On Ramsey Numbers of Forests Versus Nearly Complete Graphs

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ABSTRACT

A formula is presented for the ramsey number of any forest of order at least 3 versus any graph G of order $n \geq 4$ having clique number $n-1$. In particular, if T is a tree of order $m \geq 3$, then $r(T, G) = 1 + (m-1)(n-2)$.

INTRODUCTION

For graphs G_1 and G_2, the *ramsey number* $r(G_1, G_2)$ is the least positive integer p such that if every edge of the complete graph K_p is arbitrarily colored red or blue, then there exists either a red G_1 (a subgraph isomorphic to G_1 all of whose edges are colored red) or a blue G_2. Equivalently, $r(G_1, G_2)$ is the least positive integer p such that if $K_p = R \oplus B$ is an arbitrary factorization of K_p [i.e., R and B have order p and $E(R) \cup E(B)$ is a partition of $E(K_p)$], then G_1 is a subgraph of R (in symbols $G_1 \subset R$) or G_2 is a subgraph of B.

There has been great activity in recent years in the determination of ramsey numbers of certain pairs of specific graphs and, in some cases, of graphs belonging to prescribed classes. One of the best-known results of this latter type is due to Chvátal [3] who showed that

$$r(T, K_n) = 1 + (m-1)(n-1),$$

for every tree T of order m and every positive integer n.

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TREES VERSUS NEARLY COMPLETE GRAPHS

In this section we present a result analogous to Chvátal's. The following lemma will be useful, where P_3 denotes the path of order 3, $\beta_1(G)$ denotes the edge-independence number of a graph G, and $\bar{G}$ is the complement of G.

Lemma 1 (Chvátal and Harary [4]). For any graph G or order m without isolated vertices,

$$r(G, P_3) = \begin{cases} m, & \text{if } \bar{G} \text{ has a 1-factor;} \\ 2m - 2\beta_1(\bar{G}) - 1, & \end{cases}$$

First we establish the ramsey number of any tree of order at least 3 vs. P_3.

Theorem 1. If T_m is any tree of order $m \geq 3$, then

$$r(T_m, P_3) = \begin{cases} m+1, & \text{if } T_m \text{ is a star and } m \text{ is even;} \\ m, & \text{otherwise.} \end{cases}$$

Proof. If T_m is a star and m is even, then $\bar{T}_m$ does not have a 1-factor and and $\beta_1(\bar{T}_m) = \frac{1}{2}(m-2)$, so, by Lemma 1, $r(T_m, P_3) = m+1$. The remaining case also follows from Lemma 1 once we show that $\beta_1(\bar{T}_m) = [\frac{1}{2}m]$. This is verified by induction on m. If $m=3$ or $m=4$, the result is immediate; thus we assume that $m \geq 5$. Since T_m is not a star of even order and since $m \geq 5$, it is possible to remove two end-vertices from T_m so that the resulting tree T_{m-2} is not a star of even order. By the inductive hypothesis, $\beta_1(\bar{T}_{m-2}) = [\frac{1}{2}(m-2)]$. Since $\beta_1(\bar{T}_m) \geq \beta_1(\bar{T}_{m-2}) + 1$, we have $\beta_1(\bar{T}_m) = [\frac{1}{2}m]$. ∎

We now present a formula for the ramsey number of a tree versus $K_n - e$, which denotes the graph obtained by deleting an edge from K_n. Portions of the proof of the next result were suggested by Burr's proof [1] of Chvátal's theorem and are similar to techniques employed by Burr and Erdös [2].

Theorem 2. For each tree T_m of order $m \geq 3$ and each integer $n \geq 4$,

$$r(T_m, K_n - e) = 1 + (m-1)(n-2).$$

Proof. Since $K_{n-1} \subset K_n - e$, it follows by Chvátal's theorem that

$$1 + (m-1)(n-2) = r(T_m, K_{n-1}) \leq r(T_m, K_n - e).$$

We establish the reverse inequality $r(T_m, K_n - e) \leq 1 + (m-1)(n-2)$, and thus the theorem, by induction on $n (\geq 4)$. Observe first, however, that Lemma 1 implies that $r(T_3, K_n - e) = 2n - 3$ for all $n \geq 4$.

We begin with the case $n=4$, which is verified by induction on $m(\geq 3)$. Note that $r(T_3, K_4-e)=5$ (which is also stated in [4], for example). Assume for $m\geq 3$, that $r(T_m, K_4-e)=2m-1$, where T_m is an arbitrary tree of order m. Let T be an arbitrary tree of order $m+1$. We show that $r(T, K_4-e)=2m+1$. Suppose, to the contrary, that there exists a factorization $K_{2m+1}=R\oplus B$ such that $T\not\subset R$ and $K_4-e\not\subset B$.

Let v be an end-vertex of T and let u be the vertex of T adjacent with v. Then $T-v$ is a tree of order m. Since, by hypothesis, $r(T-v, K_4-e)=2m-1$, we may assume that $T-v\subset R$. Let S denote the set of $m+1$ vertices of K_{2m+1} that do not belong to the given red $T-v$. Since $T\not\subset R$, each edge joining u and a vertex of S is blue. If T is not a star of even order, then $r(T, P_3)=m+1$ by Theorem 1, so that the induced subgraph $\langle S\rangle$ contains a blue P_3, implying that $K_4-e\subset B$, a contradiction.

Assume next that T is a star of even order $m+1$. If $\langle S\rangle$ contains fewer than $\frac{1}{2}(m+1)$ (necessarily independent) blue edges, then S may be partitioned as $S_1\cup S_2$, where $|S_1|=1$, $|S_2|=m$, and all blue edges of $\langle S\rangle$ belong to $\langle S_2\rangle$. Thus, a red T is a subgraph of $\langle S\rangle$, which produces a contradiction. Otherwise, $\langle S\rangle$ contains exactly $\frac{1}{2}(m+1)$ blue edges. In this case, the red subgraph of $\langle S\rangle$ is $(m-1)$-regular. If only blue edges join S and the vertices of $T-v$, a blue K_4-e is produced; otherwise, a red T is produced. In either case, we arrive at a contradiction.

Next assume that

$$r(T_m, K_n-e)=1+(m-1)(n-2),$$

for a fixed but arbitrary integer $n\geq 4$ and for each $m\geq 3$. We prove that

$$r(T_m, K_{n+1}-e)=1+(m-1)(n-1),$$

for every $m\geq 3$ by induction on m. We have already seen that $r(T_3, K_{n+1}-e)=2n-1$. Hence we may assume now that

$$r(T_m, K_{n+1}-e)=1+(m-1)(n-1),$$

for a fixed $m\geq 3$ and prove that

$$r(T_{m+1}, K_{n+1}-e)=1+m(n-1).$$

Proceeding as above, we let T be an arbitrary tree of order $m+1$ and assume there exists a factorization $K_{1+m(n-1)}=R\oplus B$ such that $T\not\subset R$ and $K_{n+1}-e\not\subset B$. Let v be an end-vertex of T and let u be the vertex of T adjacent with v. Then $T-v$ is a tree of order m. By the inductive hypothesis on m,

$$r(T-v, K_{n+1}-e)=1+(m-1)(n-1),$$

which implies that $T-v\subset R$.

Let S denote the set of vertices of $K_{1+m(n-1)}$ that do not belong to the fixed red $T-v$. Observe that $|S|=1+m(n-2)$. By the inductive hypothesis on n,

$$r(T, K_n-e)=1+m(n-2),$$

implying that $\langle S\rangle$ contains a blue K_n-e. If any edge joining vertex u of $T-v$ and a vertex of K_n-e is red, a red T is produced; while if all such edges are blue, a blue $K_{n+1}-e$ is produced. Thus, a contradiction arises, completing the proof of the theorem. ∎

The conclusion in Theorem 2 may also be given in the form

$$r(T_m, K_{n-2}+\bar{K}_2)=1+(m-1)(n-2).$$

Bounds and some specific results on the ramsey numbers $r(T_m, K_l+\bar{K}_r)$ have been obtained by Rousseau and Sheehan [5].

The *clique number* of a graph G is the maximum order of a complete subgraph of G. Since $K_{n-1}\subset G\subset K_n-e$ for every graph G of order n and clique number $n-1$, we apply Theorem 2 and Chvátal's theorem to obtain the following result.

Corollary 2a. If T is any tree of order $m\geq 3$ and G is any graph of order $n\geq 4$ having clique number $n-1$, then

$$r(T, G)=1+(m-1)(n-2).$$

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Chvátal's theorem was extended in another direction by Stahl [6] who showed that if F is a forest, then

$$r(F, K_n)=\max_{1\leq j\leq m}\left\{(j-1)(n-2)+\sum_{i=j}^{m} i\cdot k_i\right\}, \tag{1}$$

where k_i is the number of components of F of order i and m is the largest order of a component of F. We now show that Stahl's proof technique can be used to extend Eq. (1). Following Stahl [6], we begin with a lemma.

Lemma 2. For each forest F of order at least 3 which consists of k trees, each of order m, and for each integer $n\geq 4$,

$$r(F, K_n-e)=mk+(m-1)(n-3).$$

Proof. By Stahl's theorem,

$$mk+(m-1)(n-3)=r(F, K_{n-1})\leq r(F, K_n-e).$$

With the aid of Theorem 2 we now show that

$$r(F, K_n - e) \leq mk + (m-1)(n-3),$$

which is verified by induction on $k(\geq 1)$. For $k = 1$, the result follows directly by Theorem 2. Assume the lemma is true for all forests consisting of $k-1$ trees $(k > 1)$, each of order $m > 3$, and let F be any forest with k trees, each of order m. Let $l = mk + (m-1)(n-3)$ and consider an arbitrary factorization $K_l = R \oplus B$ such that $K_n - e \not\subset B$. We show that $F \subset R$.

Let T be a component of F. By employing Theorem 2, we conclude that $T \subset R$. Observe that

$$K_l - V(T) = [(K_l - V(T)) \cap R] \oplus [(K_l - V(T)) \cap B]$$

is a factorization of the complete graph of order

$$l - n = m(k-1) + (m-1)(n-3).$$

By the inductive hypothesis,

$$F - V(T) \subset (K_l - V(T)) \cap R.$$

Hence $F \subset R$.

We now proceed to the case of an arbitrary forest of order at least 3. For a forest F, recall that k_i denotes the number of components of order i and that m is the largest order of a component of F.

Theorem 3. For each forest F of order at least 3 and each integer $n \geq 4$,

$$r(F, K_n - e) = \max_{1 \leq j \leq m} \left\{ (j-1)(n-3) + \sum_{i=j}^{m} i \cdot k_i \right\}.$$

Proof. By Stahl's theorem

$$\max_{1 \leq j \leq m} \left\{ (j-1)(n-3) + \sum_{i=j}^{m} i \cdot k_i \right\} = r(F, K_{n-1}) \leq r(F, K_n - e).$$

With the aid of Lemma 2 we verify the reverse inequality

$$r(F, K_n - e) \leq \max_{1 \leq j \leq m} \left\{ (j-1)(n-3) + \sum_{i=j}^{m} i \cdot k_i \right\}. \tag{2}$$

For $1 \leq j \leq m$, it is convenient to let F_j denote the subforest of F consisting of all components of order at least j. Observe that F_j has order $\sum_{i=j}^{m} i \cdot k_i$.

Suppose that the maximum in Eq. (2) is assumed for $j = j_0$ and let

$$p_0 = \sum_{i=j_0}^{m} i \cdot k_i.$$

The maximum in Eq. (2) is then $l = p_0 + (j_0 - 1)(n-3)$.

Let $K_l = R \cup B$ be an arbitrary factorization where $K_n - e \not\subset B$. We show that $F = F_1 \subset R$ by descending induction on j. Note that $F_m \subset R$ by Lemma 2. Assume that $F_{j+1} \subset R$, where $1 \leq j < m$. We show that $F_j \subset R$. If $F_j = F_{j+1}$, the result is obvious. Otherwise, $F_j - V(F_{j+1})$ consists of k_j trees, each of order j. Note that

$$K_l - V(F_{j+1}) = [(K_l - V(F_{j+1})) \cap R] \oplus [(K_l - V(F_{j+1})) \cap B]$$

is a factorization of the complete graph of order $l - \sum_{i=j+1}^{m} i \cdot k_i$. By the definition of l, it follows that

$$l - \sum_{i=j+1}^{m} i \cdot k_i \geq j \cdot k_j + (j-1)(n-3).$$

By Lemma 2,

$$r(F_j - V(F_{j+1}), K_n - e) = j \cdot k_j + (j-1)(n-3).$$

Hence

$$F_j - V(F_{j+1}) \subset (K_l - V(F_{j+1})) \cap R,$$

so that $F_j \subset R$.

Thus, by induction, $F_1 = F \subset R$, completing the proof. ∎

Theorem 3 and Stahl's theorem now imply the following.

Corollary 3a. If F is any forest of order at least 3 and G is any graph of order $n \geq 4$ having clique number $n-1$, then

$$r(F, G) = \max_{1 \leq j \leq m} \left\{ (j-1)(n-3) + \sum_{i=j}^{m} i \cdot k_i \right\}.$$

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