Generalized Degrees and Short Even Cycles

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Abstract

Let G be a graph of order n such that $|E(G)| > c_k n^{\frac{k+1}{k}}$. In 1963 Erdös showed that this implies G contains a C_{2k} . He conjectured that this edge density condition implies G contains a C_{2l} for every integer $l \in [k, n^{\frac{1}{k}}]$. In 1974 Bondy and Simonivits proved the conjecture with $c_k = 100k$.

The purpose of this paper is to provide a generalized degree analogue to this classic result of Erdös. Here we use the following idea of generalized minimum degree. Let

$$\delta_k(G) = min|N(u_1) \cup N(u_2) \cup \cdots N(u_k)|$$

where the minimum is taken over all independent sets of vertices $\{u_1, \cdots, u_k\}$ of size k and $N(u_i)$ denotes the neighborhood of the vertex u_i. We call $\delta_k(G)$ the minimum generalized k-degree for G.

1 Introduction

All graphs considered in this paper are finite and simple. For a vertex $x \in V(G)$ we denote the degree of x as $d(x)$. We define the neighborhood of a vertex x to be the set $N(x)$ where

$$N(x) = \{y \in V(G) \mid xy \in E(G)\}.$$

For terms not defined here, see [5].

Many results in extremal theory are based on edge density conditions, for example, a minimum degree requirement. Turan's [8] classic result on the existence of complete subgraphs is a prime example of such an edge density result.

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Over the last few years a wide variety of results using various forms of generalized degree conditions have been found (see for example [6] and [7]). In this paper we are concerned with the following minimum generalized degree

$$\delta_k(G) = min|N(u_1) \cup N(u_2) \cup \cdots \cup N(u_k)|$$

where the minimum is taken over all independent sets of size k.

In 1989, Noga Alon, Ralph Faudree and Zoltan Furedi [1] proved the following theorem based on this minimum generalized degree, again a form of edge density condition.

Theorem 1 *Let G be a graph of order n. Let k and d be integers where $k \geq 1$ and $d \geq 2$. Let*

$$\delta_k(G) = min|N(u_1) \cup N(u_2) \cup \cdots N(u_k)|$$

where the minimum is taken over all independent sets of size k. If G satisfies the condition

$$\delta_k(G) \geq \frac{d-2}{d-1}n,$$

then for sufficiently large n, G contains a K_d.

Their result, as stated, can be viewed as the $\delta_k(G)$ analogue of Turan's Theorem. Note that when $k = 1$ we have $\delta(G) = \delta_1(G)$.

The purpose of this paper is to provide another generalized degree analogue, this time to an edge density result due to Erdös [3]. In particular, Erdös proved the following theorem.

Theorem 2 *There exists a c_k and an $n_0(k)$ such that if*

$$|E(G)| > c_k n^{1+1/k}$$

and $n > n_0$, then C_{2k} is contained in G.

Erdös [3] conjectured that this edge density condition implies G contains a C_{2l} for every integer $l \in [k, n^{\frac{1}{k}}]$. Bondy and Simonivits [2] proved this conjecture with $c_k = 100k$. Our analogue of Theorem 2 is the following.

Theorem 3 *Let G be a graph of order n. Let k and p be integers greater than or equal to one. If G satisfies the generalized minimum degree condition*

$$\delta_k \geq cn^{\frac{1}{p}}$$

for some real number c, then for sufficiently large n, G contains a C_{2p}.

2 Results

We first consider the case where G is a bipartite graph. Consider $u_0 \in V(G)$ such that $d(u_0) \geq c_0 n^{\frac{1}{p}}$. Let $U_i = N^i(u_0)$ denote the set of all vertices $V(G)$ whose minimum distance from u_0 is i, that is,

$$U_i = \{x \in V(G) \mid d(x, u_0) = i\}.$$

Note that $U_0 = \{u_0\}$ and that $U_1 = N(u_0)$.

Lemma 1 *Let p and k be a fixed integers greater than 1. Let G be a bipartite graph of order n that satisfies the generalized minimum degree condition*

$$\delta_k(G) \geq cn^{\frac{1}{p}}$$

for some real number c. If G does not contain a C_{2p}, then for sufficiently large n, $|U_i| \geq c_i n^{\frac{1}{p}}$ for $1 \leq i \leq p$ and some real number c_i.

Proof: The proof is by induction on i. Note that the claim is true for $i = 0$ with $c_0 = 1$ as in this case $U_0 = \{u_0\}$ and $c_0 n^{0/p} = 1$. Now assume for some j, $0 < j < p$, that the claim is true for all $0 \leq i \leq j$.

Given any collection of p vertices in U_j, there must exist at least two vertices in the collection, say u and v, such that

$$(1) \qquad |N^*_{j+1}(u) \bigcap N^*_{j+1}(v)| < p - 1$$

where $N^*_{j+1}(u) = N(u) \bigcap U_{j+1}$. If this were not the case, given any collection of p vertices in U_j, the intersection would be at least $p - 1$, that is,

$$|N^*_{j+1}(u) \cap N^*_{j+1}(v)| \geq p - 1.$$

In order to see that is is true, let z_m be a vertex in U_m, $0 \leq m \leq j - 1$, such that there exists vertex-disjoint paths from z_m to u and v and let m be maximal with respect to this property. Note that such a z_m must exist as u and v are both at distance j from u_0. Then there exist a path of length $j - m$ from z_m to u, a path of length $2(p - j + m)$ from u to v via the vertices in U_j and U_{j+1} (using the fact that any two such vertices have at least $p - 1$ common neighbors), and a path from v to z_m of length $j - m$. Hence, we have a cycle of length $2p$ in G obtained by combining the path from z_m to u, the path from u to v and the path from v to z_m. Therefore, given any collection of size p in U_j, the two vertices u and v that satisfy (1) must exist.

This implies that at most $p - 1$ vertices in U_j can cover the same neighborhood set S in U_{j+1} if $|S| > p - 1$. This implies

$$|U_{j+1}| \geq \frac{\binom{c_j n^{\frac{j}{p}}}{p}}{\binom{c_j n^{\frac{j}{p}}}{p-1}} c' n^{\frac{1}{p}} = c_{j+1} n^{\frac{j+1}{p}}$$

for some real number c_{j+1}, hence completing the proof. $\diamond$

With the use of Lemma 1 we can now prove a version of our main result for bipartite graphs.

Lemma 2 *Let $k \geq 1$ and $p \geq 1$ be integers. Let G be a bipartite graph of order n that satisfies the generalized minimum degree condition*

$$\delta_k(G) \geq cn^{\frac{1}{p}}$$

for some real number c. Then for n sufficiently large, G contains a C_{2p}.

Proof: Let G be a graph that satisfies the conditions of the lemma and does not contain a C_{2p}. We shall show that this leads us to a contradiction. Let $u_0 \in V(G)$ such that $d(u_0) \geq \frac{c}{k}n^{\frac{1}{p}}$. Such a vertex must exist or our condition on $\delta_k(G)$ would be violated. Then, by Lemma 1, $|N^p(u_o)| \geq c_0 n$ for some real number c_0. Let $u_1 \in N(u_0)$ such that $d(u_1) \geq \frac{c}{k}n^{\frac{1}{p}}$. Then again by Lemma 1, $|N^p(u_1)| \geq c_1 n$ for some real number c_1 and since G is bipartite, $N^p(u_0) \cap N^p(u_1) = \phi$. For each $t \in Z^+$, we may choose $u_t \in N(u_{t-1})$. Note, for example, that if $x \in N^p(u_0) \cap N^p(u_2)$, then the minimum distance from x to u_0 would be both p and $p+2$. Thus, in fact, we obtain a sequence of pairwise disjoint subsets of $V(G)$, each of order $c_t n$ for some real number c_t. Since n is finite, this process must terminate, which is a contradiction to Lemma 1. $\diamond$

We are now able to prove our main result.

Proof of Main Result: Let G be a graph of order n and $\delta_k \geq cn^{\frac{1}{p}}$ for some real number c. There can be at most $k-1$ vertices whose degree is less than $\frac{c}{k}n^{\frac{1}{p}}$. Erdös [4] showed G contains a spanning bipartite subgraph H where each vertex $x \in V(H)$ has degree at least half its degree in G, that is,

$$deg_H x \geq \frac{deg_G x}{2}.$$

Such a spanning subgraph satisfies the generalized minimum degree condition $\delta_k \geq c'n^{\frac{1}{p}}$ for some real number c'. Hence, by Lemma 2, H contains a C_{2p}, and thus, so does G. $\diamond$

3 Future Directions

We can now ask the same question posed originally by Erdös: Do all cycles C_{2t} exist for $t \in [p, n^{1/p}]$? If so, for what constants can we verify this result?

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