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Generalized Degree Conditions for Graphs with Bounded Independence Number

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ABSTRACT

We consider a generalized degree condition based on the cardinality of the neighborhood union of arbitrary sets of r vertices. We show that a Dirac-type bound on this degree in conjunction with a bound on the independence number of a graph is sufficient to imply certain hamiltonian properties in graphs. For $K_{1,m}$-free graphs we obtain generalizations of known results. In particular we show:

Theorem. Let $r \geq 1$ and $m \geq 3$ be integers. Then for each non-negative function $f(r,m)$ there exists a constant $C = C(r,m,f(r,m))$ such that if G is a graph of order n ($n \geq r, n > m$) with $\delta_r(G) \geq (n/3) + C$ and $\beta(G) \leq f(r,m)$, then

(a) G is traceable if $\delta(G) \geq r$ and G is connected;
(b) G is hamiltonian if $\delta(G) \geq r + 1$ and G is 2-connected;
(c) G is hamiltonian-connected if $\delta(G) \geq r + 2$ and G is 3-connected.

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Dirac [2] proved that if G is a graph of order $n \geq 3$ with $\delta(G) \geq n/2$, then G is hamiltonian. In [5], Matthews and Sumner lowered the minimum degree condition for hamiltonicity by imposing the condition that G be clawfree (i.e., G contains no induced subgraph isomorphic to $K_{1,3}$).

Theorem A [5]. If G is a 2-connected $K_{1,3}$-free graph of order $n \geq 3$ with $\delta(G) \geq (n-2)/3$, then G is hamiltonian.

Recently, Markus [4] obtained similar results for $K_{1,m}$-free graphs, $m \geq 3$.

Theorem B [4]. If G is a 2-connected $K_{1,m}$-free graph of order $n \geq 3$ with $\delta(G) \geq (n+m-2)/3$, then G is hamiltonian.

Both of the previous theorems have analogs for traceable graphs and hamiltonian-connected graphs.

The idea of minimum degree can be generalized as follows. For a graph G of order n and $r \leq n$, define

$$\delta_r(G) = \min_{\substack{S \subseteq V(G) \\ |S|=r}} | \cup_{u \in S} N(u)| .$$

Then, of course, $\delta(G) = \delta_1(G)$. In [3], the following results involving $\delta_2(G)$ were established.

Theorem C [3]. If G is connected $K_{1,3}$-free graph of order n such that $\delta_2(G) \geq (n+1)/3$, then for n sufficiently large G is traceable.

Theorem D [3]. If G is a 2-connected $K_{1,3}$-free graph of order n such that $\delta_2(G) \geq (n+1)/3$, then for n sufficiently large G is hamiltonian.

Theorem E [3]. If G is a 3-connected $K_{1,3}$-free graph of order n such that $\delta_2(G) \geq (n+24)/3$, then for n sufficiently large G is hamiltonian-connected.

Here we will prove results that in some sense incorporate and generalize Theorems A–E. Undefined terms and notations can be found in [1]. We begin with Theorem 1, which establishes sufficient conditions for traceability, hamiltonicity, and hamiltonian–connectedness based on $\delta_r(G)$ and the independence number $\beta(G)$ of a graph G.

Theorem 1. Let $r \geq 1$ and $m \geq 3$ be integers. Then for each non-negative function $f(r,m)$ there exists a constant $C = C(r,m,f(r,m))$ such that if G is a graph of order n ($n \geq r, n > m$) with $\delta_r(G) \geq (n/3) + C$ and $\beta(G) \leq f(r,m)$ then

(a) G is traceable if $\delta(G) \geq r$ and G is connected;

(b) G is hamiltonian if $\delta(G) \geq r + 1$ and G is 2-connected;
(c) G is hamiltonian-connected if $\delta(G) \geq r + 2$ and G is 3-connected.

Proof. We proceed by induction on n and assume that (a), (b), and (c) have been established for all graphs of order less than n. (The proof is anchored by selecting C large.) Let G be a graph of order n such that $\delta_r(G) \geq (n/3) + C$ and $\beta(G) \leq f(r,m)$. Assume that G satisfies the hypotheses of (a), (b), or (c). We first show that

(i) if G satisfies the hypotheses of (a), then G has a path of order at least $(2n/3) - (2r/3)$;
(ii) if G satisfies the hypotheses of (b), then G has a cycle of order at least $(2n/3) - (2r/3)$;
(iii) if G satisfies the hypotheses of (c), then G has a $u - v$ path of order at least $(2n/3) - (2r/3)$ for each pair $u, v \in V(G)$.

Let X denote a longest path of G, longest cycle of G, or longest $u - v$ path of G depending on whether we are in (i), (ii), or (iii). We first show that $|V(X)| \geq n/6r$. Since $\delta_r(G) \geq (n/3) + C$, for C sufficiently large every vertex of G with at most $r - 1$ exceptions has degree at least $(n/3) + r - 1$. Let S be the set of vertices of degree less than $(n/3r) + r - 1$ and let $H = \langle V(G) - S \rangle$. Then every vertex of H has degree at least $n/3r$ (in H). Let P be a longest path in H, with initial vertex w. Then every adjacency of w in H is on P so that one of these adjacencies together with a segment of P forms a cycle C in H with at least $n/3r$ vertices. This cycle (or path) is also in G. It is straightforward to use this cycle to show that in the hamiltonian-connected case, any two vertices u and v can be joined by a path using at least half the vertices of the cycle. Thus, in all cases, $|V(X)| \geq n/6r$.

Next, if L denotes the vertices of G not on X of degree less than C/r, then $|L| \leq r - 1$. Thus, if $V(G) = V(X) \cup L$, then $|V(X)| \geq n - r + 1 \geq (2n/3) - (2r/3)$. Assume, then, that $V(G) \neq V(X) \cup L$.

We wish to show that the removal of l vertices from $G - V(X) - L$, $0 \leq l \leq 2$, results in at most two components, and each such component H satisfies

$$|V(H)| \geq \frac{n}{3} + C - f(r,m) - r - 2 \tag{1}$$

$$\delta(H) \geq r + 2 \tag{2}$$

$$\delta_r(H) \geq \frac{|V(H)|}{3} + C\,. \tag{3}$$

To do so, let H be such a component and $w \in V(H)$. Then $\deg_G w \geq C/r$. Suppose G satisfies the hypotheses of (a) and let X: $v_1, v_2, \ldots, v_k$. If w is adjacent to v_i and v_j, $1 \leq i < j < k$, then $v_{i+1}v_{j+1} \notin E(G)$; otherwise, the path

$$X'\colon v_1, v_2, \ldots, v_i, w, v_j, v_{j-1}, \ldots, v_{i+1}, v_{j+1}, \ldots, v_k$$

has order greater than X. Thus, since $\beta(G) \leq f(r,m)$ we have that $\deg_X w \leq \beta(G) + 1 \leq f(r,m) + 1$ (where the extra 1 is only needed in the hamiltonian-connected case). Similarly, if G satisfies the hypotheses of (b) or (c), then $\deg_X w \leq f(r,m) + 1$. Thus,

$$\deg_H w \geq \frac{C}{r} - f(r,m) - 1 - (r-1) - l \geq r + 2$$

for C sufficiently large. Thus, $\delta(H) \geq r + 2$. Let S be a set of r vertices of H. Then

$$|N_G(S)| \geq \frac{n}{3} + C\,.$$

However, since H is connected and $\beta(G) \leq f(r,m)$ we have that $|N_X(S)| \leq f(r,m) + 1$. Thus

$$\begin{aligned} |N_H(S)| &\geq \frac{n}{3} + C - f(r,m) - 1 - (r-1) - l \\ &\geq \frac{n}{3} + C - f(r,m) - r - 2\,. \end{aligned}$$

Thus, $|V(H)| \geq (n/3) + C - f(r,m) - r - 2 > n/3$ for C sufficiently large, so the removal of l vertices from $G - V(X) - L$ results in at most two components. Since $\delta_r(H) \geq (n/3) + C$, it follows that $n \geq C$. Thus, by choosing C at least $18rf(r,m) + 18r^2 + 36r$, we have that

$$\frac{n}{18r} \geq f(r,m) + r + 2$$

so that

$$\frac{n}{3} + C - f(r,m) - r - 2 \geq \frac{n}{3} - \frac{n}{18r} + C \geq \frac{|V(H)|}{3} + C\,.$$

Since each component H of $G - V(X) - L$ satisfies (1), (2), and (3) and has independence number at most $f(r,m)$, it follows by induction that each such component is traceable. Furthermore, any 2-connected component is hamiltonian and any 3-connected component is hamiltonian-connected. Also, if $|V(X)| > (n/3) - 2(C - f(r,m) - r - 2)$, then $G - V(X) - L$ consists of one component, which is necessarily 3-connected.

Assume now that G satisfies the hypotheses of (a). We wish to show that $|V(X)| \geq (2n/3) - (2r/3)$. Since each component H of $G - V(X) - L$ is traceable and X is a longest path, we conclude that

$$|V(X)| \geq |V(H)| \geq \frac{n}{3} + C - f(r,m) - r - 2$$
$$> \frac{n}{3} - 2(C - f(r,m) - r - 2)$$

for C sufficiently large. Thus, $G - V(X) - L$ is hamiltonian-connected.

Let X: $v_1, v_2, \ldots, v_k$. Since G is connected there is a path from $V(G) - V(X) - L$ to some vertex v_i on X. Let P_1 be a shortest such path and let w be the vertex of $G - V(X) - L$ on P_1. Let z be any other vertex of $G - V(X) - L$ and let P_2 be any hamiltonian $z - w$ path in $G - V(X) - L$. Finally, let P_3 denote the longer of the subpaths $v_1, v_2, \ldots, v_i$ and $v_i, v_{i+1}, \ldots, v_k$ of X. Then

$$P_2, P_1, P_3$$

is a path of G of order at least

$$n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}$$

Since, by assumption, X is a longest path in G, it follows that

$$|V(X)| \geq n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}.$$

and so $|V(X)| \geq (2n/3) - (2r/3)$.

Assume next that G satisfies the hypotheses of (b). If $G - V(X) - L$ is 2-connected, then $G - V(X) - L$ is hamiltonian. If $\kappa(G - V(X) - L) \leq 1$, then the removal of 0 or 1 vertices results in two 2-connected components, each of order at least $(n/3) + C - f(r,m) - r - 2$. In either case, we obtain a hamiltonian subgraph of G of order at least $(n/3) + C - f(r,m) - r - 2$. Since X is a longest cycle of G, we conclude that $|V(X)| \geq (n/3) + C - f(r,m) - r - 2$. Thus, $G - V(X) - L$ is hamiltonian-connected for C sufficiently large.

Let X: $v_1, v_2, \ldots, v_k, v_1$. Since G is 2-connected, there are two vertex-disjoint paths, the first from $V(G) - V(X) - L$ to $V(X)$ and the second from $V(X)$ to $V(G) - V(X) - L$. Let P_1, P_2 be a shortest pair of such paths. Assume, without loss of generality, that P_1 intersects $V(X)$ at v_i and P_2 intersects $V(X)$ at v_j, with $i < j$. Let w be the initial vertex of P_1 and let z be the final vertex of P_2. Let P_3 be any hamiltonian $z - w$ path of $G - V(X) - L$, and finally, let P_4 denote the longer of the subpaths $v_i, v_{i+1}, \ldots, v_j$ and $v_i, v_{i-1}, \ldots, v_j$ of X. Then

$$P_1, P_4, P_2, P_3$$

is a cycle of G of order at least

$$n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}.$$

Since, by assumption, X is a longest cycle in G, it follows that

$$|V(X)| \geq n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}.$$

and so $|V(X)| \geq (2n/3) - (2r/3)$.

Next, assume that G satisfies the hypotheses of (c). In this case, G also satisfies the hypotheses of (b). Thus, a longest cycle of G has order at least $(2n/3) - (2r/3)$. This implies that

$$|V(X)| \geq \frac{n}{3} - \frac{r}{3} > \frac{n}{3} - 2(C - f(r,m) - r - 2)$$

for C sufficiently large, and so $G - V(X) - L$ is hamiltonian-connected. Since G is 3-connected, there are three vertex-disjoint paths from $V(G) - V(X) - L$ to $V(X)$. Using two of these paths, a hamiltonian path in $G - V(X) - L$, and all but an appropriate segment of X we conclude that $|V(X)| \geq (2n/3) - (2r/3)$.

Thus, we have established that if G satisfies the hypotheses of (a), (b), or (c), then G has a path, cycle or $u - v$ path, respectively, of order at least $(2n/3) - (2r/3)$. If G satisfies the hypotheses of (a), let α denote the maximum number of vertices of degree less than C/r on a path of order at least $(2n/3) - (2r/3)$, and let Y be a longest path containing α vertices of degree less than C/r. Define α similarly if G satisfies the hypotheses of (b) or (c) and obtain either a longest cycle Y or a longest $u - v$ path Y containing α vertices of degree less than C/r.

If $G - V(Y)$ has a vertex w such that $\deg_G w \geq C/r$, then in a manner analogous to earlier arguments, we can show that $G - V(Y)$ has a component H with $|V(H)| \geq (n/3) + C - f(r,m) - 1$ that, for C sufficiently large, contradicts the fact that $|V(Y)| \geq (2n/3) - (2r/3)$. Thus, every vertex of G of degree at least C/r lies on Y. We complete the proof by showing that every vertex of G of degree less than C/r also lies on Y. Assume, to the contrary, that there are $\gamma > 0$ vertices of degree less than C/r that do not lie on Y. Since the number of vertices of G of degree less than C/r is at most $r - 1$, we have $\alpha + \gamma \leq r - 1$ and $r \geq 2$.

Assume first that G satisfies the hypotheses of (a). Let Y: $v_1, v_2, \ldots, v_k$ and let $w \in V(G) - V(Y)$. Since $\delta(G) \geq r$, we have $\deg_G w \geq r$. Thus, $\deg_Y w \geq r - (\gamma - 1) = (r - 1) - \gamma + 2 \geq \alpha + 2$. Furthermore, by the definition of α, neither v_1 nor v_k is adjacent to w. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_{\alpha+2}}$ be $\alpha + 2$ adjacencies of w on Y, $i_1 \leq i_2 \leq \cdots \leq i_{\alpha+2}$.

Let $I_0 = \{v_1, v_2, \ldots, v_{i_1-1}\}$, let $I_{\alpha+2} = \{v_{i_{\alpha+2}+1}, v_{i_{\alpha+2}+2}, \ldots, v_k\}$ and for $j = 1, 2, \ldots, \alpha + 1$ let

$$I_j = \{v_{i_j+1}, v_{i_j+2}, \ldots, v_{i_{j+1}-1}\}.$$

Since Y contains exactly α vertices of degree less than C/r, it follows that three of the sets $I_0, I_1, \ldots, I_{\alpha+2}$ contain no vertices of degree less than C/r. Let I_s be the smallest such set. If $1 \leq s \leq \alpha + 1$, let

$$P\colon v_1, v_2, \ldots, v_{i_s}, w, v_{i_{s+1}}, v_{i_{s+1}+1}, \ldots, v_k.$$

If $s = 0$, let

$$P\colon v_{i_2-1}, v_{i_2-2}, \ldots, v_{i_1}, w, v_{i_2}, v_{i_2+1}, \ldots, v_k.$$

If $s = \alpha + 2$, let

$$P\colon v_1, v_2, \ldots, v_{i_{\alpha+1}}, w, v_{i_{\alpha+2}}, v_{i_{\alpha+2}-1}, \ldots, v_{i_{\alpha+1}+1}.$$

Then P contains $\alpha + 1$ vertices of degree less than C/r. By the choice of α, then, this means that P has order less than $(2n/3) - (2r/3)$. However,

$$\begin{aligned}
|V(P)| &\geq n - \left[(\gamma - 1) + \frac{n - (\gamma - 1) - (\alpha + 2)}{3}\right] \\
&= n - \left(\frac{n + 2\gamma - \alpha - 4}{3}\right) \\
&\geq n - \left(\frac{n + 2\gamma - 4}{3}\right) \\
&\geq n - \left(\frac{n + 2(r - 1) - 4}{3}\right) \\
&= n - \left(\frac{n + 2r - 6}{3}\right) = \frac{2n}{3} - \frac{2r}{3} + 2,
\end{aligned}$$

which gives a contradiction. Thus, Y contains every vertex of degree less than C/r, which completes the proof in the case that G satisfies the hypotheses of (a).

If G satisfies the hypotheses of (b) or (c), the proof is completed in an analogous manner. In these cases, we have $\delta(G) \geq r + 1$ or $\delta(G) \geq r + 2$, respectively, so that every vertex $w \in V(G) - V(Y)$ has $\deg_X w \geq \alpha + 3$ or $\deg_X w \geq \alpha + 4$. In either case, we are able to contradict the choice of α. This completes the proof of Theorem 1. ∎

An immediate corollary of Theorem 1 provides the result that in some sense generalizes Theorems A–E.

Corollary. Let $r \geq 1$ and $m \geq 3$ be integers. Then there exists a constant $C = C(r, m)$ such that if G is a $K_{1,m}$-free graph of order n ($n \geq r, n > m$) with $\delta_r(G) \geq (n/3) + C$, then

(a) G is traceable if $\delta(G) \geq r$ and G is connected;
(b) G is hamiltonian if $\delta(G) \geq r + 1$ and G is 2-connected;
(c) G is hamiltonian-connected if $\delta(G) \geq r + 2$ and G is 3-connected.

Proof. It suffices to show that if G is a $K_{1,m}$-free graph of order n and $\delta_r(G) > n/3$, then $\beta(G) \leq 3(m-1)r$. Let $t = \beta(G)$. If $t < r$ then we are done. Otherwise, let T be a set of t independent vertices of G and let $S = V(G) - T$. Since G is $K_{1,m}$-free, each vertex of S is adjacent to at most $m - 1$ vertices of T. Thus, the number of edges from S to T is at most $(m-1)(n-t)$. However, if T' is a set of r vertices of T, then $|N_G(T')| > n/3$. Thus, the number of edges from T' to S is greater than $n/3$. It follows that the number of edges from T to S is greater than

$$\frac{\binom{t}{r}\left(\frac{n}{3}\right)}{\binom{t-1}{r-1}}.$$

Thus, $(m-1)(n-t) > \binom{t}{r}(n/3)/\binom{t-1}{r-1}$. This however, implies that $t \leq 3(m-1)r$, which completes the proof of the corollary. ∎

Since Theorems A–D are best possible with respect to the bounds on $\delta_1(G)$ and $\delta_2(G)$, the bound given on $\delta_r(G)$ in the corollary is of the correct order of magnitude. The graph G of Figure 1 indicates that a minimum degree condition of at least $r - 1$ is required in (a). The connected $K_{1,m}$-free graph G satisfies $\delta_r(G) \geq (n - r + 1)/2$ and $\delta(G) = r - 2$. However, G is not traceable.

The graph G of Figure 2 indicates that a minimum degree condition of at least $r - 1$ is also required in (b) for $r \geq 4$. The 2-connected $K_{1,m}$-free

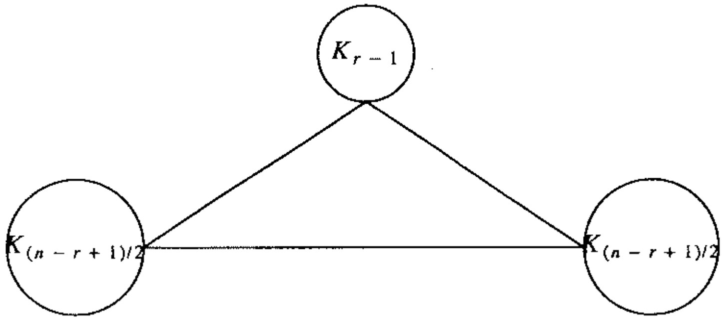

FIGURE 1

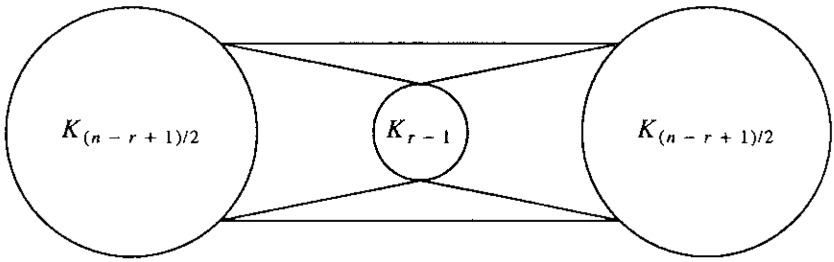

FIGURE 2

graph G satisfies $\delta_r(G) \geq (n - r + 1)/2$ and $\delta(G) = r - 2$. However, G is not hamiltonian.

In our next result we restrict ourselves to lower bounds on $\delta_3(G)$ in $K_{1,3}$-free graphs. Here we can lower the minimum degree conditions for traceable, hamiltonian and hamiltonian-connected from $r = 3$, $r + 1 = 4$ and $r + 2 = 5$ to 2, 3, and 4 respectively. We observe that in this case, the property of being $K_{1,3}$-free is used heavily throughout the proof. Furthermore, the constant C in the statement of Theorem 2 must be chosen so that n is sufficiently large for Theorem E to be applicable.

Theorem 2. There exists a constant C such that if G is a $K_{1,3}$-free graph of order n with $\delta_3(G) \geq (n/3) + C$, then

(a) G is traceable if $\delta(G) \geq 2$ and G is connected;
(b) G is hamiltonian if $\delta(G) \geq 3$ and G is 2-connected;
(c) G is hamiltonian-connected if $\delta(G) \geq 4$ and G is 3-connected.

Proof. We proceed by induction on n and assume that (a), (b), and (c) have been established for all graphs of order less than n. Let G be a $K_{1,3}$-free graph of order n such that $\delta_3(G) \geq (n/3) + C$, and assume that G satisfies the hypotheses of (a), (b), or (c). Since G is $K_{1,3}$-free, $\beta(G) \leq 18$.

In a manner analogous to the proof of Theorem 1, we can show that G has a path, cycle, or $u - v$ path of order at least $(2n/3) - 2$, depending on whether G satisfies the hypotheses of (a), (b), or (c). This, however, implies that G has a path, cycle, or $u - v$ path X that contains all vertices of G of degree at least $C/3$. Thus, $|V(X)| \geq n - 2$. To complete the proof, we show that G has a path, cycle or $u - v$ path Y of order at least $(2n/3) - 2$ that contains all vertices of G of degree less than $C/3$.

Suppose, first, that G has exactly one vertex y of degree less than $C/3$. If y is on X, then let $Y = X$. If y is not on X, then since $\deg_G y$ is at least 2, 3, or 4 depending on whether G satisfies the hypotheses of (a), (b), or (c), we can delete an appropriate segment of X and add y together with two adjacent edges to obtain the required Y. Thus, we assume that G has two vertices x and y of degree less than $C/3$. If both x and y are on X, let $Y = X$. Suppose, then, that at least one of x and y are not on X.

I. Assume that G satisfies the hypotheses of (c).

Case 1. Suppose $xy \notin E(G)$ and that exactly one of x and y, say x, is on X. If $\deg_G y > 4$, then we can delete an appropriate segment of X and add y to obtain the required $u - v$ path Y. Thus, we may assume that $\deg_G y = 4$.

Let $X: u = x_1, x_2, \ldots, x_{n-1} = v$ and suppose $N_G(y) = \{x_i, x_j, x_k, x_l\}$, where $i < j < k < l$. Let $x = x_t$. We may assume $i < t < l$; otherwise, we can easily obtain the desired Y. Then (by symmetry) either $j < t < k$ or $k < t < 1$.

Subcase (i). Suppose $j < t < k$. Then $j \geq i + \lfloor n/3 \rfloor + 4$; otherwise, let

$$Y: u = x_1, x_2, \ldots, x_i, y, x_j, x_{j+1}, \ldots, x_{n-1} = v\,.$$

Similarly, $l \geq k + \lfloor n/3 \lfloor + 4$. Furthermore, since G is $K_{1,3}$-free and $\deg_G y = 4$, it follows that $x_{j-1}x_{j+1} \in E(G)$. Consider the vertex x_j. Since $\delta_3(G) \geq (n/3) + C$, we have that $|N_G\{x, y, x_j\}|(n/3) + C$. Since $\deg_G x$ and $\deg_G y$ are less than $C/3$, it follows that $\deg_G x_j > n/3$. Thus $x_jx_p \in E(G)$ for some p with $i + 1 \leq p \leq i + \lceil n/3 \rceil$ or $l - \lceil n/3 \rceil \leq p \leq l - 1$. Thus we have either

$$Y: u = x_1, x_2, \ldots, x_i, y, x_j, x_p, x_{p+1}, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{n-1} = v$$

or

$$Y: u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p, x_j, y, x_l x_{l+1} \ldots, x_{n-1} = v\,.$$

Subcase (ii). Suppose $k < t < l$. Then, necessarily, $j \geq i + \lfloor n/3 \rfloor + 4$ and $k \geq j + \lfloor n/3 \rfloor + 4$. Furthermore, since G is $K_{1,3}$-free, $x_{j-1}x_{j+1} \in E(G)$. As in the previous case, $\deg_G x_j > n/3$. Thus, $x_j, x_p \in E(G)$ for some p with $i + 1 \leq p \leq i + \lceil n/3 \rceil$ or $k - \lceil n/3 \rceil \leq p \leq k - 1$. Thus we have either

$$Y: u = x_1, x_2, \ldots, x_i, y, x_j, x_p, x_{p+1}, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{n-1} = v$$

or

$$Y: u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_p, x_j, y, x_k, x_{k+1} \ldots, x_{n-1} = v\,.$$

Case 2. Suppose $xy \notin E(G)$ and that neither x nor y is on X. Since $\deg_G x \geq 4$, we obtain a $u - v$ path X' of order at least $(2n/3) - 2$ that contains x. If we choose a longest such path X' then X' contains all vertices of degree at least $C/3$. If x and y are on X', let $Y = X'$ and if only x is on X' we may proceed as in Case 1.

Case 3. Suppose $xy \in E(G)$ and one of x and y, say x is on X. Since $xy \in E(G)$ and $\deg_G y \geq 4$, we can clearly add y and delete an appropriate segment of X to obtain the required $u - v$ path Y.

Case 4. Suppose $xy \in E(G)$, neither x nor y is on X and $|N_X(\{x,y\})| \geq 4$. We then obtain a $u - v$ path X' of order at least $(2n/3) - 2$ that contains one or both of x and y. A longest such path X' contains all vertices of degree at least $C/3$. If x and y are on X', let $Y = X'$; otherwise, we may proceed as in Case 3.

Case 5. Suppose $xy \in E(G)$, neither x nor y is on X and $|N_X(\{x,y\})| = 3$. Let

$$X\colon u = x_1, x_2, \ldots, x_{n-2} = v$$

and suppose $N_X(\{x,y\}) = \{x_i, x_j, x_k\}$ where $i < j < k$. We may assume that $j \geq i + \lfloor n/3 \rfloor + 4$ and $k \geq j + \lfloor n/3 \rfloor + 4$ since otherwise we can easily obtain the desired $u - v$ path Y. As in Case 1, $x_{j-1}x_{j+1} \in E(G)$ and $\deg_G x_j > n/3$. Thus, $x_j x_p \in E(G)$ for some p with $i + 1 \leq p \leq i + \lceil n/3 \rceil$ or $k - \lceil n/3 \rceil \leq p \leq k - 1$. Thus, we have either

$$Y\colon u = x_1, x_2, \ldots, x_i, x, y, x_j, x_p, x_{p+1}, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{n-2} = v$$

or

$$Y\colon u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_p, x_j, x, y, x_k x_{k+1} \ldots, x_{n-2} = v\,.$$

II. Assume that G satisfies the hypotheses of (b).

Case 1. Suppose $xy \notin E(G)$ and that exactly one of x and y, say x, is on X. If $\deg_G y > 3$, then we can delete an appropriate segment of X and add y to obtain the required cycle X. Thus, we may assume that $\deg_G y = 3$.

Let $X\colon x_1, x_2, \ldots, x_{n-1}, x_1$ and suppose $N_G(y) = \{x_i, x_j, x_k\}$, where $i < j < k$. Without loss of generality, we may assume that $x = x_t$, where $k < t \leq n - 1$. As in previous cases, we may assume that $j \geq i + \lfloor n/3 \rfloor + 4$, $k \geq j + \lfloor n/3 \rfloor + 4$, $x_{j-1}x_{j+1} \in E(G)$, and $\deg_G x_j > n/3$. Thus $x_j x_p \in E(G)$ for some p with $i + 1 \leq p \leq i + \lceil n/3 \rceil$ or $k - \lceil n/3 \rceil \leq p \leq k - 1$. In either case, we obtain the desired cycle Y.

Case 2. Suppose $xy \notin E(G)$ and that neither x nor y is on X. Since $\deg_G x \geq 3$, we may proceed as in I, Case 2.

Case 3. Suppose $xy \in E(G)$ and that exactly one of x and y is on X, say x. Since $xy \in E(G)$ and $\deg_G x \geq 3$, we may proceed as in I, Case 3.

Case 4. Suppose $xy \in E(G)$, neither x nor y is on X and $|N_X(\{x, y\})| \geq 3$. Here we may proceed as in I, Case 4.

Case 5. Suppose $xy \in E(G)$, neither x nor y is on X and $|N_X(\{x, y\})| = 2$. Let

$$X: x_1, x_2, \ldots, x_{n-2}, x_1$$

and assume, without loss of generality, that $N_X(\{x, y\}) = \{x_1, x_j\}$, where $j < n - 2$. We may also assume $j \geq \lfloor n/3 \rfloor + 5$ and $n - 2 \geq j + \lfloor n/3 \rfloor + 3$; otherwise we easily obtain the desired cycle Y. As in previous cases, $x_{j-1}, x_{j+1} \in E(G)$ and $\deg_G x_j > n/3$. Thus, $x_j x_p \in E(G)$ for some p with $2 \leq p \leq \lceil n/3 \rceil + 1$ or $n - \lceil n/3 \rceil - 1 \leq p \leq n - 2$. In either case, we obtain the desired cycle Y.

III. Assume that G satisfies the hypotheses of (a).

Cases 1–4 follow exactly as they did in I and II. We list them without proof.

Case 1. Suppose $xy \notin E(G)$ and exactly one of x and y, say x, is on X.

Case 2. Suppose $xy \notin E(G)$ and that neither x nor y is on X.

Case 3. Suppose $xy \in E(G)$ and exactly one of x and y is on X.

Case 4. Suppose $xy \in E(G)$, neither x nor y is on X, and $|N_X(\{x, y\})| \geq 2$

Case 5. Suppose $xy \in E(G)$, neither x nor y is on X, and $|N_X(\{x, y\})| = 1$. Consider the connected graph $G' = G - \{x, y\}$. Since each of x and y has degree 2 in G it follows that $\delta(G') > n/3$. If G' is 2-connected then, by the Matthews-Sumner result G' is hamiltonian and we obtain a hamiltonian path Y in G. If G' has a cutvertex w, consider $G' - w$. Then, since $\delta(G' - w) > n/3$ and, consequently, $\delta_2(G' - w) > n/3$, it follows that $G' - w$ has exactly two components, both of which are 3-connected and hence hamiltonian-connected by Theorem E. But then since G is $K_{1,3}$-free, G has a hamiltonian path Y and the proof is complete. ∎

The graph G of Figure 1, with $r = 3$, indicates that for the traceable case, $\delta(G) \geq 2$ is a necessary condition.

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