m_k(\sum_{1\leq i\leq k-1} m_i + s)}{2} \\ &= \frac{4(3t + 4(k-1-t) + s)}{2} \\ &= 2(3t+s) + 8(k-t-1). \end{aligned}$$

Combining the above two inequalities, we obtain the following

$$e\left(C_k, \bigcup_{1\leq i\leq t} C_i\right) \geq 2(3t+s) > 8t.$$

Since $s > t$, there is a cycle C_i with $1 \leq i \leq t$ such that $e(C_i, C_k) \geq 9$ by the Pigeonhole Principle. By Lemma 5, $G(V(C_i, C_k))$ contains two vertex disjoint triangles, a contradiction. ∎

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