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Graph spectra

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Abstract

The *k-spectrum* $s_k(G)$ of a graph G is the set of all positive integers that occur as the size of an induced k-vertex subgraph of G. In this paper we determine the minimum order and size of a graph G with $s_k(G) = \{0, 1, \ldots, \binom{k}{2}\}$ and consider the more general question of describing those sets $S \subseteq \{0, 1, \ldots, \binom{k}{2}\}$ such that $S = s_k(G)$ for some graph G.

1. Introduction

In [2] it was shown that for every positive integer k there is an integer $N(k)$ such that every connected graph of order at least $N(k)$ contains either a complete graph of order k or an induced tree of order k. On the other hand, by Ramsey's theorem every graph of sufficiently large order contains either a complete graph of order k or an independent set of k vertices. It follows, then, that every connected graph of sufficiently large order contains either an induced subgraph of order k and size $\binom{k}{2}$ or two induced subgraphs of order k, one of size 0 and one of size $k-1$. In this paper we consider the set of sizes of *all* induced subgraphs of a fixed order k in a graph G. In particular, we define the *k-spectrum* $s_k(G)$ of a graph G by

$$s_k(G) = \{\, j \mid G \text{ contains an induced subgraph of order } k \text{ and size } j \,\}.$$

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Thus $s_k(G) \subseteq \{0, 1, \dots, \binom{k}{2}\}$. Furthermore, from the remarks above we can say that if G is a connected graph of sufficiently larger order then either $\binom{k}{2} \in s_k(G)$ or $0, k-1 \in s_k(G)$. In Section 2 we establish two extremal results regarding graphs G for which $s_k(G) = \{0, 1, \dots, \binom{k}{2}\}$. In Section 3 we consider the more general problem of describing those sets $S \subseteq \{0, 1, \dots, \binom{k}{2}\}$ such that $S = s_k(G)$ for some graph G.

2. Extremal results

If $s_k(G) = \{0, 1, \dots, \binom{k}{2}\}$ we will say that the graph G has a *complete k-spectrum.* In Theorem 1 we determine the minimum order among all graphs with complete k-spectra.

Theorem 1. *The minimum number of vertices in a graph with a complete k-spectrum is* $2k-1$.

Proof. If G is any graph with a complete k-spectrum then $0, \binom{k}{2} \in s_k(G)$. Thus G contains K_k and $\bar{K}_k$ as induced subgraphs. Since these subgraphs can have at most one vertex in common it follows that G has order at least $2k-1$.

We complete the proof of describing a graph $G(k)$ of order $2k-1$ that has a complete k-spectrum. Let $V(G(k)) = \{w_1, w_2, \dots, w_k, x_1, x_2, \dots, x_{k-1}\}$, where $\langle\{w_1, w_2, \dots, w_k\}\rangle$ is a complete subgraph of $G(k)$ and $\langle\{x_1, x_2, \dots, x_{k-1}\}\rangle$ is an empty subgraph of $G(k)$. Furthermore, $x_i w_j \in E(G(k))$ if and only if $j > i$. Then $G(k)$ has order $2k-1$ and clearly $0, \binom{k}{2} \in s_k(G(k))$. In order to verify that $G(k)$ has a complete k-spectrum, let t be any integer satisfying $0 < t < \binom{k}{2}$. We show that $G(k)$ contains an induced k-vertex subgraph of size t. Let ℓ be the largest integer for which $\binom{\ell}{2} \leqslant t$ and let $r = t - \binom{\ell}{2}$. Note that $0 \leqslant r \leqslant \ell - 1$. Then

$$\langle\{w_1, w_2, \dots, w_\ell, x_{\ell-r}, x_{\ell+1}, x_{\ell+2}, \dots, x_{k-1}\}\rangle$$

has order k and size $\binom{\ell}{2} + r = t$. $\square$

The graph with a complete k-spectrum constructed in Theorem 1 has size

$$\binom{k}{2} + (k-1) + (k-2) + \cdots + 1 = 2\binom{k}{2}.$$

It is reasonable to ask if there is a graph with a complete k-spectrum and size less than $2\binom{k}{2}$. In Theorem 2 we determine the minimum size of a graph with a complete k-spectrum. We will write $H \prec G$ to mean that H is an *induced subgraph* of G.

Theorem 2. *For k sufficiently large, the minimum number of edges in a graph with a complete k-spectrum is*

$$\binom{k}{2} + k \log k - O(k \log \log k).$$

Proof. We begin by constructing a graph $S(k)$ that has a complete k-spectrum and size

$$\binom{k}{2} + k\lceil \log k \rceil + \binom{\lceil \log k \rceil}{2} - (2^{\lceil \log k \rceil} - 1).$$

Let $V(S(k)) = \{w_1, w_2, \ldots, w_k, x_1, x_2, \ldots, x_{\lceil \log k \rceil}, y_1, y_2, \ldots, y_k\}$, where $deg\, y_i = 0$ $(1 \leqslant i \leqslant k)$. Furthermore, $\langle\{w_1, w_2, \ldots, w_k\}\rangle$ and $\langle\{x_1, x_2, \ldots, x_{\lceil \log k \rceil}\}\rangle$ are complete subgraphs of $S(k)$. Finally, $x_i w_j \in \mathrm{E}(S(k))$ if and only if $j > 2^{i-1}$. Then $S(k)$ has size

$$\binom{k}{2} + \binom{\lceil \log k \rceil}{2} + k\lceil \log k \rceil - (2^0 + 2^1 + \cdots + 2^{\lceil \log k \rceil - 1})$$
$$= \binom{k}{2} + k\lceil \log k \rceil + \binom{\lceil \log k \rceil}{2} - (2^{\lceil \log k \rceil} - 1).$$

We show, by induction on k, that $S(k)$ has a complete k-spectrum. Certainly $S(2)$ has a complete 2-spectrum. Assume, for some $k \geqslant 3$, that $S(k-1)$ has a complete $(k-1)$-spectrum, and consider $S(k)$. Since $S(k-1) \prec S(k)$ it follows that $S(k)$ contains induced $(k-1)$-vertex subgraphs having sizes $0, 1, \ldots, \binom{k-1}{2}$ and containing at most $k-1$ of the isolated vertices of $S(k)$. Thus $S(k)$ contains induced k-vertex subgraphs of sizes $0, 1, \ldots, \binom{k-1}{2}$. It remains to show that $S(k)$ contains k-vertex subgraphs of size $\binom{k}{2} - i$ for $0 \leqslant i \leqslant k-2$. Since $K_k \prec S(k)$ we may assume $i \geqslant 1$.

For fixed i satisfying $1 \leqslant i \leqslant k-2$, let

$$i = b_1 2^0 + b_2 2^1 + \cdots + b_{\lceil \log k \rceil} 2^{\lceil \log k \rceil - 1}$$

be the binary expansion of i, let $J = \{j \mid b_j = 1\}$ and let $m = \max\{j \mid j \in J\}$. Then $|J| \leqslant \lceil \log k \rceil \leqslant k$. Let $V(i) = \{x_j \mid j \in J\} \cup \{w_1, w_2, \ldots, w_{k-|J|}\}$. Then $|E(V(i))| = \binom{k}{2} - i$ provided $k - |J| \geqslant 2^{m-1}$. If $|J| = 1$ then $k - |J| = k - 1 \geqslant 2^{m-1}$. If, on the other hand, $|J| \geqslant 2$ then

$$k \geqslant i + 2 \geqslant 2^0 + 2^1 + \cdots + 2^{|J|-2} + 2^{m-1} + 2$$
$$= 2^{|J|-1} - 1 + 2^{m-1} + 2 \geqslant 2^{m-1} + |J|.$$

We complete the proof by showing that for k sufficiently large, every graph with a complete k-spectrum has at least $\binom{k}{2} + k \log k - 2k \log\log k$ edges. Let G be such a graph with $S \subseteq V(G)$ such that $|S| = k$ and $\langle S \rangle$ is complete.

Assume first that there exists $S' \neq S$ such that $|S'| = k$ and $|E(\langle S' \rangle)| \geqslant \binom{k}{2} - k$ and $|S' - S| = \ell > \log k$. Then

$$|E(G)| \geqslant \binom{k}{2} + \binom{\ell}{2} + \ell(k - \ell) - k.$$

The function $f(\ell) = \binom{\ell}{2} + \ell k - \ell^2 + k - \lceil k \log k \rceil$ is nonnegative at $\ell = \lceil \log k \rceil$, and it is an increasing function of ℓ for $\log k < \ell \leqslant k - 1$. Therefore,

$$|E(G)| \geqslant \binom{k}{2} + k \log k - 2k,$$

for k sufficiently large. Thus we may assume that if $S \neq S'$ and $|S'| = k$ and $|E(\langle S' \rangle)| \geqslant \binom{k}{2} - k$ then $|S' - S| \leqslant \log k$.

Let S_1 be the vertex set of an induced k-vertex subgraph of G of size $\binom{k}{2} - 1$. Then $1 \leqslant |S_1 - S| \leqslant \log k$. Let $v_1 \in S_1 - S$. Since $|E(\langle S_1 \rangle)| = \binom{k}{2} - 1$ it follows that $deg_{\langle S_1 \rangle} v_1 \geqslant (k-1) - 1 = k - 2$. Thus v_1 is adjacent to at least $k - 2 - (\log k - 1) = k - (\log k + 1)$ vertices of S. Let S_2 be the vertex set of an induced k-vertex subgraph of G of size $\binom{k}{2} - (\log k + 2)$. Since every induced k-vertex subgraph of $\langle S \cup \{v_1\} \rangle$ contains at least $\binom{k}{2} - (\log k + 1)$ edges, it follows that $|S_2 - S - \{v_1\}| \geqslant 1$. Furthermore, since $\log k + 2 \leqslant k$ for k sufficiently large, $|S_2 - S| \leqslant \log k$. Let $v_2 \in S_2 - S - \{v_1\}$. Since $|E(\langle S_2 \rangle)| = \binom{k}{2} - (\log k + 2)$, it follows that $deg_{\langle S_2 \rangle} v_2 \geqslant (k-1) - (\log k + 2) = k - (\log k + 3)$. Thus v_2 is adjacent to at least $k - (\log k + 3) - (\log k - 1) = k - (2 \log k + 2)$ vertices of S. In general, suppose that for some $\ell \leqslant \lfloor \log(k/\log k) \rfloor$ we have selected distinct vertices $v_1, v_2, \ldots, v_{\ell-1} \notin S$ such that for $1 \leqslant i \leqslant \ell - 1$, the vertex v_i is adjacent to at least $k - (2^{i-1} \log k + 2^i - i)$ vertices of S. Observe that for $i \leqslant \ell$ we have

$$2^{i-1} \log k + 2^i - i \leqslant k/2 + k/\log k \leqslant k,$$

for k sufficiently large. Every induced k-vertex subgraph of $\langle S \cup \{v_1, v_2, \ldots, v_{\ell-1}\} \rangle$ contains at least

$$\binom{k}{2} - \left(\sum_{i=1}^{\ell-1} (2^{i-1} \log k + 2^i - i) + \binom{\ell - 1}{2} \right)$$

edges, i.e., at least

$$\binom{k}{2} - ((2^{\ell-1} - 1) \log k + 2^{\ell} - \ell - 1)$$

edges. Let S_ℓ be the vertex set of an induced k-vertex subgraph of G of size

$$\binom{k}{2} - ((2^{\ell-1} - 1) \log k + 2^{\ell} - \ell).$$

Then $|S_\ell - S - \{v_1, v_2, \ldots, v_{\ell-1}\}| \geqslant 1$. Let $v_\ell \in S_\ell - S - \{v_1, v_2, \ldots, v_{\ell-1}\}$. Then

$$deg_{\langle S_\ell \rangle} v_\ell \geqslant (k-1) - ((2^{\ell-1} - 1) \log k + 2^{\ell} - \ell).$$

Furthermore, $|S_\ell - S| \leqslant \log k$ and so v_ℓ is adjacent to at least

$$k - ((2^{\ell-1} - 1) \log k + 2^{\ell} - \ell + 1) - (\log k - 1)$$

$$= k - (2^{\ell-1} \log k + 2^{\ell} - \ell)$$

vertices of S. Thus there exist distinct vertices $v_1, v_2, \ldots, v_\ell \notin S$, where $\ell = \lfloor \log(k/\log k) \rfloor$, such that v_i is adjacent to at least $k - (2^{i-1} \log k + 2^i - i)$ vertices of S for $i = 1, 2, \ldots, \ell$. Therefore,

$$|E(G)| \geqslant \binom{k}{2} + \sum_{i=1}^{\ell} (k - 2^{i-1} \log k - 2^i + i)$$

$$\geqslant \binom{k}{2} + \sum_{i=1}^{\ell} (k - 2^i \log k)$$

$$\geqslant \binom{k}{2} + k \log(k/\log k) - (2 \log k)(k/\log k - 1)$$

$$\geqslant \binom{k}{2} + k \log k - 2k \log \log k. \qquad \square$$

In [3] Erdős and Spencer defined the *size spectrum* $s(G)$ of a graph G by

$$s(G) = \{j \mid G \text{ has an induced subgraph of size } j\}.$$

Thus $s(G) = \bigcup_{k=1}^{|V(G)|} s_k(G)$. They showed that if M_n is the largest cardinality among the size spectra of graphs of order n, then $M_n \leqslant \binom{n}{2} - O(n \log \log n)$. It follows from the construction of the graph $S(k)$ in Theorem 2 (by considering $n = \log(k + k)$ that $M_n \geqslant \binom{n}{2} - n \log n$.

Corollary 1. *Let M_n be the largest cardinality among the size spectra of graphs of order n. Then*

$$\binom{n}{2} - n \log n \leqslant M_n \leqslant \binom{n}{2} - O(n \log \log n).$$

3. Properties of k-spectra of graphs

For a fixed integer k, every graph of sufficiently large order n has at least one of 0 and $\binom{k}{2}$ in its k-spectrum. This follows, of course, by choosing n to be at least as large as the diagonal Ramsey number $r(k, k)$. We will say that a set S of integers is *k-realizable* if there is an integer N_k such that for every $n \geqslant N_k$ there is a graph G of order n for which $s_k(G) = S$. Thus two necessary conditions for S to be k-realizable are that $S \subseteq \{0, 1, \ldots, \binom{k}{2}\}$ and that either 0 or $\binom{k}{2}$ is in S. As a corollary of our next result we determine a necessary condition for a set $S \subseteq \{0, 1, \ldots, \binom{k}{2}\}$ containing *both* 0 and $\binom{k}{2}$ to be k-realizable.

For disjoint graphs G and H, let $G \cup H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. By adding all edges to $G \cup H$ between the vertices of G and those of H we obtain the graph $G + H$.

Theorem 3. *Let I_k denote the set of all integers that are in the k-spectrum of every graph G of order $n \geqslant r(k2^k+1, k2^k+1)$ for which $0, \binom{k}{2} \in s_k(G)$. Then*

$$I_k = \left(\bigcap_{\ell=0}^{k} s_k(A_\ell(k))\right) \cap \left(\bigcap_{\ell=0}^{k} s_k(\overline{A_\ell(k)})\right)$$

where

$$A_\ell(k) = (K_k + \bar{K}_\ell) \cup \bar{K}_{k-\ell}.$$

Proof. We first observe that $A_\ell(k)$ is an induced subgraph of $(K_{n-k} + \bar{K}_\ell) \cup \bar{K}_{k-\ell}$, for every $n \geqslant 2k$. Furthermore, $s_k(A_\ell(k)) = s_k((K_{n-k} + \bar{K}_\ell) \cup \bar{K}_{k-\ell})$. Similarly, $\overline{A_\ell(k)}$ is an induced subgraph of $(\bar{K}_{n-k} \cup K_\ell) + K_{k-\ell}$ for every $n \geqslant 2k$ and $s_k(\overline{A_\ell(k)}) = s_k((\bar{K}_{n-k} \cup K_\ell) + K_{k-\ell})$. Since $0, \binom{k}{2} \in s_k(A_\ell(k))$ and $0, \binom{k}{2} \in s_k(\overline{A_\ell(k)})$ for $0 \leqslant \ell \leqslant k$, it follows that if $x \in I_k$, i.e., if x is in the k-spectrum of every graph of order $n \geqslant r(k2^k+1, k2^k+1)$ that has 0 and $\binom{k}{2}$ in its k-spectrum, then

$$x \in \left(\bigcap_{\ell=0}^{k} s_k(A_\ell(k))\right) \cap \left(\bigcap_{\ell=0}^{k} s_k(\overline{A_\ell(k)})\right).$$

Thus,

$$x \subseteq \left(\bigcap_{\ell=0}^{k} s_k(A_\ell(k))\right) \cap \left(\bigcap_{\ell=0}^{k} s_k(\overline{A_\ell(k)})\right).$$

We complete the proof by showing that if G is a graph of order $n \geqslant r(k2^k+1, k2^k+1)$ such that $0, \binom{k}{2} \in s_k(G)$ then G contains either $A_\ell(k)$ or $\overline{A_\ell(k)}$ as an induced subgraph for some ℓ satisfying $0 \leqslant \ell \leqslant k$. Thus, either

$$s_k(A_\ell(k)) \subseteq s_k(G) \text{ or } s_k(\overline{A_\ell(k)}) \subseteq s_k(G),$$

which implies

$$\left(\bigcap_{\ell=0}^{k} s_k(A_\ell(k))\right) \cap \left(\bigcap_{\ell=0}^{k} s_k(\overline{A_\ell(k)})\right) \subseteq I_k.$$

Since $n \geqslant r(k2^k+1, k2^k+1)$, G contains either a complete graph of order $k2^k+1$ or an independent $(k2^k+1)$-set of vertices. Suppose first that G contains a complete graph of order $k2^k+1$. Thus G contains disjoint sets A and B such that $\langle A \rangle = K_{k2^k}$ and $\langle B \rangle = \bar{K}_k$. Let $S_1, S_2, \ldots, S_{2^k}$ denote the distinct subsets of B and, for $1 \leqslant i \leqslant 2^k$, let $T_i = \{v \in A \mid N_B(v) = S_i\}$. Then $\bigcup_{i=1}^{2^k} T_i = A$ and, since $|A| = k2^k$, it follows that $|T_j| \geqslant k$ for some j. But then

$$A_\ell(k) \prec \langle T_j \cup B \rangle,$$

where $\ell = |S_j|$. The case in which G contains an independent $(k2^k+1)$-set of vertices follows from a symmetric argument. □

Corollary 2. *If S is k-realizable and* $0, \binom{k}{2} \in S$, *then* $I_k \subseteq S$.

It is worth noting that $s_k(A_\ell(k))$ and $s_k(\overline{A_\ell(k)})$, are straightforward to calculate. Thus, I_k can be determined for small k.

By definition, $\{0, \binom{k}{2}\} \subseteq I_k$. It is easy to check that for some values of k ($k = 5$, for example), $I_k = \{0, \binom{k}{2}\}$. In such a case, Corollary 2 gives no new information. The case $k = 5$ follows from our next result.

Propositon 1. *If k is an integer for which* $(k-1)^2 + k^2$ *is prime, then* $I_k = \{0, \binom{k}{2}\}$.

Proof. We first note that $s_k(A_k(k)) = \{\binom{k}{2} - \binom{b}{2}\colon 1 \leqslant b \leqslant k\}$ and $s_k(\overline{A_k(k)}) = \{\binom{a}{2}\colon 1 \leqslant a \leqslant k\}$ for every positive integer k. Thus $s_k(A_k(k)) \cap s_k(\overline{A_k(k)}) - \{0, \binom{k}{2}\} \neq \emptyset$ for some k if and only if there are integers $1 < a < k$ and $1 < b < k$ for which

$$\binom{k}{2} = \binom{a}{2} + \binom{b}{2}. \tag{1}$$

Setting $n = 2k - 1$, $x = 2a - 1$ and $y = 2b - 1$, Eq. (1) becomes

$$n^2 + 1 = x^2 + y^2. \tag{2}$$

Since every odd prime divisor of $n^2 + 1$ is of the form $4q + 1$ (see [4, Theorem 3.1], for example), it follows that the prime decomposition of $n^2 + 1$ is

$$n^2 + 1 = 2^\alpha \prod_{i=1}^{t} p_i^{\alpha_i}, \tag{3}$$

where $p_i \equiv 1 \pmod 4$. It follows from Eq. (3) that Eq. (2) has precisely $4\prod_{i=1}^{t}(\alpha_i + 1)$ ordered pairs (x, y) of integer solutions. Thus Eq. (2) has only the eight trivial solutions $(x, y) = (\pm n, \pm 1)$ and $(\pm 1, \pm n)$ if and only if $n^2 + 1 = 2^\alpha p_1$. However,

$$n^2 + 1 = 2((k-1)^2 + k^2),$$

where $(k-1)^2 + k^2$ is odd. Thus Eq. (2) has only the eight trivial solutions if and only if $(k-1)^2 + k^2$ is prime. Therefore, if $(k-1)^2 + k^2$ is prime, then $a = \frac{1}{2}(x+1) = 1$ and $b = \frac{1}{2}(y+1) = \frac{1}{2}(n+1) = k$ are the only integers $1 \leqslant a \leqslant b \leqslant k$ satisfying Eq. (1) and, consequently, $I_k = s_k(A_k(k)) \cap s_k(\overline{A_k(k)}) = \{0, \binom{k}{2}\}$. □

From Proposition 1 we see that $I_k = \{0, \binom{k}{2}\}$ for $k = 2, 3, 5, 8, \ldots$. However, it is unknown whether $(k-1)^2 + k^2$ is prime for infinitely many k and, consequently, we do not know if $I_k = \{0, \binom{k}{2}\}$ for infinitely many k. But it is worth noting that Proposition 1 does not give a necessary condition for $I_k = \{0, \binom{k}{2}\}$. For example, $I_7 = \{0, 21\}$ even though $6^2 + 7^2 = 85$, which is not prime.

If $I_k \neq \{0, \binom{k}{2}\}$, then Corollary 2 gives a nontrivial necessary condition for a set $S \subseteq \{0, 1, \ldots, \binom{k}{2}\}$ containing 0 and $\binom{k}{2}$ to be k-realizable. As our next result shows, there are infinitely many k for which this happens.

Proposition 2. *For infinitely many k,* $I_k \neq \{0, \binom{k}{2}\}$.

Proof. Let k be an integer such that $\binom{k}{2} = 2\binom{a}{2}$, for some $a \geqslant 3$. We first show that $\binom{a}{2} \in I_k$. Note that $|E(\bar{K}_{k-a} \cup K_a)| = \binom{a}{2}$, and $|E(K_{k-a} + \bar{K}_a)| = \binom{k}{2} - \binom{a}{2} = \binom{a}{2}$.

Let $0 \leqslant \ell \leqslant k$ be fixed. If $a \leqslant \ell$, then

$$K_{k-a} + \bar{K}_a \prec (K_k + \bar{K}_\ell) \cup \bar{K}_{k-\ell} = A_\ell(k),$$

and

$$\bar{K}_{k-a} \cup K_a \prec (\bar{K}_k \cup K_\ell) + K_{k-\ell} = \overline{A_\ell(k)}.$$

Thus $\binom{a}{2} \in s_k(A_\ell(k) \cap s_k(\overline{A_\ell(k)})$. If, on the other hand, $a > \ell$, that is, $k - a < k - \ell$, then

$$\bar{K}_{k-a} \cup K_a \prec \bar{K}_{k-\ell} \cup K_k \prec (K_k + \bar{K}_\ell) \cup \bar{K}_{k-\ell} = A_\ell(k),$$

and

$$K_{k-a} + \bar{K}_a \prec K_{k-\ell} + \bar{K}_k \prec (\bar{K}_k \cup K_\ell) + K_{k-\ell} = \overline{A_\ell(k)}$$

implying that $\binom{a}{2} \in s_k(A_\ell(k)) \cap s_k(\overline{A_\ell(k)})$. Hence, by Theorem 3, $\binom{a}{2} \in I_k$. Since $0 < \binom{a}{2} < \binom{k}{2}$, it follows that $I_k \neq \{0, \binom{k}{2}\}$ for every value of k satisfying $\binom{k}{2} = 2\binom{a}{2}$.

We conclude the proof by showing that this last equation, or, equivalently, $2a^2 - 2a - k^2 + k = 0$ has infinitely many positive integer solutions (k, a). Solving this equation for a, we see that a is a positive integer if $(k-1)^2 + k^2$ is a perfect square.

Consider the Pell-equation $x^2 - 2y^2 = 1$, which has infinitely many positive integer solutions (x, y). (See [4], for example.) For any such solution $x > y > 0$, set $k = 2y(x - y)$. Since $2y^2 + 1 = x^2$, we have $k - 1 = y^2 - (x - y)^2$. Therefore, $(k-1)^2 + k^2 = (y^2 - (x-y)^2)^2 + 4y^2(x-y)^2 = (y^2 + (x-y)^2)^2$, and the proof is complete. □

4. *k*-realizable sets for $k \leqslant 5$

It is an open problem to characterize k-realizable sets for general k. For $k \leqslant 4$, however, the k-realizable sets have been characterized. The case $k \leqslant 2$ is trivial. For $k = 3$, the results are summarized in Table 1. As can be seen, there are only four 'missing' sets, namely $\{1\}$, $\{2\}$, $\{1, 2\}$ and $\{0, 3\}$ for $k = 3$. The first three of these fail to be 3-realizable since Ramsey's theorem says that a 3-realizable set contains either 0 or 3. Interestingly, each of these sets is the 3-spectrum of at least one graph G. In particular, $s_3(P_3) = \{2\}$, $s_3(K_1 \cup K_2) = \{1\}$ and $s_3(P_4) = \{1, 2\}$. Finally, it follows from the 'Gap Theorem', whose proof can be found in [1], that $\{0, 3\}$ is the 3-spectrum of no graph.

Table 1

3-realizable sets S	Graphs G with $s_3(G) = S$
$\{0\}$	$\bar{K}_n$ (unique)
$\{3\}$	K_n (unique)
$\{0,1\}$	$tK_2 \cup (n-2t)K_1$ (unique)
$\{0,2\}$	$K_{r,n-r}$ (unique)
$\{1,3\}$	$K_r \cup K_{n-r}$ (unique)
$\{2,3\}$	$K_n - tK_2$ (unique)
$\{0,1,2\}$	P_n
$\{0,1,3\}$	$K_r \cup K_s \cup K_t$, $r+s+t=n$
$\{0,2,3\}$	$\overline{K_r \cup K_s \cup K_t}$, $r+s+t=n$
$\{1,2,3\}$	$\overline{P_n}$
$\{0,1,2,3\}$	See Theorem 1

Gap Theorem. *If $S = \{a_1 < a_2 < \cdots < a_t\}$ is the k-spectrum of some graph then $a_{i+1} - a_i \leqslant k-2$ except when $a_i = a_1 = 0$ or $a_{i+1} = a_t = \binom{k}{2}$. In these latter cases, $a_{i+1} - a_i \leqslant k-1$.*

For the case $k = 4$, some preliminary results are helpful in our analysis. Since $\binom{4}{2} = 2\binom{3}{2}$, the proof of Proposition 2 gives that $3 \in I_4$. Thus, if G is a graph of sufficiently large order with $0, 6 \in s_4(G)$ then $3 \in s_4(G)$. This result can be strengthened to include *all* graphs with $0, 6 \in s_4(G)$.

Theorem 4. *If G is a graph for which $0, 6 \in s_4(G)$ then $3 \in s_4(G)$.*

Proof. Let G be a graph for which $0, 6 \in s_4(G)$. Since $6 \in s_4(G)$, it follows that G has a triangle. If G is not connected then G contains $K_3 \cup K_1$ as an induced subgraph and $3 \in s_4(G)$. Thus we may assume that G is connected. Furthermore, we may assume that the distance between any pair of vertices of G is 1 or 2; for otherwise, G contains P_4 as an induced subgraph and $3 \in s_4(G)$.

Let $\{v_1, v_2, v_3, v_4\}$ be a set of 4 independent vertices of G. Then for each pair v_i, v_j $(i \neq j)$ there is a vertex $x \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that $xv_i, xv_j \in E(G)$. Moreover, we may assume that if $xv_i, xv_j \in E(G)$ then $xv_k \notin E(G)$ for $k \neq i, j$; for otherwise, G contains $K_{1,3}$ as an induced subgraph and hence $3 \in s_4(G)$. Thus there are vertices x and y such that v_1, x, v_2, y, v_3 is a path in G and for which the only possible chord is xy. Then G contains either P_4 or $K_3 \cup K_1$ as an induced subgraph depending on whether $xy \in E(G)$. In either case, $3 \in s_4(G)$. □

The proof of Theorem 4 depends only on the fact that G contains K_3 and $\bar{K}_3$ as induced subgraphs. Thus we also have the following results.

Corollary 3. *If G is a graph for which at least one of 0, 1 and one of 5, 6 is in $s_4(G)$ then $3 \in s_4(G)$.*

Using Ramsey's Theorem, the Gap Theorem, Corollary 3 and case-by-case analysis we can describe precisely the situation for 4-spectra. In what follows we will use, for example, 013 to denote the set $\{0, 1, 3\}$. Also, $\overline{013}$ will denote the complement of 013, that is, $\{2, 4, 5, 6\}$.

4-realizable sets:

0 6 01 03 36 56 012 013 023 034 035 136 236 346 356 456 0123 0124

0136 0234 0235 0345 0356 1236 1346 2346 2456 3456 $\overline{01}$ $\overline{02}$ $\overline{04}$ $\overline{05}$ $\overline{12}$ $\overline{15}$

$\overline{16}$ $\overline{25}$ $\overline{26}$ $\overline{45}$ $\overline{46}$ $\overline{56}$ $\overline{0}$ $\overline{1}$ $\overline{2}$ $\overline{4}$ $\overline{5}$ $\overline{6}$ 0123456

Sets that are the 4-spectrum of some graph but are not 4-realizable:

1 2 3 4 5 12 13 23 24 34 35 45 123 124 134 234 235 245 345 1235

1345 2345 1234 $\overline{06}$

The remaining subsets of $\{0, 1, 2, 3, 4, 5, 6\}$ are the 4-spectra of no graphs.

Using similar techniques, although much more detailed, to those used in Theorem 4 we can show the following.

(a) If G is a graph for which $0, 8 \in s_5(G)$ then $4 \in s_5(G)$.

(b) If G is a graph for which $0, 1, 9, 10 \in s_5(G)$ then $5 \in s_5(G)$.

(c) If G is a graph of sufficiently large order for which $0, 14 \in s_6(G)$ then $5 \in s_6(G)$.

We close with three open questions based on our knowledge of k-spectra for $k \leqslant 5$. According to Theorem 4 and (a) above, $\{0, 1, \ldots, \binom{k}{2}\} - \{k-1\}$ is not k-realizable for $k = 4, 5$.

Question 1. *For $k \geqslant 4$, is $\{0, 1, \ldots, \binom{k}{2}\} - \{k-1\}$ k-realizable?*

As mentioned earlier, the proof of Theorem 4 depends only on the fact that G contains K_3 and $\bar{K}_3$ as induced subgraphs. Therefore, if $0, 3 \in s_3(G)$ then $3 \in s_4(G)$. We can also show that if $0, 4, 6 \in s_4(G)$ then $4 \in s_5(G)$. Thus for $k = 4, 5$ we have that if G has a complete $(k-1)$-spectrum, then $k - 1 \in s_k(G)$.

Question 2. *For $k \geqslant 4$, if G has a complete $(k-1)$-spectrum, is $k - 1 \in s_k(G)$?*

Finally, for $k \leqslant 4$ we know that if S is the k-spectrum of at least one graph and either 0 or $\binom{k}{2}$ is in S, then, in fact, S is k-realizable.

Question 3. *For $k \geqslant 1$, if S is the k-spectrum of at least one graph and either 0 or $\binom{k}{2}$ is in S, then, is S k-realizable?*

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