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Degree conditions and cycle extendability ☆

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Abstract

A non-Hamiltonian cycle C in a graph G is extendable if there is a cycle C' in G with $V(C') \supset V(C)$ with one more vertex than C. For any integer $k \geqslant 0$, a cycle C is k-chord extendable if it is extendable to the cycle C' using at most k of the chords of the cycle C. It will be shown that if G is a graph of order n, then $\delta(G) > 3n/4 - 1$ implies that any proper cycle is 0-chord extendable, $\delta(G) > 5n/9$ implies that any proper cycle is 1-chord extendable, and $\delta(G) > \lfloor n/2 \rfloor$ implies that any proper cycle is 2-chord extendable. Also, each of these results is sharp in the sense that the minimum degree condition cannot, in general, be lowered.

1. Introduction

Only finite graphs without loops or multiple edges will be considered. The degree of vertex v of G will be denoted by $d_G(v)$ or just $d(v)$, and $d_S(v)$ will represent the degree relative to a subset S of vertices. The neighborhood of v in S (vertices of S adjacent to v) will be denoted by $N_S(v)$ or just $N(v)$ when $S = V(G)$. The minimum degree of G will be denoted by $\delta(G)$.

In [2, 3] Hendry introduced the concept of cycle extendability in graphs and directed graphs. His definition of a cycle being extendable is the following.

Definition 1. A cycle C in a graph G is *extendable* if there is a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$.

☆ Dedicated to the memory of our friend and colleague, George R.T. Hendry.

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Hendry, in studying extendability, considered, among other things, degree conditions on a graph G that imply that a graph is Hamiltonian, like the following classical condition of Dirac (see [1]).

Theorem 1 (Dirac [1]). *If G is a graph of order $n \geqslant 3$ with $\delta(G) \geqslant n/2$, then G is Hamiltonian.*

In [2] Hendry showed that the condition of Dirac implies that each cycle of the graph can be extended except for cycles in some special classes of graphs. We will consider a more restricted form of cycle extendability by placing requirements on the number of edges in the original cycle that remain in an extended cycle. A *chord* of a cycle C is any edge between vertices in the cycle that is not in the cycle.

Definition 2. For any integer $k \geqslant 0$, a cycle C is *k-chord extendable* if it is extendable to a cycle C' with the cycle C' using at most k of the chords of the cycle C. A graph G is *k-chord extendable* if each non-Hamiltonian cycle of G is k-chord extendable.

We will consider Dirac type conditions that imply that a graph is k-chord extendable. For any integer $k \geqslant 0$, the following result, which determines the minimum degree in a graph that implies that the graph is k-chord extendable, will be proved.

Theorem 2. *Let G be a graph of order $n \geqslant 3$. Then,*
$\delta(G) > 3n/4 - 1$ implies that G is 0-chord extendable,
$\delta(G) > 5n/9$ implies that G is 1-chord extendable, and
$\delta(G) > \lfloor n/2 \rfloor$ implies that G is 2-chord extendable.
Also, each result is sharp for infinitely many integers n.

Note that there exist graphs of order n and minimum degree $n/2$ with cycles that are not extendable, for example a complete bipartite graph. Therefore, no degree condition less restrictive than $\delta(G) > \lfloor n/2 \rfloor$ will alone imply k-chord extendability for any k.

2. 0-chord extendable graphs

A cycle C is 0-chord extendable if and only if there is a vertex $x \notin V(C)$ that is adjacent to two consecutive vertices on the cycle. We next determine the minimum degree condition that insures that each cycle of length m in a graph of order n is 0-chord extendable. First we will describe examples that give lower bounds on the minimum degree required to imply that a cycle is 0-chord extendable.

Example 1. For any $m = 2k < n$, consider the graph H_n (n even) of order n obtained from a C_4 by replacing the four vertices (in cyclic order) by complete graphs of order k, k, $\lfloor (n-2k)/2 \rfloor$, and $\lceil (n-2k)/2 \rceil$, respectively, and by replacing each edge of the C_4 by an appropriate complete bipartite graph. Consider any cycle C_m in the complete bipartite graph $K_{k,k}$ of H_n. This cycle is not 0-chord extendable since each vertex not in C_m is adjacent only to alternating vertices on the cycle C_m, and so is not adjacent to two consecutive vertices of the cycle. There are at most three distinct degrees of the vertices of H_n, and so it is easy to verify that

$$\delta(H_n)=\min\left\{n-\frac{m}{2}-1,\left\lfloor\frac{n+m-2}{2}\right\rfloor\right\}.$$

Example 2. Similar examples exist when the length of the cycle m is odd. In this case, start with the disjoint union of two complete graphs $K_m \cup K_{n-m}$. Fix a cycle C_m of length m in the K_m. Let H_n be the graph obtained by adding edges between the K_m and the K_{n-m} such that each vertex not in the C_m is adjacent to $(m-1)/2$ vertices of the cycle that are not pairwise consecutive. Also, the edges are added such that the degrees of the vertices in C_m do not differ by more than 1. Thus, the vertices not on the cycle have degree $n-m-1+(m-1)/2=n-(m+3)/2$, and the vertices on the cycle have degree either $m-1+\lfloor(n-m)(m-1)/(2m)\rfloor$ or $m-1+\lceil(n-m)(m-1)/(2m)\rceil$. Thus, H_n is not 0-chord extendable and has minimum degree

$$\delta(H_n)=\min\left\{n-\frac{m+3}{2},\left\lfloor\frac{(m-1)(n+m)}{2m}\right\rfloor\right\}.$$

Theorem 3. *Let G be a graph of order $n\geqslant 3$, and $0\leqslant m<n$ an integer. Then for m even,*

$$\delta(G)>\min\left\{n-\frac{m}{2}-1,\left\lfloor\frac{n+m-2}{2}\right\rfloor\right\}.$$

implies that each C_m is 0-chord extendable, and for m odd,

$$\delta(G)>\min\left\{n-\frac{m+3}{2},\left\lfloor\frac{(m-1)(n+m)}{2m}\right\rfloor\right\}$$

implies that each C_m is 0-chord extendable. Also, the result is sharp.

Proof. Let C_m be a cycle that is not 0-chord extendable in a graph G of order n that satisfies the minimum degree condition of Theorem 3. First consider the case of m even. Thus, each vertex not on the cycle is adjacent to at most $m/2$ vertices on the cycle, and so has degree at most $n-m-1+m/2=n-m/2-1$. Also, two consecutive vertices on the cycle cannot have a common adjacency off of the cycle, and so there is a vertex on C_m that has degree at most $m-1+\lfloor(n-m)/2\rfloor$. This contradicts the minimum degree condition on G.

When m is odd, each vertex not on the cycle C_m is adjacent to at most $(m-1)/2$ vertices on the cycle. Thus, each vertex not on the cycle has degree at most $n-m-1+(m-1)/2$, and there are at most $(n-m)(m-1)/2$ edges between the vertices on the cycle and off of the cycle. Hence, some vertex on the cycle C_m has degree at most

$m-1+\lfloor(n-m)(m-1)/2m\rfloor$. This contradicts the minimum degree condition on G and completes the proof that the stated minimum degree conditions are sufficient for 0-chord extendability.

Examples 1 and 2 verify that Theorem 3 is sharp. □

An immediate consequence of Theorem 3 is the following corollary.

Corollary 1. *If G is a graph of order $n \geqslant 3$ with $\delta(G) > 3n/4 - 1$, then G is 0-chord extendable. Also, the result is sharp.*

3. 2-chord extendable graphs

A cycle $C_m = (x_1, x_2, \ldots, x_m, x_1)$ of length m is 2-chord extendable in a graph G if and only if there is a vertex $x \notin V(C_m)$ such that there is a cycle of length $m+1$ containing C_m and x, and using at most two chords of C_m. For example, this is true if there exists integers $i<j<k$ (taken modulo m) such that xx_i, xx_j, $x_{i+1}x_k$, $x_{j+1}x_{k+1} \in E(G)$, since

$$C_{m+1} = (x, x_i, x_{i-1}, \ldots, x_{k+1}, x_{j+1}, x_{j+2}, \ldots, x_k, x_{i+1}, x_{i+2}, \ldots, x_{j-1}, x_j, x)$$

is a cycle of length $m+1$ using only two chords of C_m, namely $x_{i+1}x_k$, $x_{j+1}x_{k+1}$. Observe that if $k=j+1$, then the cycle C_{m+1} uses only one chord of C_m and we get 1-chord extendability, and in the case $k=j=i+1$, then C_{m+1} is a 0-chord extension of C_m. If there exist integers $i<k<j$ (taken modulo m) such that $xx_i, xx_j, x_{i+1}x_k$, $x_{j+1}x_{k-1} \in E(G)$, then C_m is also 2- chord extendable. There are other configurations that give 2-chord extendability, but the configurations just described will be sufficient for our purposes.

Example 3. For n even, the complete bipartite graph $K_{n/2,n/2}$ has minimum degree $n/2$, and no proper even cycle C_m is extendable since there are no odd cycles. If n is odd, then consider the graph H_n obtained from the complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ by adding a single edge e into the large part. The graph H_n has minimum degree $(n-1)/2$, and any even cycle C_m that does not contain both endvertices of the added edge e is not extendable, since any odd cycle must contain the edge e.

Theorem 4. *A graph G of order $n \geqslant 3$ is extendable (in fact 2-chord extendable) if $\delta(G) > \lfloor n/2 \rfloor$. Also, the minimal degree condition cannot be lowered without losing 2-chord extendability.*

Proof. Assume that G is a graph of order n with $\delta(G) > \lfloor n/2 \rfloor$, and $C = C_m$ is a proper cycle that is not 2-chord extendable. Let A be the vertices of G not in C. If each vertex of A has at most one adjacency in C, then there will be at most $n-m$ edges between C and A. This implies that each vertex of A has degree at most $n-m$, and some vertex of C has degree at most $m-1+(n-m)/m$. The minimum degree condition implies that

$n-m>\lfloor n/2\rfloor$, which gives that $n>2m$. On the other hand, the minimum degree condition implies that $m-1+(n-m)/m\geqslant n/2$,and this gives $n\leqslant 2m$. This gives a contradiction, so we can assume that there is a vertex in A that is adjacent t at least two vertices of C.

Let $x\in A$ such that xx_i, $xx_j\in E(G)$ with $i<j$. If $j=i+1$, then C is 0-chord extendable, so we can assume that $j>i+1$. If $x_{i+1}x_{j+1}\in E(G)$, then

$$(x, x_i, x_{i-1}, \ldots, x_{j+1}, x_{i+1}, x_{i+2}, \ldots, x_j, x)$$

is a cycle of length $m+1$, which implies that C is 1-chord extendable, a contradiction. Thus, we can assume that $x_{i+1}x_{j+1}\notin E(G)$.

If there is an integer k with $i<j<k$ such that $x_{i+1}x_k, x_{j+1}x_{k+1}\in E(G)$, then we have already observed that C is 2-chord extendable in this case. Also, if there is an integer k with $i<k<j$ such that $x_{i+1}x_k, x_{j+1}x_{k-1}\in E(G)$, then C is 2-chord extendable. These two observations imply that

$$d_C(x_{i+1})+d_C(x_{j+1})\leqslant m.$$

If x_{i+1} and x_{j+1} have no common adjacency in A, then $d_A(x_{i+1})+d_A(x_{j+1})\leqslant n-m$, which implies that $d(x_{i+1})+d(x_{j+1})\leqslant n$ and there is a vertex of degree at most $\lfloor n/2\rfloor$, a contradiction. Hence, we can assume there is a $y\in A$ commonly adjacent to x_{i+1} and x_{j+1}. Note that x_i and x_j play the same role as x_{i+1} and x_{j+1}, so $d_C(x_i)+d_C(x_j)\leqslant m$. Since C is not 0-chord extendable, $d_A(x_i)+d_A(x_{i+1})\leqslant n-m$ and $d_A(x_j)+d_A(x_{j+1})\leqslant n-m$. This implies that

$$d(x_i)+d(x_{i+1})+d(x_j)+d(x_{j+1})\leqslant 2(n-m)+2m=2n,$$

which implies the existence of a vertex of degree at most $\lfloor n/2\rfloor$. This contradiction completes the positive proof of Theorem 4.

The graphs in Example 3 verify that the condition in Theorem 4 cannot be improved. □

Theorem 4 cannot be improved because, for example, no even cycle C_m of $K_{n/2,n/2}$ is extendable. For small odd cycles C_m the minimum degree needed for 2-chord extendability can be reduced to $\delta>\lfloor (n-m)(m-1)/(2m)+2\rfloor$ and this is sharp. We shall see, however, in Section 4 that in fact this minimum degree condition implies that C_m is 1-chord extendable, so we delay the proof until Section 4.

4. 1-chord extendable graphs

A cycle $C_m=(x_1, x_2, \ldots, x_m, x_1)$ is 1-chord extendable if and only if there exists a vertex x not in C_m and an i such that xx_i and $xx_{i+1}\in E(G)$ (C_m is 0-chord extendable) or there exist integers i and j such that xx_i, xx_j, and $x_{i+1}x_{j+1} \in E(G)$, so that there is the cycle

$$C_{m+1} = (x, x_i, x_{i-1}, \ldots, x_{j+1}, x_{i+1}, x_{i+2}, \ldots, x_j, x).$$

Let $\delta_n(m)$ denote the minimum integer such that if G is a graph of order n with $\delta(G) > \delta_n(m)$, then any cycle C_m of length m is 1-chord extendable. In this section, bounds on the function δ_n will be given.

We start with some examples that give lower bounds for the function δ_n. Recall from Example 3 that no proper even cycle C_m is extendable in $K_{n/2,n/2}$, and in $K_{(n-1)/2,(n+1)/2}+e$ (the edge e is added to the larger part) no proper even cycle not containing both endvertices of e is extendable. Therefore, $\delta_n(m) \geqslant \lfloor n/2 \rfloor$ for m even.

Example 4, For n divisible by 3 and $k<n/3$, consider the graph $K_{n-3k} \cup K_{k,k,k}$, a disjoint union of a complete graph of order $n-3k$ and a complete tripartite graph with parts of order k. Let A_1, A_2 and A_3 denote the vertices in the parts of the tripartite graph, and partition the vertices of the K_{n-3k} into three sets B_1, B_2 and B_3, each with $n/3-k$ vertices. Let $H_n(3k)$ denote the graph obtained from the graph $K_{n-3k} \cup K_{k,k,k}$ by adding the edges between A_i and B_i for $1 \leqslant i \leqslant 3$. For $m=3k$, let C_m denote a cycle of length m in the $K_{k,k,k}$ in which every third vertex is from the same A_i. It is easy to verify that in the graph $H_n(3k)$ the cycle C_m is not 1-chord extendable (although it is 2-chord extendable), and $\delta(H_n(3k)) = \min\{n/3+k,\ n-2k-1\}$. In terms of the length of the cycle C_m,

$$\delta(H_n(m)) = \min\left\{\frac{n+m}{3},\ n-\frac{2m}{3}-1\right\}.$$

Therefore, $\delta_n(m) > \min\{(n+m)/3,\ n-2m/3-1\}$. Note that for $n/2 \leqslant m \leqslant 3(n-2)/4$ we have $\min\{(n+m)/3,\ n-2m/3-1\} \geqslant n/2$, and so this gives an improved lower bound for $\delta_n(m)$ in this interval. When $(n+m)/3 = n-2m/3-1$ (i.e. when $m=(2n-3)/3$), the maximum value of this minimum is attained, and it is $(5n-3)/9$. Hence, any minimum degree condition implies that the 1-chord extendability of all proper cycles must exceed $(5n-3)/9$.

Example 5. Let m be an odd integer with $3 \leqslant m < n$. Start with the graph $C_m \cup K_{n-m}$, and let R_n be the graph obtained by making each vertex of the K_{n-m} adjacent to precisely $(m-1)/2$ nonconsecutive vertices of the C_m in such a way that the degree of any two vertices in the C_m differs by at most 1. Each vertex in the K_{n-m} has degree $(2n-m-3)/2$, and each vertex in the C_m has degree either $\lceil (n-m)(m-1)/(2m)+2 \rceil$ or $\lfloor (n-m)(m-1)/(2m)+2 \rfloor$. Clearly in R_n the minimum degree is $\lfloor (n-m)(m-1)/(2m)+2 \rfloor$ for $n \geqslant 8$, and the cycle C_m is not 1-chord extendable. It is easily verified that the minimum degree $\lfloor (n-m)(m-1)/(2m)+2 \rfloor$ of Example 5 is greater than or equal to the minimum degree $(n+m)/3$ of Example 4 precisely when $m \leqslant n/5$. Thus, Example 5 gives a better lower bound for $\delta_n(m)$ when $m \leqslant n/5$ and m is odd.

Example 6. For integers m and n with $m>n/2\geqslant 2$ start with the graph $C_m\cup K_{n-m}$. Select $m-\lfloor n/2\rfloor$ vertices on the cycle by starting with some vertex and selecting every other vertex along the cycle, and denote this set by A. Let B be the $m-\lfloor n/2\rfloor+1$ neighbors of the vertices of A along the cycle. Form a graph S_n by adding to $C_m\cup K_{n-m}$ all of the edges between the K_{n-m} and A, and all of the edges between the vertices in the C_m except for those between vertices in B. The vertices in the K_{n-m} have degree $\lceil n/2\rceil-1$ and the vertices in B have degree $\lfloor n/2\rfloor-1$. Thus, the minimum degree in S_n is $\lfloor n/2\rfloor-1$, and the cycle C_m cannot be 1-chord extended. If m is an even cycle, then the bipartite graph of Example 1 gives a better lower bound for $\delta_n(m)$, but for odd cycles C_m the bound $\lfloor n/2\rfloor-1$ of this example is greater than the bound $n-2m/3-1$ of Example 4 if $m\geqslant(3n+3)/4$.

The examples give lower bound for the function δ_n. We now determine some upper bounds for this function. We start with extending small cycles.

Theorem 5. *If G is a graph of order $n\geqslant 3$ with $\delta(G)>\lfloor n/2\rfloor$ and $m\leqslant n/3$, then any cycle C_m of length m is 1-chord extendable.*

Proof. Let $C=C_m=(x_1,x_2,,\ldots,x_m,x_1)$ be a cycle that is not 1-chord extendable, and we will show that this leads to a contradiction. Denote the vertices not in the cycle C by A. Select consecutive vertices $u=x_1$ and $v=x_2$ on the cycle C.

Since C is not 1-chord extendabled, u and v have no common adjacency off of the cycle, so $d_A(u)+d_A(v)\leqslant n-m$. Thus, by assumption, $d_C(u)+d_C(v)>m$. This implies that there is a vertex $x_k\in C$ such that ux_k and $vx_{k+2}\in E(G)$; for otherwise, if $ux_k\in E(G)$, then $vx_{k+2}\notin E(G)$, which implies that $d_C(v)\leqslant m-d_C(u)$, a contradiction. Then, let $w=x_{k+1}$. Note that if u and w have a common adjacency, say x off of the cycle C, then C is 1-chord extendable by the following cycle:

$$(x,u,x_m,x_{m-1},\ldots,x_{k+2},v,x_3,\ldots,x_k,w,x).$$

Thus, when u and v have the '*skipped crossing pattern*' that produced the vertex w, we can assume that u and w have no common adjacencies off of the cycle, and likewise the same is true for v and w. This implies that the neighborhoods of u, v and w in A are pairwise disjoint.

Two cases will be considered, when $d_C(u)+d_C(v)>(n+1)/2$ and when $m<d_C(u)+d_C(v)\leqslant(n+1)/2$. We consider the latter case first. Since $d_C(u)+d_C(v)>m$, u and v have a 'skipped crossing pattern', so there is a vertex w on C such that the neighborhood of u, v, and w in A are pairwise disjoint. Therefore

$$\frac{3n}{2}<d(u)+d(v)+d(w)\leqslant(n-m)+\frac{n+1}{2}+(m-1)\leqslant\frac{3n-1}{2},$$

a contradiction.

We are left with the case $d_C(u)+d_C(v)>(n+1)/2$. If u and v have a '*double skipped crossing pattern*' (u is adjacent to x_k and x_{k+1} and v is adjacent to x_{k+2} and x_{k+3}), then $w=x_{k+1}$ and $w'=x_{k+2}$ are both vertices in a 'skipped crossing pattern'. This gives that u, v, w, and w' have pairwise disjoint neighborhoods in A, which implies that

$$2n<d(u)+d(v)+d(w)+d(w')\leqslant(n-m)+4(m-1)\leqslant 2n-4,$$

a contradiction. Hence, we can assume that we do not have a 'double skipped crossing pattern' with u and v. Thus, in any interval $(x_i,x_{i+1},x_{i+2},x_{i+3})$ of C, and for any i with $3\leqslant i\leqslant m-3$, one of $ux_i, ux_{i+1}, vx_{i+2}, vx_{i+3}$ is not in $E(G)$. Therefore, counting multiplicities, there will be at least $m-5$ edges from u or v not in $E(G)$. No edge will be counted more than twice in the missing collection, so there are at least $(m-5)/2$ missing edges, and $d_C(u)+d_C(v)\leqslant 2(m-1)-(m-5)/2=(3m+1)/2$. This gives

$$\frac{n+1}{2}<d_C(u)+d_C(v)\leqslant\frac{3m+1}{2}\leqslant\frac{n+1}{2},$$

a contradiction, which completes the proof of this case and of Theorem 5. □

For small odd cycles the minimum degree needed for 1-chord extendability is less than that required in Theorem 5, as the following theorem verifies.

Theorem 6. *If G is a graph of order $n\geqslant 3$ with $\delta(G)>\lfloor(n-m)(m-1)/(2m)+2\rfloor$, where m is an odd integer satisfying $3\leqslant m\leqslant\sqrt{n/3}$, then any cycle C_m is 1-chord extendable.*

Proof. Let C be a cycle of length m that is not 1-chord extendable. Denote the vertices not in C by A and let v be a vertex in C such that $d_C(v)=t$ is a maximum. If $t=2$ then C has no chords.

Since C is not 1-chord extendable, each vertex of A is adjacent to at most $(m-1)/2$ vertices of C. Thus, there are at most $(n-m)(m-1)/2$ edges between C and A. Hence, some vertex x of C has degree at most $\lfloor(n-m)(m-1)/2+2\rfloor$ in G, contradicting the minimum degree condition. Thus $t\geqslant 3$.

Since $t\geqslant 3$, there is a chord vz such that one of the paths from v to z of C together with the chord vz of C is an odd cycle C' of order at most $m+2-2\lceil t/2\rceil$. Let $C'=(v,v_0,w_0,v_1,w_2,\ldots,v_l,w_l=z,v)$, where then $l\leqslant\lceil(m-t)/2\rceil-1$.

Let $\delta=\lfloor(n-m)(m-1)/(2m)+3\rfloor$. Then for each vertex x of C, we have $d_A(x)\geqslant\delta-t$. Let X, Y, and Z be sets of $\delta-t$ vertices each such that $X\subset N_A(v_0)$, $Y\subset N_A(w_0)$ and $Z\subset N_A(v_1)$. Note that X and Y are disjoint as are Y and Z, since G is not 1-chord extendable. Let $S=A-X-Y$ and $S'=A-Y-Z$. For $1\leqslant i\leqslant l$, define

$$X_i=N_A(v_i)\cap X,\qquad Y_i=N_A(w_i)\cap Y,\qquad Z_i=N_A(v_i)\cap Z.$$

Consider X_1. Since $d_A(v_1)\geqslant\delta-t$ and $N_A(v_1)\cap Y=\emptyset$, it follows that $N_A(v_1)\subset S\cup X$ and so $|X_1|\geqslant\delta-t-|S|$. Similarly, considering w_1 and Y, we see that $N_A(w_1)\subset S'\cup Y$, implying that $|Y_1| \geqslant \delta - t - |S'| = \delta - t - |S|$. Finally, considering v_2 and Z, we see that $|Z_2| \leqslant \delta - t - |S|$. More generally, for $i \leqslant 2$ we have that

$$N_A(v_i) \subset S \cup (Y - Y_{i-1}) \cup X$$

and

$$N_A(w_i) \subset S' \cup (Z - Z_i) \cup Y.$$

Thus,

$$|X_i| \geqslant \delta - t - |S| - |Y - Y_{i-1}| = |Y_{i-1}| - |S|$$

and

$$|Y_i| \geqslant \delta - t - |S| - |Z - Z_i| = |Z_i| - |S|.$$

Consequently,

$$|X_2| \geqslant |Y_1| - |S| \geqslant \delta - t - 2|S|$$

and

$$|Y_2| \geqslant |Z_2| - |S| \geqslant \delta - t - 2|S|.$$

Furthermore, since v_3 and Z have the same relationship as do v_2 and X, it follows that $|Z_3| \geqslant \delta - t - 2|S|$. In general, we have

$$|X_i| \geqslant \delta - t - i|S|, \qquad |Y_i| \geqslant \delta - t - i|S|, \qquad |Z_i| \geqslant \delta - t - (i-1)|S|.$$

Let w be the vertex in $V(C) - V(C')$ that is adjacent to w_l. Then, as in the previous argument, if $W = N_A(w) \cap X$, then $|W| \geqslant \delta - t - (l+1)|S|$. We will show that $|W| > 0$. Now,

$$\delta - t = \left\lfloor \frac{(n-)(m-1)}{2m} + 3 - t \right\rfloor > \left(\frac{n}{m}\right)\left(\frac{m-1}{2}\right) + 2 - t - \frac{m-1}{2}.$$

Also,

$$|S| = n - m - 2(\delta - t) < \frac{n}{m} + 2t - 5.$$

Thus,

$$(l+1)|S| \leqslant \left\lceil \frac{m-t}{2} \right\rceil |S| < \left(\frac{m-t+1}{2}\right)\left(\frac{n}{m} + 2t - 5\right).$$

Therefore $\delta - t > (l+1)|S|$ if

$$\left(\frac{n}{m}\right)\left(\frac{m-1}{2}\right) + 2 - t - \frac{m-1}{2} > \left(\frac{m-t+1}{2}\right)\left(\frac{n}{m} + 2t - 5\right)$$

or, equivalently, if

$$\left(\frac{n}{m}\right)\left(\frac{t-2}{2}\right) > \left(\frac{m-t+1}{2}\right)(2t-5) + t + \frac{m-1}{2} - 2.$$

Since $3 \leqslant t \leqslant m-1$ and $m < \sqrt{n/3}$, we conclude that $|W| > 0$. Let $w' \in W$. Then $w', v_0, w_0, v_1, w_1, \ldots, v_l, w_l = z, v, P, w, w'$ is a 1-chord extension of C, where P is a $v-w$ subpath of C disjoint from C' except for v. This contradiction completes the proof of Theorem 6. □

Note that Example 5 shows that the condition in Theorem 6 is the best possible. Also, since $\lfloor (n-m)(m-1)/(2m) + 2 \rfloor < \lfloor n/2 \rfloor$, Theorem 6 is an improvement of Theorem 5.

The same proof techniques used in Theorem 5 can be used to verify the following.

Theorem 7. *If G is a graph of order $n \geqslant 3$ with $\delta(G) > (n+3m)/4$ and $m \geqslant n/3$, then each cycle C_m is 1-chord extendable.*

Proof. As was done in the proof of Theorem 5, let $C = C_m = (x_1, x_2, \ldots, x_n, x_1)$ be a cycle that is not 1-chord extendable, denote the vertices not in the cycle C by A, and select consecutive vertices $u = x_1$ and $v = x_2$ on the cycle C. Since u and v have no common adjacencies in A, $d_C(u) + d_C(v) > 2(n+3m)/4 - (n-m) = (5m-n)/2 \geqslant m$ because $n \leqslant 3m$.

We will first consider the case $m < d_C(u) + d_C(v) \leqslant (3m+2)/2$. From previous arguments we know that u and v have a 'skipped crossing pattern', so there is a $w \in C$ such that u, v, and w have pairwise disjoint neighborhoods in A. This implies that

$$\frac{3(n+3m)}{4} < d(u) + d(v) + d(w) \leqslant (n-m) + \frac{3m+2}{2} + (m-1) = n + \frac{3m}{2}.$$

However, this inequality is equivalent to $m < n/3$, a contradiction.

Next we consider the case when $d_C(u) + d_C(v) > (3m+2)/2$. In an argument of Theorem 5 it was shown that this degree condition is sufficient to imply u and v have a 'doubly skipped crossing pattern'. Thus there exist vertices w and w' in C such that u, v, w, and w' have pairwise disjoint neighborhoods in A. Thus, we have

$$n + 3m < d(u) + d(v) + d(w) + d(w') \leqslant (n-m) + 4(m-1) = n + 3m - 4,$$

a contradiction, which completes the proof of this case and of Theorem 7. □

The upper bound on the function $\delta_n(m)$ from the previous result can be improved significantly for large values of m. The next result is such an improvement for $m \geqslant n/2$.

Theorem 8. *If G is a graph of order $n \geqslant 3$ with $\delta(G) > (3n-m)/4$, then any cycle C_m is 1-chord extendable.*

Proof. Again, let $C = C_m = (x_1, x_2, \ldots, x_m, x_1)$ be a cycle that is not 1-chord extendable, and denote the vertices not in the cycle C by A. If there were no edges between C and A, then in one of the components of G there would be a vertex of degree less than $n/2$, a contradiction. Select consecutive vertices $u=x_1$ and $v=x_2$ on the cycle X such that at least one of these vertices is adjacent to a vertex in A. Since u and v have no common adjacency in A, $d_A(u)+d_A(v)\leqslant n-m$, so we can assume that $d_A(v)\leqslant(n-m)/2$ and u is adjacent to a vertex $w\in A$.

Consider the nonadjacent pair v, w of vertices, and note that if vx_k and $wx_{k-1}\in E(G)$, then C_m is 1-chord extendable by the cycle

$$C_{m+1}=(w, u, x_2, \ldots, x_k, v, x_3, \ldots, x_{k-1}, x).$$

We can assume that this does not occur, so $d_C(w)\leqslant m-d_C(v)$, since each adjacency of v on C forces a nonadjacency of w on C. Hence, we have

$$(3n-m)/2<d(u)+d(v)\leqslant(n-m)/2+(n-m-1)+m=(3n-m)/2-1,$$

a contradiction, which completes the proof of Theorem 8. □

The next two lemmas will be needed in the proof of the next theorem.

Lemma 1. *Let C be a cycle in a graph G of order $n\geqslant 3$ that is not 1-chord extendable. If $\delta(G)\geqslant n/2$, then each vertex of C is adjacent to a vertex not in C.*

Proof. Let $C=(x_1, x_2, \ldots, x_m, x_1)$ be a cycle that is not 1-chord extendable, and denote the vertices not in the cycle C by A. Assume that there is a vertex in C with no adjacency in A. If there were no edges between C and A, then in one of the components of G there would be a vertex of degree less than $n/2$, a contradiction.

With no loss of generality we can assume that $u=x_1$ has no adjacency in A but x_2 is adjacent to $v\in A$. If $vx_k\in E(G)$, then $ux_{k-1}\notin E(G)$, because the following cycle C' would imply that C was 1-chord extendable.

$$C'=(u, x_{k-1}, x_{k-2}, \ldots, x_2, v, x_k, x_{k+1}, \ldots, x_m, u).$$

Therefore, $d_C(u)\leqslant m-d_C(v)$. Since u has no adjacencies in A, it follows that $d(u)+d(v)<n$, and so one of u or v has degree less than $n/2$. This contradiction completes the proof of Lemma 1. □

Proof techniques similar to those used in the proof of Lemma 1 can be used to prove the following more specialized lemma.

Lemma 2. *Let G be a graph of order $n\geqslant 3$ with $\delta(G)>n/2+ln$ for some positive number l, and let C be a cycle of length $n/2+pn$ for some p that is not 1-chord extendable. If there is a vertex $u\in A$ with $d_C(u)=rn$, then there is a vertex v in C with $d_A(v)>(r+l-\mathrm{p})n$. Also, for each vertex $w\in C$, $d_A(w)>2ln+1$.*

Proof. Let $C=(x_1, x_2, \ldots, x_m, x_1)$ be the cycle that is not 1-chord extendable with $m=n/2+pn$. Let u be a vertex in A with $d_C(u)=rn$, let R be the rn neighbors of u in C, and let R^+ be the successors of the set R along the cycle C. If v is a vertex in R^+, then v is not adjacent to any vertex in R^+, for if so, then C would be 1-chord extendable. Thus, $d_C(v) \leqslant n/2 + pn - rn$, and so $d_A(v) > n/2 + ln - (n/2 + pn - rn) = (r + l - p)n$.

Each vertex u' in A has $d_C(u') > n/2 + ln - (n/2 - pn - 1) = ln + pn + 1$. Also, each vertex $w \in C$ is the successor along C of some neighbor in C of a vertex $u' \in A$, since by Lemma 1 each vertex of C has a neighbor in A. A repeat of the argument of Lemma 1 with u' replacing u and w replacing v implies that $d_A(w) > (ln + pn + 1) + ln - pn = 2ln + 1$. This completes the proof of Lemma 2. □

Note that $l > 0$ in the previous lemma, but p can be negative as well as positive. With these two lemmas, we are ready to prove the following.

Theorem 9. *If G is graph of order $n \geqslant 3$ with $\delta(G) > 5n/9$, then any proper cycle C of G is 1-chord extendable. Also, the minimal degree condition $\delta(G) \geqslant (5n-3)/9$ will not insure 1-chord extendability.*

Proof. Example 2 shows that $\delta(G) \geqslant (5n-3)/9$ will not insure 1-chord extendability. For the positive proof, let $C = C_m = (x_1, x_2, \ldots, x_m, x_1)$ be a cycle that is not 1-chord extendable, with $m = n/2 + pn$ for some number p. We will show that this leads to a contradiction.

Let u be a vertex in C such that $d_A(u) = tn$ is a maximum. Let v be a neighbor of u along the cycle C; in fact, we can assume that $u = x_1$ and $v = x_2$. By Lemma 2, each vertex in C has at least $2(5n/9 - n/2) + 1 = (n+9)/9$ adjacencies in A. Thus $d_A(v) = (n+9)/9 + rn$ for some nonnegative number r. Since C is not 1-chord extendable, the neighborhoods of u and v in A are disjoint, and their union contains $tn + (n+9)/9 + rn$ vertices. This implies that $tn + (n+9)/9 + rn \leqslant (1/2 - p)n$, and so $p + t < 7/18$.

Let A' be the vertices of A that are not adjacent to either u or v. Hence,

$$|A'| = n/2 - pn - (tn + (n+9)/9 + rn) = (7/18 - p - t - r)n - 1.$$

Also,

$$d_C(u) + d_C(v) \geqslant 2\left(\frac{5n}{9}\right) - tn - \frac{n+9}{9} - rn = (1 - t - r)n - 1.$$

This implies that, using the same counting techniques as in Theorem 5, there are at least $(1 - t - r)n - 1 - (n/2 + pn) = (1/2 - t - p - r)n - 1$ different 'skipped crossing patterns' from u and v (i.e. ux_k and $vx_{k+2} \in E(G)$ and the *central* vertex x_{k+1} has no common adjacencies with either u or v in A). Let B be the set of central vertices in the 'skipped crossing patterns'. Thus $|B| \geqslant (1/2 - t - p - r)n - 1$.

Each vertex of B has at least $(n+9)/9$ adjacencies in A, and all of these adjacencies must be in A'. Therefore, the number of edges between A' and B is at least $((n+9)/9)|B|$. Thus, there is some vertex in $w \in A'$ with $d_C(w) \geqslant ((n+9)/9)|B|/|A'|$. By the maximality of $tn = d_A(u)$ and by Lemma 2, we have $d_C(w) \leqslant tn - (n/18 - pn) = (t + p - 1)/18)n$. Hence,

$$\binom{\frac{n+9}{9}}{}\binom{(\frac{1}{2}-p-t-r)n-1}{(\frac{7}{18}-p-t-r)n-1} \leqslant \left(t+p-\frac{1}{18}\right)n.$$

The inequality,

$$\binom{\frac{n+9}{9}}{}\binom{(\frac{1}{2}-p-t)n}{(\frac{7}{18}-p-t)n} \leqslant \left(t+p-\frac{1}{18}\right)n,$$

which is independent of r, follows from the previous inequality, since

$$\binom{(\frac{1}{2}-p-t-r)n-1}{(\frac{7}{18}-p-t-r)n-1} > \binom{(\frac{1}{2}-p-t)n}{(\frac{7}{18}-p--t)n}.$$

If we substitute $x = p + t$, replace $(n+9)/9$ by just $n/9$, and divide by n, we have the more compact inequality

$$\left(\frac{1}{9}\right)\binom{\frac{1}{2}-x}{\frac{7}{178}-x} < x - \frac{1}{18}.$$

However, since $x < 7/18$, this last inequality is equivalent to

$$18(1-2x) < (18x-1)\,(7-18x),$$

which is equivalent to $(18x-5)^2 < 0$, a contradiction which completes the proof that $\delta(G) > 5n/9$ implies that any proper cycle is 1-chord extendable.

Example 4 implies that the minimum degree condition cannot be decreased to $(5n-3)/9$, and completes the proof of Theorem 9. □

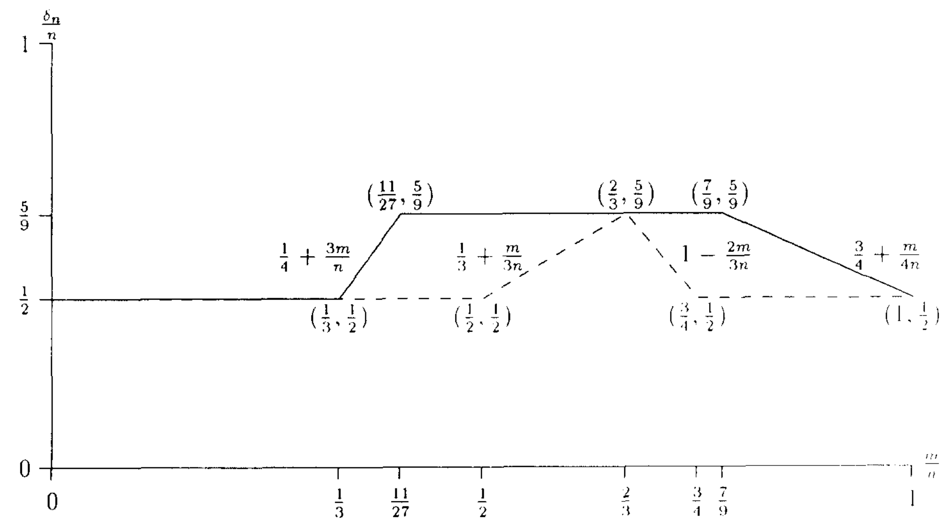

Fig. 1. Bounds on $\delta_n(m)$.

The diagram in Fig. 1 illustrates the upper and lower bound on $\delta_n(m)$ given by the previous theorems, except for some small odd cycles C_m.

5. Questions

We have investigated Dirac type (minimum degree) conditions that imply k-chord extendability. It is natural to consider any condition that implies that a graph G is Hamiltonian, and ask what is the corresponding condition that implies G in k-chord extendable. In particular, it would be interesting to know that the nature of degree sum conditions, neighborhood conditions or generalized degree conditions that imply k-chord extendability, and we have begun the study of such conditions.

A particular problem left unanswered in this paper is the minimum degree condition in a graph G of order n that implies 1-chord extendability for small odd cycles. For small even cycles $\delta(G) > \lfloor n/2 \rfloor$ implies that C_m is 1-extendable if $m \leqslant n/3$ and even, and this condition is sharp. However, for odd cycles a smaller minimum degree is needed as Theorem 6 indicates. Perhaps Theorem 6 can be extended to all odd cycles of length at most $n/5$.

Of course it would be nice to determine precisely the function $\delta_n(m)$.

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