On p-Intersection Representations

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ABSTRACT

For a graph $G = (V, E)$ and integer p, a p-intersection representation is a family $\mathcal{F} = \{S_x: x \in V\}$ of subsets of a set S with the property that $|S_u \cap S_v| \geq p \Leftrightarrow \{u, v\} \in E$. It is conjectured in [1] that $\theta_p(G) \leq \theta(K_{n/2,\,n/2})(1 + o(1))$ holds for any graph with n vertices. This is known to be true for $p = 1$ by [4]. In [1], $\theta(K_{n/2,\,n/2}) \geq (n^2 + (2p - 1)n)/4p$ is proved for any n and p. Here, we show that this is asymptotically best possible.

Further, we provide a bound on $\theta_p(G)$ for all graphs with bounded degree. In particular, we prove $\theta_p(G) \leq O(n^{1/p})$ for any graph G with the maximum degree bounded by a constant.

Finally, we also investigate the value of θ_p for trees. Improving on an earlier result of M. Jacobson, A. Kézdy, and D. West, (The 2-intersection number of paths and bounded-degree trees, preprint), we show that $\theta_2(T) \leq O(d\sqrt{n})$ for any tree T with maximum degree d and $\theta_2(T) \leq O(n^{3/4})$ for any tree on n vertices. We conjecture that our result can be further improved and that $\theta_2(T) \leq O(\sqrt{n})$ as long as $\Delta(T) \leq \sqrt{n}$. If this conjecture is true, our method gives $\theta_2(T) \leq O(n^{2/3})$ for any tree T which would be the best possible. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

For terms not defined here see [8].

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Graphs have long been used as models for a wide variety of different systems. In particular, the representation of a graph as the intersections of sets has drawn considerable attention. More precisely, given a graph $G = (V, E)$ with vertex set V and edge set E, a family $\mathcal{F} = \{A_x : x \in V\}$ of (not necessarily distinct) sets is called an *intersection representation* of G if

$$A_x \cap A_y \neq \emptyset \Longleftrightarrow \{x, y\} \in E$$

for every pair x, y of distinct vertices of G; conversely G is called an *intersection graph* of $\mathcal{F}$. Unlike other types of graph representations, every graph is the intersection graph for some family of sets.

Erdös, Goodman, and Pósa [4] introduced the question of finding the minimum number of elements in the set $S = \cup_{x \in V} A_x$ used to represent G. They showed the following:

Theorem 1. If G is any graph with n vertices, then there is a set S with $[n^2/4]$ elements and a family of n subsets of S that represent G. Further, $[n^2/4]$ is the smallest such number.

Their proof introduced an important way of working with intersection representations, namely as edge clique coverings. An *edge clique cover* is a family, C, of cliques such that for all $e \in E$, there exists $C \in C$ such that $e \subset C$.

Suppose $\mathcal{F} = \{A_x : x \in V\}$ is an intersection representation of G and let $|\cup \mathcal{F}| = t$ denote the cardinality of the union of all sets in $\mathcal{F}$. Then $C_i = \{x : i \in A_x\}$ provides a bijection between $[t] = \{0, 1, \ldots, t - 1\}$ and $C = \{C_i : i \in [t]\}$ and C is an edge clique cover. Also, if C is an edge clique cover with $|C| = t$, an intersection representation is given by $\mathcal{F} = \{A_x : x \in V\}$ where $A_x = \{j : x \in C_j\}$.

Over the years the original intersection question has been altered in a variety of ways. One refinement has recently drawn considerable attention. Given a positive integer p, and graph $G = (V, E)$, a family $\mathcal{F} = \{A_x : x \in V\}$ of (not necessarily distinct) sets is called a *p-intersection representation* of G if

$$|A_x \cap A_y| \geq p \Longleftrightarrow \{x, y\} \in E$$

for every pair x, y of distinct vertices of G. Define $\theta_p(G)$ to be

$$\theta_p(G) = \min_{\mathcal{F}}(|\cup \mathcal{F}|)$$

where $\mathcal{F}$ is taken over all p-intersection representations of G. Then $\theta_p(G)$ is called the *p-intersection number* of G.

Also, for an integer n, let

$$\theta_p(n) = \max\{\theta_p(G) : |V(G)| = n\}.$$

Clearly, for any graph G and integer $p \geq 2$, a p-intersection representation of G exists since we know an intersection representation of G exists and given a $(p - 1)$-intersection representation we may add 1 element to the universe and include it in each set. This gives

$$\theta_p(G) \leq \theta_{p-1}(G) + 1 \leq \theta_1(G) + p - 1. \tag{1}$$

Analogous to the characterization of intersection representations by edge clique covers, there is a characterization of p-intersection representations.

We define a *p-edge cover* (also known as *p-generator*) of a graph $G = (V, E)$ to be a family $\mathcal{P}$ of subsets of V, such that

$$\{x, y\} \in E \Longleftrightarrow |\{P \in \mathcal{P}\colon \{x, y\} \subset P\}| \geq p .$$

In a manner similar to before, if $\mathcal{F} = \{A_x\colon x \in V\}$ is a p-intersection representation of $G = (V, E)$ with $| \cup \mathcal{F}| = t$ then $P_i = \{x\colon i \in A_x\}$ gives a bijection between $[t]$ and $\mathcal{P} = \{P_i\colon i \in [t]\}$. Thus,

$$\theta_p(G) = \min_{\mathcal{P}}(|\mathcal{P}|) \tag{2}$$

where the minimum is taken over all p-edge covers, $\mathcal{P}$, of G.

From the proof of Theorem 1, the graph which attains the maximum θ_1 value is $K_{n/2, n/2}$, so we may say

$$\theta_1(G) \leq \theta_1(K_{n/2, n/2}) \leq [n^2/4]. \tag{3}$$

It is natural to ask if $K_{n/2, n/2}$ also provides the maximum value for $\theta_p(G)$ when $p > 1$. Chung and West [1] conjecture that

$$\theta_2(G) \leq \theta_2(K_{n/2, n/2}), \tag{4}$$

for all G. Thus, the problem of determining $\theta_p(K_{n/2, n/2})$ for $p > 1$ takes on added significance. In Section 2, we will show that for any $\varepsilon > 0$, there exists an n_o such that $n > n_0$ implies

$$\theta_p(K_{n/2, n/2}) = \frac{n^2}{4p}(1 \pm \varepsilon) .$$

In Section 3, we discuss the relationship between $\theta_1(G)$ and $\theta_p(G)$, $p \geq 2$ and give bounds on $\theta_p(G)$ for graphs G without 3-cycles.

In Section 4 we provide a general bound on θ_p for graphs of bounded degree.

In Section 5, we answer a conjecture of Jacobson, Kézdy and West [9] concerning θ_2 for trees of bounded degree. Further, we provide a general upper bound for $\theta_2(T)$ where T is any tree.

2. p-INTERSECTION NUMBERS FOR $K^{(r)}_{n/r, \ldots, n/r}$

As was mentioned earlier, the problem of finding bounds on $\theta_p(G)$ has led to consideration of the value of $\theta_p(K_{n/2, n/2})$. Theorems 2 and 3 are due to M. Chung and D. West [1].

Theorem 2. For $G = K_{n/2, n/2}$, and $p \geq 2$,

$$\theta_p(G) \geq \frac{n^2 + 2(2p - 1)n}{4p} .$$

Theorem 3. For n odd, $n \neq 0 \pmod 3$,

$$\theta_2(K_{n/2,n/2}) = \frac{n^2 + 6n}{8}.$$

Here we give the value of θ_p for a larger class of graphs. For each r, $K^{(r)}_{n,\cdots,n}$ is the complete r-partite graph with n vertices in each independent set. It is mentioned in [1] that P. Frankl and V. Rödl's result [6], given below as Theorem 4, provides an asymptotic upper bound for $\theta_p(K_{n/2,n/2})$. We generalize that result to $K^{(r)}_{n/r,\ldots,n/r}$ in Theorem 5. (Note: We have recently learned that a similar result has been shown by Z. Furedi [7], so we shall only sketch our proof here. A complete proof can be found in [2].)

A hypergraph $\mathcal{H}$ is an ordered pair, $(X, \mathcal{E})$, of vertices X and edges $\mathcal{E}$ such that $\mathcal{E} \subset \mathcal{P}(X)$. In a d-uniform hypergraph, each edge has order d. The *point covering number* $t(\mathcal{H})$ is the minimum number t so that there exist t edges of $\mathcal{H}$ whose union is the whole set X. The following theorem by P. Frankl and V. Rödl applies the probabilistic approach of [11] to show that for a wide class of d-uniform hypergraphs, *near perfect* coverings exist. Also note that for a set A, $\deg A = |\{E: A \subset E\}|$ and we use the notation $[A]^2$ to be the 2-element subsets of A.

Theorem 4. Let $\epsilon > 0$ be arbitrary, $\mathcal{H}$ a d-uniform hypergraph on X, $|X| = n$, and $a > 3$ a real number. If there exists a positive real $\delta = \delta(\epsilon)$ such that if for some D one has $(1 - \delta)D < \deg(u) < (1 + \delta)D$ for all $u \in X$ and $\deg(\{u, v\}) < D/(\log n)^a$ holds for all distinct $u, v \in X$, then for all $n > n_0(\delta)$,

$$t(\mathcal{H}) \leq n(1 + \epsilon)/d \text{ holds.}$$

Theorem 5. For $\epsilon > 0$ and $p \geq 2$, there exists an integer n_0 such that for $n > n_0$,

$$\theta_p(K^{(r)}_{n/r,\cdots,n/r}) \leq \frac{n^2}{r^2 p}(1 + \epsilon).$$

Proof. (*Sketch*) Consider $K^{(r)}_{n/r,\cdots,n/r}$ with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_r$. In order to demonstrate a p-edge cover of $K^{(r)}_{n/r,\cdots,n/r}$, we construct a $(rp/2)$-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$ satisfying the hypothesis of Theorem 4. The edge of $\mathcal{H}$ will correspond to sets in a p-edge cover of $K^{(r)}_{n/r,\cdots,n/r}$.

To provide a p-edge cover of $K^{(r)}_{n/r,\cdots,n/r}$ we take subsets of size rp of the vertex set V, with p vertices in each of the r independent sets, making sure that we cover the *inside pairs*, pairs with each vertex within some V_i, at most $p - 1$ times each and the *crossing pairs*, pairs with each vertex from different independent sets V_i, V_j, $i \neq j$, at least p times.

Thus we construct the vertex set X of $\mathcal{H}$ by taking $p - 1$ copies of the $r\binom{n/r}{2}$ inside pairs together with p copies of the $\binom{r}{2}(n/r)^2$ crossing pairs. Thus,

$$|X| = r\binom{n/r}{2}(p - 1) + \binom{r}{2}\frac{n^2}{r^2}p.$$

The edges of our $\binom{rp}{2}$-uniform hypergraph consist of all sets E that can be formed in the following way. For each i, $1 \leq i \leq r$, select a subset $P_i \subset V_i$, $|P_i| = p$, then $E \in \mathcal{E}$ will consist of $\binom{rp}{2}$ vertices in X selected by choosing a copy of each pair in $\cup P_i$.

It is straightforward to see that

$$|\mathcal{E}| = \binom{n/r}{p}^r (p-1)^{r\binom{p}{2}} p^{\binom{r}{2}p^2}.$$

Each vertex of $\mathcal{H}$ has one of two possible degrees based on whether it was derived from an inside pair or a crossing pair. If A is the degree of a vertex from an inside pair and B that of an outside pair, a direct count shows that

$$\binom{n/r-2}{p-2}\binom{n/r}{p}^{r-1} (p-1)^{r\binom{p}{2}-1} p^{\binom{r}{2}p^2} = A$$

while

$$\binom{n/r-1}{p-1}^2\binom{n/r}{p}^{r-2} (p-1)^{r\binom{p}{2}} p^{\binom{r}{2}p^2-1} = B$$

and hence $\lim_{n\to\infty} A/B = 1$. Thus, $\mathcal{H}$ satisfies the regularity condition for degrees of vertices of the hypothesis of Theorem 4 with $D \sim A \sim B$ where

$$D = c_{p,r} n^{rp-2}$$

where $c_{p,r}$ is a constant depending only on p and r. Applying Theorem 4 completes the proof. ∎

Theorem 6 gives a lower bound on the p-intersection number for $K^{(r)}_{n/r,\ldots,n/r}$ and extends Theroem 2 to the r-partite case. We first state a lemma whose proof is elementary and hence omitted.

Lemma 1. If $a = (a_1, a_2, \ldots, a_r)$ is a vector and such that $(1/r)\sum_{i=1}^{r} a_i = s$, then

$$\sum_{i=1}^{r}\binom{a_i}{2} \ge r\binom{s}{2} \quad \text{and} \quad \sum_{i<j} a_i a_j \le \binom{r}{2}s^2.$$

Using this Lemma the following result can be obtained. We omit the proof here, but it can be found in [2]. We note that a slightly weaker bound then that of Theorem 6 (but of the same order of magnitude) can be obtained directly using Theorem 2.

Theorem 6. For $G = K^{(r)}_{n/r,\ldots,n/r}$, and $p \ge 2$,

$$\theta_p(G) \ge \frac{(n + rp - r)^2}{r^2 p}.$$

As can be seen by the last two results, $\theta_p(K^{(r)}_{n/r,\ldots,n/r}) \sim n^2/(r^2 p)$.

3. COMPARING θ_p AND θ_1

An easy comparison between $\theta_p(G)$ and $\theta_1(G)$ for any graph G gives $\theta_p(G) \leq \theta_1(G) + p - 1$, see (1). There exist graphs G with equality holding, such as $G = K_n$. But this is not always the case. Theorem 7, gives the lower bound for $\theta_p(G)$ as $O(\theta_1(G)^{1/p})$. We will see that this lower bound is attained for some classes of graphs, see (5), (6) and Proposition 1.

Theorem 7. For $p \geq 2$ and any graph G on n vertices,

$$\binom{\theta_p(G)}{p} \geq \theta_1(G)).$$

Proof. Let $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ be a p-edge clique cover of G where $t = \theta_p(G)$. Let $\mathcal{P} \subset \mathcal{V}$ such that $|\mathcal{P}| = p$ and consider the set $C = \cap \mathcal{P}$. This intersection must be a clique since only pairs of vertices that are edges can be contained in p different sets. Also, each edge in G must be in the intersection of some choice of p sets in $\mathcal{V}$. Therefore, the set of all such intersections forms an edge clique cover for G. There are $\binom{t}{p}$ such intersecting sets, thus

$$\binom{t}{p} \geq \theta_1(G). \quad \blacksquare$$

We turn our attention to a smaller class of graphs. For graphs G with no 3-cycles, $\theta_1(G) = |E|$, since in an edge clique cover, we must use each edge separately. So, as a corollary to Theorem 7, we have

Corollary 8. Let $p \geq 2$ and $G = (V, E)$ be a graph which contains no 3-cycles. Then,

$$\binom{\theta_p(G)}{p} \geq |E|.$$

Corollary 9. Let $G = (V, E)$ be a graph which contains no 3-cycles. Then,

$$\theta_2(G) > \sqrt{2|E|}.$$

This lower bound is known to be obtained for stars. It is an easy exercise to show that for S_n, the star on n vertices,

$$\left\lceil \sqrt{2(n-1)} \right\rceil = \theta_2(S_n). \tag{5}$$

Note that, in [9] it was shown that for paths P_n,

$$\theta_2(P_n) \sim 2\sqrt{n}. \tag{6}$$

4. THE p-INTERSECTION NUMBER OF GRAPHS WITH BOUNDED DEGREE

In Section 3 we have shown that for $p > 1$, and all graphs G, $\theta_p(G) \geq c\theta_1(G)^{1/p}$ where c is an absolute constant. Notice that for a d-regular triangle free graph, this gives $\theta_p(G) \geq c(dn)^{1/p}$.

Theorem 11 gives an upper bound on the value of $\theta_p(G)$ for $p > 1$ and graphs G of bounded degree. We show that if $\Delta(G) \leq d$ then $\theta_p(G) \leq C(dn)^{1/p}$ where C is a constant dependent on both p and d. Thus this is the best possible up to the constant c. The techniques used in the proof of Theorem 11 are similar to those found in [3]. In the proof, we make use of the well-known Lovaśz Local Lemma [5].

Theorem 10. Let $G = (V, E)$ be a graph with maximum degree D and vertices $v_1, v_2, \ldots, v_n$. For each v_i let us associate an event A_i and suppose that A_i is totally independent of the set of events

$$\{A_j \colon \{v_i, v_j\} \notin E\}.$$

Also suppose

$$\mathrm{Prob}(A_i) \leq \frac{1}{4D}.$$

Then

$$\mathrm{Prob}(\overline{A}_1 \wedge \overline{A}_2 \wedge \cdots \wedge \overline{A}_n) > 0.$$

Theorem 11. Let G be a graph on n vertices with $\Delta(G) \leq d$ and $p > 1$ be an integer, then

$$\theta_p(G) \leq Cn^{1/p}$$

where $C = 3epd^2(d + 1)^{1/p}$.

Proof. Let

$$C = 3epd^2(d + 1)^{1/p}$$

and

$$m = Cn^{1/p}.$$

Let $[m]^p$ be the set of all p-element subsets of [m] and consider the uniform probability space of all $1-1$ assignments $\phi \colon E(G) \mapsto [m]^p$. We will show that there exists one such assignment ϕ such that the function $S_\phi \colon V(G) \mapsto \mathcal{P}(m)$ defined by:

$$u \mapsto S_u = \bigcup\nolimits_{\{u,v\} \in E(G)} \phi(\{u, v\})$$

is a p-intersection representation of G.

Clearly, for any function ϕ, S_ϕ has the property that

$$\forall \{u, v\} \in E(G), |S_u \cap S_v| \geq p.$$

We need only show that there exists a map ϕ such that

$$\forall \{u, v\} \notin E(G), |S_u \cap S_v| < p\,.$$

Let $u, v \in V(G)$ be such that $\{u, v\}$ is not an edge. Suppose

$$S_u = B_1 \cup B_2 \cup \cdots \cup B_s \quad \text{and} \quad S_v = C_1 \cup C_2 \cup \cdots \cup C_t$$

where $s, t \leq d$. Then,

$$|S_u| \leq dp \quad \text{and} \quad |S_v| \leq dp\,.$$

We can give an upper bound on $\text{Prob}(|S_u \cap S_v| \geq p)$ by considering $|S_u|$ and $|S_v|$ to be as large as possible. Since for all edges $e_1, e_2 \in E(G)$ the events $\phi(e_1) = B$ and $\phi(e_2) = C$ are independent, we may consider S_u and S_v to be arbitrary subsets of $[m]$ of size dp. Thus,

$$\begin{aligned}
\text{Prob}(|S_u \cap S_v| \geq p) &\leq \frac{\binom{dp}{p}\binom{m}{dp-p}}{\binom{m}{dp}} \\
&\leq (ed)^p \left(\frac{dp}{m - dp + p} \right)^p .
\end{aligned} \tag{7}$$

Let I be the set of all nonedges of G, so that I is the set of all independent pairs of vertices of G. For $\{u, v\} \in I$, let $E_{u,v}$ be the event that $|S_u \cap S_v| \geq p$. We will use the Lovaśz Local Lemma, Theorem 10 to show that

$$\text{Prob}\left(\bigwedge_{\{u,v\} \in I} \overline{E_{u,v}} \right) > 0\,.$$

Let $\{u, v\}$, $\{w, x\} \in I$. Then events $E_{u,v}$ and $E_{w,x}$ are dependent if and only if there exists $z \in \{u, v\}$ and $z' \in \{w, x\}$ such that $\{z, z'\} \in E$ or $|\{u, v, w, x\}| \leq 3$. For each $\{u, v\} \in I$, there are at most $2dn$ pairs $\{w, x\} \in I$ that meet the first criteria and at most $2n$ pairs that meet the second. On the other hand, $E_{u,v}$ is independent of all other events $E_{y,z}$. Thus, the maximum degree D of the dependency graph of the Lovaśz Local Lemma, is $\max_{E_{u,v}} \deg(E_{u,v})$ which is at most $2dn + 2n$. We need only show that for all $\{u, v\} \in I$,

$$\text{Prob}(E_{u,v}) \leq \frac{1}{4D}\,,$$

or from (7),

$$\left(\frac{ed^2 p}{m - dp + p} \right)^p \leq \frac{1}{8(d+1)n}\,. \tag{8}$$

As we chose

$$m = 3epd^2(d+1)^{1/p} n^{1/p}\,,$$

we have that

$$m \geq (8(d+1)^{(1)/p} e d^2 p n^{1/p} + p(d-1)$$

which is equivalent to Eq. (8). ∎

5. THE 2-INTERSECTION NUMBER FOR TREES

Jacobson, Kézdy and West [9] as well as Scheinerman [13] conjecture that for order n trees T with maximum degree bounded by a constant, $\theta_2(T) \leq O(\sqrt{n})$, thus meeting the lower bound obtained in Corollary 9. In [9] it was shown that for any $\epsilon > 0$, there exists an n_0 such that $n \geq n_0$ implies $\theta_2(T) \leq c(\sqrt{n})^{1+\epsilon}$ and that for paths P_n, $\theta_2(P_n)$ is asymptotic to $2\sqrt{n}$. Theorem 13 improves this result for trees with maximum degree d and answers the conjecture positively when d is a constant.

We also provide an upper bound of order $n^{3/4}$ for general trees in Theorem 14. We believe, however, that the correct bound for trees of order n that have maximum degree $\sqrt{n}$ is on the order $\sqrt{n}$. If true, the approach of this paper yields that for any tree, T, $\theta_2(T) \leq cn^{2/3}$ where c is an absolute constant. To support these conjectures we offer Proposition 1 concerning the class $\mathcal{T}_\gamma = \{T$: T is a rooted tree on $n+1$ vertices, with root of degree $n^{1-\gamma}$ and each vertex adjacent to the root is adjacent to n^γ leaves$\}$. Also note, in [9] it is shown that for $T \in \mathcal{T}_{2/3}$, $\theta_2(T) \geq cn^{2/3}$.

We will also make use of Theorem 12 by Sauer and Spencer [12]. First consider the following definition.

Definition 1. Let G_1 and G_2 be two graphs on n vertices. A packing of G_1 and G_2 consists of an embedding of $E(G_1 \cup G_2)$ into K_n.

Theorem 12. If $2\Delta(G_1)\Delta(G_2) < n$, then there is a packing of G_1 and G_2.

Proposition 1. Let $0 \leq \gamma \leq 1$ and $T \in \mathcal{T}_\gamma$, then there exist constants c_1, c_2, c_3, c_4, and c_5 such that

$$\theta_2(T) \leq \begin{cases} c_1 n^{(1+\gamma)/2} & 0 \leq \gamma < 1/6 \\ c_2 n^{2/3-\gamma/2} & 1/6 \leq \gamma < 1/3 \\ c_3 n^{1/2} & 1/3 \leq \gamma < 1/2 \\ c_4 n^\gamma & 1/2 \leq \gamma < 2/3 \\ c_5 n^{1-\gamma/2} & 2/3 \leq \gamma \leq 1 . \end{cases}$$

The proof of this Proposition is long and uses ideas on partial affine planes similar to those applied in the proof of Theorem 14, hence we omit it. The interested reader may find it in [2].

Notice that when $\gamma = 1/2$, we have an example of a tree T with $\Delta(T) = \sqrt{n}$ and $\theta_2(T) = O(\sqrt{n})$ which supports our second conjecture. Also note that as $\gamma \to 1$ the tree $T_\gamma \to K_{1,n}$ and $\theta_2(T_\gamma) \to cn^{1/2}$ (as it should).

We begin the proof of Theorem 13 by giving several lemmas, the first of which appears in [9].

Lemma 1. Let T_n be a tree of order n. If $\Delta(T_n) \leq d$, then we can remove one edge of T_n, disconnecting it into two components T_1 and T_2 such that

$$1 \leq \frac{|T_1|}{|T_2|} \leq d .$$

Lemma 2. Given a tree T_n of order n with $\Delta(T_n) \leq d$, there exists a partition $V(T_n) = S_1 \cup \cdots \cup S_t$ with

$$\sqrt{n} \geq |S_1| \geq |S_2| \geq \cdots \geq |S_t| \geq \frac{1}{d+1}\sqrt{n}$$

and so

$$\sqrt{n} < t < (d+1)\sqrt{n},$$

where the induced graph on S_i is a tree for each i and at most $t-1$ edges are not contained in the sets $S_1, S_2, \ldots, S_t$.

Proof. We apply Lemma 1 successively to break T_n into smaller components. We stop as soon as a component has order smaller than $\sqrt{n}$. If A and B are components obtained by a split of a component of size greater than $\sqrt{n}$ with $|A| \geq |B|$, then by Lemma 1

$$\frac{\sqrt{n} - |B|}{|B|} \leq \frac{|A|}{|B|} \leq d$$

so that

$$\frac{\sqrt{n}}{d+1} \leq |B| .$$

Since the classes of our partition $S_1, S_2, \ldots, S_t$ are trees and therefore connected, there can be no more than $t-1$ edges of T_n that are not contained in one of these classes. ∎

Theorem 13. Let T be a tree with $\Delta(T) \leq d$. Then

$$\theta_2(T_n) \leq (11d+3)\sqrt{n} .$$

Proof. We construct a 2-edge cover C for T, that is, we show how to find a collection of sets covering each edge at least twice and each nonedge at most once.

First we apply Lemma 2 to break T into

$$t \leq (d+1)\sqrt{n} \tag{9}$$

subtrees $S_1, S_2, \ldots, S_t$, where each has order at most $\sqrt{n}$, leaving at most $(d+1)\sqrt{n}$ edges not contained in any one of these subtrees. Note that $V(T) = V(S_1) \cup V(S_2) \cup \cdots \cup V(S_t)$. For each edge e, not contained in one of these subtrees, we consider a copy of e, say e'. Let

$$M = \left\{e \colon e \notin \bigcup_{i=1}^{t} E(S_i)\right\}, \quad M' = \{e' \colon e \in M\} .$$

Then

$$|M \cup M'| \leq 2(d + 1)\sqrt{n}. \tag{10}$$

The 2-edge cover C will contain the individual edges in $M \cup M'$, and the sets $S_1, \ldots, S_t$. That is,

$$C \supset \{V(S_1), V(S_2), \ldots, V(S_t)\} \cup M \cup M'.$$

Thus, we need only cover the edges in each subtree S_i one more time; noting that no two edges from the same tree S_i can appear again in the same set (since there are no 3-cycles).

We form a new partition $V(T) = F_1 \cup F_2 \cup \cdots \cup F_r$ of forests by forming the union of the subtrees S_i until there are at most $r \leq \sqrt{n}/d$ forests and the order of each forest is less than $(d + 1)\sqrt{n}$. This can be done since if we continue to combine trees S_i while there are less than $d\sqrt{n}$ vertices in the resulting forest, we can add another tree S_i without exceeding $(d + 1)\sqrt{n}$ vertices.

Next, we devise a way to select one edge from each of $F_1, F_2, \ldots, F_r$ to form a new set in the cover, but we must guard against the following: Given the situation shown in Figure 1, if, in order to cover the edges $\{y_i, x_i\}$, $\{y_i', x_i\}$, $\{y_j, x_j\}$, and $\{y_j', x_j\}$ we include the vertices y_i, x_i, y_j, x_j together in a set in the cover, and the vertices y_i', x_i, y_j', x_j together in another set, then we would cover a nonedge (namely $\{x_i, x_j\}$) twice, destroying our cover.

We are going to construct the cover C with the following properties:

(1) Each edge of $F_1 \cup F_2 \cup \cdots \cup F_r$ appears in one set $C \in C$.
(2) $|E(F_i) \cap C| \leq 1$ for each $C \in C$ and all $i = 1, 2, \ldots, r$.
(3) If $h_i, h_j \in C$ and h_i' and h_j' are incident to h_i and h_j respectively (see Figure 1), then $h_i' h_j' \in C'$ for no $C' \in C$.

FACT. For integers $1 \leq d \leq n$ and $r \leq \sqrt{n}/d$ let $L_1, \ldots, L_r$ be graphs with order at most $(d + 1)\sqrt{n}$ and maximum degree at most $2d$. Then these graphs can be packed on a set H of size at most $8d\sqrt{n}$. That is, there exist graphs $L_1', \ldots, L_r'$ with

(i) $V(L_i') \subset H$ for each $i = 1, 2, \ldots, r$.
(ii) $L_i \cong L_i'$ for each $i = 1, 2, \ldots, r$.
(iii) $E(L_i') \cap E(L_j') = \varnothing$ for each $1 \leq i, j \leq r$, $i \neq j$.

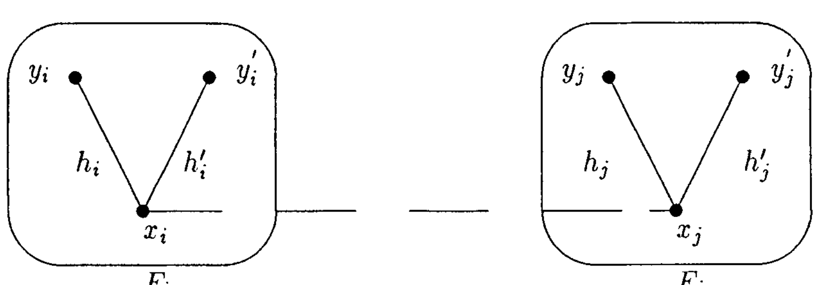

FIGURE 1. The situation to avoid.

Proof of FACT. It is an application of Theorem 12. Suppose that for $s < r$ we found graphs $L'_1, \dots, L'_s$ satisfying properties (i)–(iii). Clearly,

$$2\Delta(L_{s+1})\Delta\left(\bigcup_{i=1}^{s} L'_i\right) \le 2(2d)(2sd) < 8d^2 r \le 8\sqrt{n}d$$

and hence, by Theorem 12, there exists L'_{s+1} which together with $L'_1, \dots, L'_s$ satisfies properties (i)–(iii). ▮

Now in order to find a cover C satisfying 1-3 consider the line graphs $L(F_i) = L_i$ for each of the forests $F_1, \dots, F_r$ and apply the FACT to obtain a packing $L'_1, \dots, L'_r$ of $L_1, \dots, L_r$ in the set H, where $|H| \le 8d\sqrt{n}$. Let ϕ_i: $L_i \to L'_i$ be an isomorphism required by (ii). To each $h \in H$ and $i = 1, 2, \dots, r$ such that $h \in V(L'_i)$ let $h_i = \{x^i_h, y^i_h\}$ be a vertex of L_i (hence, an edge of F_i) such that $\phi(h_i) = h$. We set

$$C_h = \bigcup_i \{\{x^i_h, y^i_h\} \in E(F_i) | h \in V(L'_i)\}$$

and set $C = \{C_h \mid h \in H\}$.

Now we verify properties (1)–(3). As each edge $\{x, y\}$ of F_i corresponds to a vertex of L_i which is packed onto some $h \in H$ we get that $\{x, y\} \in C_h$, hence property (1) holds.

Similarly, property (2) follows from the fact that $E(F_i) \cap C_h$ is empty if $h \notin V(L'_i)$ and $E(F_i) \cap C_h = \{\phi(h)\}$ otherwise.

Finally, if $h_i = \{x_i, y_i\}$, $h_j = \{x_j, y_j\} \in C_h$ and $\overline{h}_i = \{x_i, \overline{y}_i\}$, $\overline{h}_j = \{x_j, \overline{y}_j\} \in C_{\overline{h}}$ for some $h, \overline{h} \in H$, then due to the fact that $\{h_i, \overline{h}_i\} \in E(L_i)$ and $\{h_j, \overline{h}_j\} \in E(L_j)$ we get that

$$\{\phi(h_i), \phi(\overline{h}_i)\} = \{\phi(h_j), \phi(\overline{h}_j)\} = \{h, \overline{h}\}$$

contradicting the fact each pair $\{h, h'\}$ can be covered by at most one edge.

Since $|H| \le 8d\sqrt{n}$, we infer that

$$\theta_2(T) \le |C| = t + |M \cup M'| + |H| = (11d + 3)\sqrt{n}. \quad ▮$$

Note that this result can be extended to forests.

We now turn our attention to all trees and give an upper bound on θ_2 regardless of the vertex degrees. We will use Theorem 13 applied to subtrees resulting from the following dismantling of the tree.

Suppose T is a tree of order n. Suppose further that T contains vertices of degree at least $d = n^\alpha$. Consider the set $X \subset V(T)$ composed of all vertices of T of degree more than d. Then $T - X$ consists of a collection of subtrees

$$T_1, T_2, \dots, T_r$$

each of which has maximum degree at most d.

We can further divide these subtrees into two collections, S, those subtrees of order less than n^β for some fixed β and $\mathcal{B}$, those subtrees of order at least n^β.

We now describe a partition of $E(T)$ based on our dismantling of T.

C-I The edges contained within subtrees in $\mathcal{B}$.

C-II To each subtree $T_i \in S$ we choose some $x \in X$ with the property that there exists some $y \in V(T_i)$ with $\{x, y\} \in E(T)$. In this way we will consider a mapping $\phi\colon S \mapsto X$ such that $\phi(T_i) = x$. We set $S(x) = \{(x, y) \in E(T)\colon y \in V(T_i)$ and $\phi(T_i) = x\}$. In a similar manner we define $T(x) = S(x) \bigcup_{x=\phi(T_i)} T_i$. We now set C-II $= \bigcup_{x \in X} E(T(x))$.

C-III All other edges. That is, any edges

(a) connecting vertices in X,
(b) connecting vertices of X to subtrees in $\mathcal{B}$,
(c) all edges $\{x, y\}$ such that $x \in X$, $y \in V(T_i)$ where $T_i \in S$ and $\phi(T_i) \neq x$, that is, edges, which were not covered in case C-II, connecting vertices in X with "small" trees.

With this partition in mind, we now turn to bounding θ_2 for all trees. We make no attempt to determine the constant in this result, since we feel the true upper bound has magnitude $O(n^{2/3})$. Thus all constants are merely referred to as c.

Theorem 14. If T is a tree of order n, then $\theta_2(T) \leq cn^{3/4}$ where c is an absolute constant.

Proof. Let T be a tree of order n. We may assume that T contains vertices of degree at least $n^{1/4}$ or we would be done by Theorem 13. Throughout the proof, let $d = n^{\alpha}$, $\alpha = \frac{1}{4}$, and $\beta = \frac{1}{4}$. From our earlier discussion we obtain a set X, two classes of subtrees S and $\mathcal{B}$ and a partition of the edges of T as described by the classes C-I, C-II, and C-III above.

For our proof we build sets corresponding to the vertices, so that the resulting sets form a 2-intersection representation of T. We consider that each vertex begins by being assigned the empty set and at each stage of the proof, we simply add elements to the already existing set corresponding to that vertex. We continue adding elements to that set until every edge incident with that vertex has been represented, that is, the sets M_x, M_y corresponding to the end vertices x, y of the edge $e = xy$ intersect in at least two elements. On the other hand, we will maintain a 2-representation, that is, $|M_x \cap M_y| < 2$ if $xy \notin E(T)$. Then $|\bigcup_{x \in T} M_x|$ will provide our bound.

Case 1. Consider the edges of C-I. Apply Theorem 13 (see the note after the proof of Theorem 13) to the union of trees in $\mathcal{B}$ with $d = n^{1/4}$. We see that the forest $F_{\mathcal{B}} = \bigcup_{T \in \mathcal{B}} T$ can be 2-represented on a set of cardinality at most $11(d + 1)\sqrt{n} = O(n^{3/4})$. In other words, to each vertex $x \in V(F_{\mathcal{B}})$ we assigned a set M_x^1 in such a way that $|\bigcup M_x^1| = O(n^{3/4})$ and $\{M_x^1\colon x \in V(F_{\mathcal{B}})\}$ represents $F_{\mathcal{B}}$.

Case 2. Let $d_x = \deg_{S(x)}(x)$ denote the degree of x in the substar $S(x)$. Further, let

$$d_{x_1} \leq d_{x_2} \leq \cdots \leq d_{x_s} < \sqrt{n} \leq d_{x_{s+1}} \leq d_{x_{s+2}} \leq \cdots \leq d_{x_\ell}$$

where $\ell = |X|$. For each i, $1 \leq i \leq \ell$ we denote by $T_1(x_i), T_2(x_i), \ldots, T_{d_{x_i}}(x_i)$ the trees belonging to S with $\phi(T_j(x_i)) = x_i$, $1 \leq j \leq d_{x_i}$. Next we distinguish two cases, depending upon whether d_x is less than $\sqrt{n}$.

Case 2.1. In this subcase we represent the C-II edges contained in the trees $T(x_1), T(x_2), \ldots, T(x_s)$. We represent these edges in two stages:

(i) we represent the edges of the stars $S(x_1), \ldots, S(x_s)$ and separately

(ii) for each $j = 1, 2, \ldots, s$ the edges of $\bigcup_{\phi(T_i)=x_j} T_i$.

In both cases we make use of a structure which we call a partial affine plane.

Recall that an affine plane $A = (X, \mathcal{L})$ of order m is a structure of m^2 points, X, and $m^2 + m$ lines $\mathcal{L}$ each containing m-points of X, with the following properties: every pair of points is contained in exactly one line, the lines can be partitioned into $m + 1$ parallel classes containing m lines each, and 2 lines from different parallel classes intersect in exactly 1 point. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_\infty \cup \cdots \cup \mathcal{L}_m$ be the partition into parallel classes of A. Consider ℓ lines from the parallel class $\mathcal{L}_1$, and Y, the $m\ell$ points on these lines. Then we call $(Y, \bigcup_{i=1}^m \mathcal{L}_i)$ a *partial affine plane* of order (m, ℓ) with $m\ell$ points and m^2 lines each containing ℓ points of Y. We also use $\mathcal{L}(\mathcal{A}(\Updownarrow, \ell))$ for the set of all lines in $A(m, \ell)$.

Let $A_1 = A(n^{3/8}, 2n^{1/4})$. This partial affine plane has $2n^{5/8}$ points and $n^{3/4}$ lines, each of length $2n^{1/4}$. Consider the $1-1$ mapping

$$\psi: \{1, 2, \ldots, s\} \mapsto \mathcal{L}(\mathcal{A}_1).$$

(This is possible as $s \leq |X| \leq n^{3/4} = |\mathcal{L}(A_1)|$.) Also, for each $j = 1, 2, \ldots, s$, consider a set system with vertex set $L = \psi(j)$ 2-representing a star $S(x_j)$. (Such a set system exists as $\theta_2(S(x_j)) \leq 2\sqrt{n^{1/2}} = 2n^{1/4} = |L|$.) For a vertex $z \in S(x_j)$, $j = 1, 2, \ldots, s$, let M_x^2 be the corresponding set from the 2-representation of $S(x_j)$.

Let $A_2 = A(\sqrt{n}, 2n^{1/4})$ be a partial affine plane with n lines of length $2n^{1/4}$. Each tree $T_j(x_i)$, $1 \leq i \leq s$, $1 \leq j \leq d_{x_i} \leq \sqrt{n}$ requires at most $2n^{1/4}$ elements in its 2-representation and there are clearly at most n such trees. Consider a $1-1$ mapping

$$\psi_2: \{(1,1), (2,1), \ldots, (d_{x_1}), \ldots, (1, s), \ldots, (d_{x_s}, s)\} \mapsto \mathcal{L}(A_2)$$

with the property

$$\psi_2(j, i) \cap \psi_2(j', i) = \emptyset \tag{11}$$

that is, lines $\psi_2(1, i), \ldots, \psi_2(d_{x_i}, i)$ are contained in a parallel class.

Also for each pair (j, i), $1 \leq i \leq s$, $1 \leq j \leq d_{x_i}$, consider a set system with vertex set $L = \psi_2(j, i)$ 2-representing the tree $T_j(x_i)$. For each vertex $z \in T_j(x_i)$, let M_z^3 be the corresponding set from the 2-representation of $T_j(x_i)$.

Note that the assignment $z \mapsto M_z^2 \cup M_z^3$ is a legal 2-representation of $T(x_1), T(x_2), \ldots, T(x_s)$. This follows by a simple inspection using Eq. (11) and the fact that the trees $T_1(x_i), \ldots, T_{d_{x_i}}(x_i)$ are represented on pairwise disjoint sets.

Also, clearly

$$\left| \bigcup_{i=1}^{s} \bigcup_{z \in T(x_i)} M_z^2 \cup M_z^3 \right| = |A_1| + |A_2| = O(n^{5/8}) + O(n^{3/4}) = O(n^{3/4}).$$

Case 2.2. In this subcase, we represent the C-II edges contained in the trees,

$$T(x_{s+1}), T(x_{s+2}), \ldots, T(x_\ell).$$

As in Case 1.1, consider first the edges of the stars $S(x_{s+1}), S(x_{s+2}), \ldots, S(x_\ell)$.

This time however, we subdivide each $S(x_i)$, $s+1 \le i \le \ell$ into smaller stars $S(x_i) = S^1(x_i) \cup \cdots \cup S^{q_i}(x_i)$, each of cardinality at least $\sqrt{n}/2$ but not bigger than $\sqrt{n}$. Clearly, $\theta_2(S^j(x_i)) \le 2n^{1/4}$ for each $i = s+1, \ldots, \ell$ and $1 \le j \le q_i$. Consider the 2-representation $z \mapsto M_z^4$ of $S^j(x_i)$ on a set K_i^j, $|K_i^j| \le 2n^{1/4}$. Let $K = \bigcup K_i^j$ be a disjoint union. As the total number of edges in $S(x_{s+1}), S(x_{s+2}), \ldots, S(x_\ell)$ is smaller than n we infer that $|K| \le (n/(\sqrt{n}/2)2n^{1/4} = 4n^{3/4} = O(n^{3/4})$.

To represent the remaining part of the trees $T(x_{s+1}), T(x_{s+2}), \ldots, T(x_\ell)$, that is, the trees T_k with $\phi(T_k) = x_i$, $i = s+1, \ldots, \ell$, consider a partial affine plane $A_3 = A(\sqrt{n}, 2n^{1/4})$. For each $i = s+1, \ldots, \ell$ and $j = 1, 2, \ldots, q_i$ let

$$T_{ij} = \{T_k \in S\colon \phi(T_k) = x_i, V(T_k) \cap V(S^j(x_i)) \ne \emptyset\}.$$

Let $|T_{ij}| = t_{ij}$. We will consider a mapping

$$\psi_3\colon \bigcup_{i=s+1}^{\ell} \bigcup_{j=1}^{q_i} T_{ij} \mapsto \mathcal{L}(A_3)$$

so that for every $i = s+1, \ldots, \ell$ and $j = 1, \ldots, q_i$ the members of each T_{ij} get mapped into a set of parallel lines.

For each $T \in T_{ij}$ let $z \mapsto M_z^5$ be a 2-representation establishing $\theta_2(T) \le 2n^{1/4}$. An easy inspection shows that $z \mapsto M_z^4 \cup M_z^5$ is a 2-representation of edges belonging to $T(x_{s+1}), \ldots, T(x_\ell)$.

Altogether, this subcase requires at most

$$\left| \bigcup_{i=s+1}^{\ell} \bigcup_{z \in T(x_i)} M_z^4 \cup M_z^5 \right| = O(n^{3/4})$$

elements.

Case 3. Finally, we consider the edges of C-III. Consider a forest F with vertex set $X \cup \mathcal{B}$, that is, each $v \in V(F)$ corresponds to either an element of X or a tree from $\mathcal{B}$. As $|V(T')| \ge n^{1/4}$ for each $T' \in \mathcal{B}$, we have $|\mathcal{B}| \le O(n^{3/4})$ and as $|X| \le O(n^{3/4})$ we see that the number of edges in F (which are precisely type (a) and (b) edges of C-III) does not exceed $O(n^{3/4})$.

We show next that the number of edges of type (c) is bounded by $O(n^{3/4})$ as well. Let $S_\infty \subset S$ be a set of trees $T^* \in S$ which are connected to at least two vertices of X. For each $T^* \in S_\infty$ let $N_X(T^*) = \{x \in X\colon y \in V(T^*), \{x, y\} \in E(T)\}$.

The sets $\{N_X(T^*)\colon T^* \in S_\infty\}$ viewed as edges of a hypergraph do not form any cycles and thus,

$$\sum_{T^* \in S_\infty} (|N_X(T^*)| - 1) \le |X| - 1 \tag{12}$$

(c.f. [10], page 77). As the number of edges of type (c) equals $\sum_{T^* \in S_\infty} (|N_X(T^*)| - 1)$, we infer that the number of edges of type (c) is bounded by $O(n^{3/4})$ as well.

We now construct a 2-representation of edges belonging to C-III. To each edge $e \in$ C-III, assign a two-element set M_e with the property that $M_e \cap M_{e'} = \emptyset$ for distinct $e, e' \in$ C-III and for each $z \in V(T)$ set

$$M_z^6 = \bigcup\{M_e\colon z \in V(T)\}.$$

Clearly, $|M_{z_1}^6 \cap M_{z_2}^6| \geq 2$ if and only if $\{z_1, z_2\} \in$ C-III and due to the fact that the total number of edges in C-III is bounded by $O(n^{3/4})$ we see that

$$\left|\bigcup_{z \in V(T)} M_z^6\right| \leq O(n^{3/4}).$$

In summary then, $z \mapsto M_z = \bigcup_{i=1}^6 M_z^i$ is a 2-representation of T, that is, $|M_{z_1} \cup M_{z_2}| \geq 2$ if and only if $\{z_1, z_2\} \in E(T)$ and $|\bigcup_{z \in V(T)}| \leq O(n^{3/4})$, establishing the bound. ∎

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