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On Hamiltonian-Connected Graphs

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ABSTRACT

One of the most fundamental results concerning paths in graphs is due to Ore: In a graph G, if $\deg x + \deg y \geq |V(G)| + 1$ for all pairs of nonadjacent vertices $x, y \in V(G)$, then G is hamiltonian-connected. We generalize this result using set degrees. That is, for $S \subset V(G)$, let $\deg S = |\cup_{x \in S} N(x)|$, where $N(x) = \{v \mid xv \in E(G)\}$ is the neighborhood of x. In particular we show: In a 3-connected graph G, if $\deg S_1 + \deg S_2 \geq |V(G)| + 1$ for each pair of distinct 2-sets of vertices $S_1, S_2 \subset V(G)$, then G is hamiltonian-connected.
Several corollaries and related results are also discussed.

1. INTRODUCTION

We use the notation and terminology of [2]. Only simple graphs are considered. Let G be a graph. A *hamiltonian path* in G is a spanning path of $V(G)$. A *hamiltonian cycle* in G is a spanning cycle of $V(G)$. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph is *hamiltonian-connected* if there is a hamiltonian path between any two vertices. Let the neighborhood of v be $N(v) = \{x \in V(G) | xv$ is an edge of $G\}$. Let $\delta(G) = \min\{\deg v : v \in V(G)\}$, and let $\deg\{x, y\} = |N(x) \cup N(y)|$. Clearly, this idea of generalized degree could easily be extended to any number of vertices; however, we will have need only for the above definitions. Now let $\delta_2(G) = \min\{\deg\{u, v\} : u, v \in V(G), u \neq v\}$. Given a path in G we denote by (a, b) the subpath from vertex a to vertex b, not including a or b, while $[a, b]$ denotes the subpath containing both end vertices. Where needed, we use the notation v^+ or v^- to indicate the successor or predecessor

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of the vertex v along some path with a given (or implied) direction. We use the notation $d(a, b)$ to denote the distance from a to b in the graph G and $d_P(a, b)$ to denote the distance from a to b along the path P.

It is a well-known result of Ore [6] that if $\deg x + \deg y \geq |V(G)| + 1$ for each pair of nonadjacent vertices x, y, then G is hamiltonian connected. A similar result also due to Ore [5] holds for G being hamiltonian. Ore's theorem generalized an earlier classic result of Dirac [1] based on minimum degrees. A generalized version of Dirac's hamiltonian result was shown in [3]:

Theorem A. If G is a 2-connected graph on n vertices with $\delta_2(G) \geq n/2$ where n is sufficiently large, then G is hamiltonian.

It is natural to wonder if Dirac's theorem for hamiltonian-connected graphs can also be generalized using set degrees. This was answered positively in [4], where the following result, using only degree conditions, was shown.

Theorem B. If G is a graph of sufficiently large order n with $\delta(G) \geq 3$ and $\delta_2(G) \geq (n + 3)/2$, then G is hamiltonian-connected.

Note that the graphs in Theorem B must be 3-connected by a result of [4]. It is now natural to ask if the generalized degree bound in Theorem B can be lowered somewhat if we begin by assuming the graph is 3-connected. Here, we prove the following result, reminiscent of Ore's Theorem.

Theorem C. If G is a 3-connected graph of order n and if all pairs of distinct 2-sets of vertices S_1 and S_2 satisfy

$$\deg S_1 + \deg S_2 \geq n + 1,$$

then G is hamiltonian-connected.

The following corollary is immediate from Theorem C and supplies the weakening of Theorem B we sought.

Corollary D. If G is a 3-connected graph of order n and each distinct pair of vertices S satisfies $\deg S \geq (n + 1)/2$, then G is hamiltonian-connected.

The following corollary is also immediate from Theorem C.

Corollary E. If G is a 3-connected graph such that for all arbitrary 2-sets of vertices S_1 and S_2,

$$\deg S_1 + \deg S_2 \geq |V(G)| + 1,$$

then G is hamiltonian-connected.

2. THE PRELIMINARIES

We will need several lemmas before we can prove the main result, Theorem C. We begin with a simple lemma used often in this area. For this reason the proof is omitted.

Lemma 1. Let G be a graph on n vertices that contains no $x - y$ hamiltonian path. Further, suppose $P : x = v_1, v_2, \ldots, v_{n-1} = y$ is a path from x to y missing exactly one vertex of G, say v. If v_i and v_j are in $N(v)$, then neither $v_{i-1}v_{j-1}$ or $v_{i+1}v_{j+1}$ are in $E(G)$. ∎

Lemma 2. Suppose G has order n and contains no hamiltonian $x - y$ path. Further, suppose $P : x = v_1, v_2, \ldots, v_{n-1} = y$ is an $x - y$ path in G missing only v and that $z \in V(P)$ with the property that for every i, if $v_i \in N(v) \cup N(z)$, then $v_{i-1}, v_{i+1} \notin N(z)$. Then $\deg\{v, z\} \leq n/2$.

Proof. From the hypothesis of Lemma 2, it is readily seen that $N(v) \cup N(z)$ contains no two consecutive vertices on the path P. Thus, $\deg\{v, z\} \leq (|V(P)| + 1)/2 = n/2$ and the lemma is verified. ∎

Lemma 3. Suppose G has order n and satisfies $\deg S_1 + \deg S_2 \geq n + 1$ for each pair of distinct pairs of vertices S_1, S_2. Let P be an $x - y$ path in G missing only v where $\deg v$ is maximum and further suppose $\deg v \geq (n + 1)/4$ and v is not adjacent to x. If G contains no hamiltonian $x - y$ path, then we may select two neighbors of v on P, say s and t, such that $d_P(s, t) \leq 3$.

Proof. Select P and v so that $\deg v$ is a maximum. If the claim fails to hold, then $d_p(s, t) \geq 4$ and thus

$$4(\deg v - 1) + 1 \leq n - 2$$

since v is not adjacent to x and all adjacencies are on P. Hence, $\deg v \leq (n + 1)/4$. But then, $\deg v = (n + 1)/4$ and G contains the subgraph of Figure 1.

Let $P : x = v_1, v_2, \ldots, v_{n-1} = y$ and let $u = v_{n-6}$ and $w = v_{n-2}$. Then,

$$\deg\{v, x\} + \deg\{u, w\} \leq n + 1 .$$

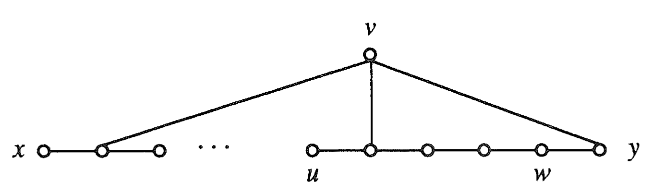

FIGURE 1

This follows since if x is not adjacent to v_{n-3} then

$$\deg\{v,x\} + \deg\{u,w\} \leq n,$$

as the predecessor of any vertex in $N(v) \cup N(x)$ is not in $N(u) \cup N(w)$ (except possibly v_{n-5}, which does not affect our count).

If x is adjacent to v_{n-3}, then by our choice of P and v,

$$\deg w \leq \deg v = \frac{n+1}{4}$$

and so w has at most $\deg v - 1$ neighbors outside $N(v)$ (since $y \in N(w) \cup N(v)$). Thus, $\deg\{v,w\} \leq (n-1)/2$. But then, $\deg\{v,x\} \geq (n+3)/2$ and $\deg\{u,w\} \geq (n+3)/2$. Hence,

$$\deg\{v,x\} + \deg\{u,w\} \geq n + 3,$$

and Lemma 3 follows. ∎

Lemma 4. If G is a graph of order n containing no hamiltonian $x - y$ path and P is an $x - y$ path in G missing only v (where v is not adjacent to x), then any two neighbors of v on P, say s and t, satisfy $d_p(s,t) \geq 3$.

Proof. Clearly, $d_p(s,t) \geq 2$ or a hamiltonian path from x to y is immediate. Thus, suppose $d_p(s,t) = 2$ and let $w = t^-$ be the vertex on P between s and t. Further, let $u = s^-$ be the predecessor (in the x to y direction) of s on P (note that $u = x$ is possible).

For each z adjacent to either v or w, both u and v are not adjacent to z^- (where $z \in (x,s]$ or $z \in [t,y]$), or else a hamiltonian $x - y$ path is easily found. Thus, (as the adjacencies of u and v are restricted to P) we see that

$$\deg\{u,v\} \leq n - 1 - \deg\{v,w\} + 1,$$

hence

$$\deg\{u,v\} + \deg\{v,w\} \leq n,$$

a contradiction. Thus, Lemma 4 holds. ∎

Suppose G is a graph of order n with no hamiltonian $x - y$ path and suppose G contains an $x - y$ path missing only one vertex that is not adjacent to the first vertex x of the path. Further, suppose this vertex has consecutive adjacencies say s and t on the path P such that $d_P(s,t) = 3$. Then, over all such paths and vertices in G, we may select a path $P = x, v_2, \ldots, v_{n-1} = y$, a missing vertex v and $s, t \in N(v)$ such that $d_P(s,t) = 3$ and $d_P(x,s)$ is a minimum. We may assume that $s = v_j$

and $t = v_k$ (where $k = j + 3$). Also, let $u = v_{j+1}$ and $w = v_{j+2}$. Finally suppose that G satisfies the condition that for each pair of distinct 2-sets S_1 and S_2, we have that $\deg S_1 + \deg S_2 \geq n + 1$. In the remaining lemmas we assume this situation holds in the graph G.

Lemma 5. In G, with vertices u, v, w defined as above, $\deg\{v, w\} + \deg\{u, v\} \leq n + 1$, hence,

$$\deg\{v, w\} + \deg\{u, v\} = n + 1.$$

Proof. For each $v_i \in N(v) \cup N(w)$, $1 \leq i < n - 1$, $i \neq j + 1$, the vertex $v_{i+1} \notin N(u) \cup N(v)$ or a hamiltonian $x - y$ path results. Thus, we see that $\deg\{u, v\} \leq (n - 1) + 1 - \deg\{v, w\} + 1$ (accounting for the nonadjacency claim failing for w and possibly v_{n-1}). Hence, the lemma follows. ∎

Since we are considering graphs of order n that satisfy the degree condition $\deg S_1 + \deg S_2 \geq n + 1$, pairs of vertices like those of Lemma 5 are in a sense critical, as our adjacency–nonadjacency count is "tight." If we could find any other vertex that is not forced to be adjacent to one pair and nonadjacent to the other pair, our degree sum count would fall below $n + 1$. More precisely, a vertex v_i is called a *bonus vertex* if $v_i \notin (N(v) \cup N(w) - \{v_{n-1}\})^+ \cup (N(u) \cup N(v))$. Note that $x \in N(u)$, otherwise x is a bonus vertex.

Lemma 6. In G, the vertex $u = v_{j+1}$ is not adjacent to v_{j-1}.

Proof. Suppose $u = v_{j+1}$ is adjacent to v_{j-1}. Then for every $z \in N(v) \cup N(w)$ where $z \in [x, v_{j-1})$ or $z \in (t^+ = v_{k+1}, y)$, we have that $z^+ \notin N(v) \cup N(v_{k+1})$ or an $x - y$ hamiltonian path exists. Further, in the interval $[v_{j-1}, v_{k+1}]$, v has only s and t as neighbors, while if w is adjacent to any of v_{j-1}, s, or v_{k+1} a hamiltonian $x - y$ path results. Finally, v_{k+1} is not adjacent to s, u, or w or again a hamiltonian $x - y$ path results. Thus, in this region, $\deg\{v, w\} = 3$ and $\deg\{v, v_{k+1}\} \leq 3$. Finally, note that y has been omitted from our count. But then,

$$\deg\{v, v_{k+1}\} \leq (n - 1) - 6 - (\deg\{v, w\} - 3) + 3 + 1,$$

hence

$$\deg\{v, w\} + \deg\{v, v_{k+1}\} \leq n,$$

again a contradiction, completing the proof of Lemma 6. ∎

Now, since w is not adjacent to v_{j-1} (or a hamiltonian $x - y$ path is immediate) and just as clearly, v is not adjacent to v_{j-1}, by an argument similar to that of Lemma 5 and Lemma 5 itself, we have that $x \neq v_{j-1}$ (or $x = v_{j-1}$ would be a bonus vertex). By Lemma 4 we also have that v is not adjacent to v_{j-2} and so from the argument of Lemma 5, u is adjacent to v_{j-2} or w is adjacent to v_{j-2}, or v_{j-1} would be a bonus vertex.

Lemma 7. In G, the vertex w must be adjacent to v_{j-2}.

Proof. Since $v_{j-1} \notin N(u) \cup N(v)$, then $v_{j-1} \in (N(v) \cup N(w) - \{v_{n-1}\})^+$. Using the fact that $v_{j-2} \notin N(v)$, we have $v_{j-2} \in N(w)$. ∎

Lemma 8. In G, x must be adjacent to v_{j-1}.

Proof. If $x = v_{j-2}$, then x is adjacent to v_{j-1}. Thus, suppose $x \neq v_{j-2}$. Suppose also that x is not adjacent to v_{j-1}. Now consider the path

$$P': x = v_1, \dots, v_{j-2}, w, u, s = v_j, v, t = v_k, \dots, y$$

with $v' = v_{j-1}$, $s' = v_{j-2}$ and $t' = v_j$. Then, P', v', s' and t' contradicts our choice of P, s, t with $d_P(x, s)$ as a minimum. Hence, Lemma 8 follows and so x is adjacent to v_{j-1}. ∎

Lemma 9. In G, $x \neq v_{j-2}$.

Proof. Suppose to the contrary that $x = v_{j-2}$. Recall, $x \in N(u)$. Now, for each i with $v_i \in N(v) \cup N(v_{j-1})$, then $v_{i-1}, v_{i+1} \notin N(v_{j-1})$. Thus, by Lemma 2, $\deg\{v, v_{j-1}\} \leq n/2$. However, from this fact, combined with Lemma 5, we either have $\deg\{v, v_{j-1}\} + \deg\{v, w\} < n + 1$ or $\deg\{v, v_{j-1}\} + \deg\{u, v\} < n + 1$, a contradiction. ∎

By Lemma 9, the vertex v_{j-3} exists. If $x = v_{j-3}$, then $x, v_{j-1}, v_{j-2}, w, u, s, v, t, \dots, y$ is a hamiltonian $x - y$ path in G, a contradiction. Hence, $x \neq v_{j-3}$. By our choice of s, t, we see that v is not adjacent to v_{j-3}. Also, since w is adjacent to v_{j-2}, then v_{j-3} is not adjacent to u (or a hamiltonian $x - y$ path would exist).

Lemma 10. In G, $v_{j-2} \notin N(u)$.

Proof. Suppose to the contrary that $v_{j-2} \in N(u)$. Then, if $v_i \in N(v) \cup N(v_{j-1})$, both v_{i-1}, $v_{i+1} \notin N(v_{j-1})$. Hence, By Lemma 2, $\deg\{v, v_{j-1}\} \leq n/2$, a contradiction when combined with Lemma 5. ∎

Now, by Lemma 10, $v_{j-2} \notin N(u)$. Then we note that $v_{j-3} \in N(w)$, for otherwise v_{j-1} is a bonus vertex, contradicting Lemma 5. Hence, we may assume that $v_l, v_{l+1}, \dots, v_{j-2} \in N(w)$ where $v_l = x$ or $v_{l-1} \notin N(w)$. However, if $x = v_l$, then $x = v_l, v_{j-1}, \dots, v_{l+1}, w, u, s, v, t, \dots, y$ is a hamiltonian $x - y$ path, a contradiction. Thus, $x \neq v_l$, hence, $l \geq 2$.

However, we also have that $v_2 \neq v_l$; otherwise $x = v_{l-1}, v_{j-1}, \ldots, v_l, w, u, s, v, t, \ldots, y$ would be a hamiltonian $x - y$ path. Hence, $l \geq 3$.

Lemma 11. In G, $v_2 \notin N(u) \cup N(v) \cup N(w)$ and $v_3 \in N(u) \cup N(v) \cup N(w)$.

Proof. If $v_2 \in N(u)$, then

$$x, v_{j-1}, v_{j-2}, w, v_{j-3}, \ldots, v_2, u, s, v, t, \ldots, y$$

is a hamiltonian $x - y$ path, a contradiction.

If $v_2 \in N(v)$, then

$$x, v_{j-1}, \ldots, v_2, v, s, \ldots, y$$

is a hamiltonian $x - y$ path, a contradiction.

We see that $v_2 \notin N(w)$, for otherwise

$$x, v_{j-1}, \ldots, v_2, w, u, s, v, t, \ldots, y$$

would be a hamiltonian path from x to y. Hence, $v_2 \notin N(u) \cup N(v) \cup N(w)$ and so, by the count in Lemma 5, $v_3 \in N(u) \cup N(v) \cup N(w)$ or else v_3 would be a bonus vertex. ∎

Note that since x is not adjacent to v, $x \in N(u) \cap N(w)$, otherwise v_2 is a bonus vertex by Lemma 11, which contradicts Lemma 5. Further, v_2 is not adjacent to v_{j-1} for otherwise

$$x, v_{j-1}, v_2, \ldots, v_{j-2}, w, u, s, v, t, \ldots, y$$

is a hamiltonian $x - y$ path, a contradiction.

Lemma 12. In G suppose that there is a path from x to y missing only v, that $\deg v \geq (n+1)/4$, and that v is not adjacent to x. Then G contains a hamiltonian $x - y$ path.

Proof. Suppose G is as above and that the result fails to hold. By Lemmas 3–11, v_3 must be adjacent to one of u, v, or w. First, suppose that $v_3 \in N(u)$.

Note that v_2 is not adjacent to v_p where $l \leq p \leq j-3$, for otherwise,

$$x, v_2, v_p, \ldots, v_3, u, w, v_{p+1}, \ldots, s, v, t, \ldots, y$$

is a hamiltonian $x - y$ path. Next, note that v_2 is not adjacent to v_{j-1}, for then

$$x, v_{j-1}, v_2, \ldots, v_{j-2}, w, u, s, v, t, \ldots, y$$

is a hamiltonian $x - y$ path in G.

Claim 1. For each i, $v_i \in N(v_2)$ implies that $v_{i-1}, v_{i+1} \notin N(v_2)$.

Proof of Claim 1. If $4 \leq i + 1 \leq l$, then v_i, $v_{i+1} \in N(v_2)$ implies that

$$x, v_{j-1}, v_{j-2}, w, v_{j-3}, \ldots, v_{i+1}, v_2, v_i, \ldots, v_3, u, s, v, t, \ldots, y$$

is a hamiltonian $x - y$ path.

If $i \geq t$, then $v_i, v_{i+1} \in N(v_2)$ implies that

$$x, v_{j-1}, v_{j-2}, w, v_{j-3}, \ldots, v_3, u, s, v, t, \ldots, v_i, v_2, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path. Thus, Claim 1 holds.

Claim 2. For every i, $v_i \in N(v)$ implies that $v_{i-1}, v_{i+1} \notin N(v_2)$.

Proof of Claim 2. Note that v is not adjacent to v_p where $l + 1 \leq p \leq j - 2$ (by Lemma 2), and that $l \leq j - 4$ if v is adjacent to v_l (by the choice of s and t). Thus, if $v_l \in N(v)$, then v_{l-1} is not adjacent to v_2, for otherwise

$$x, v_2, v_{l-1}, \ldots, v_3, u, w, v_l, \ldots, s, v, t, \ldots, y$$

is a hamiltonian $x - y$ path.

For $i < l$, $v_{i-1} \in N(v_2)$ implies

$$x, v_2, v_{i-1}, \ldots, v_3, u, s, v_{j-1}, v_{j-2}, w, v_{j-3}, \ldots, v_i, v, t, \ldots, y$$

is a hamiltonian $x - y$ path. While $v_{i+1} \in N(v_2)$ implies that

$$x, v_2, v_{i+1}, \ldots, v_l, w, v_{l+1}, \ldots, u, v_3, \ldots, v_i, v, t, \ldots, y$$

is a hamiltonian $x - y$ path.

For $i \geq k$, $v_{i-1} \in N(v_2)$ implies that

$$x, w, \ldots, v_2, v_{i-1}, \ldots, t, v, v_i, \ldots, y$$

is a hamiltonian $x - y$ path. (Since v_2 is not adjacent to w, $i = k$ is obvious.)

Also, $v_{i+1} \in N(v_2)$ implies that

$$x, u, w, t, \ldots, v_i, v, s, \ldots, v_2, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path. Thus, Claim 2 follows.

Now by Claims 1 and 2, $\deg\{v, v_2\} \le n/2$. Thus, $\deg\{u, v\} \ge (n/2) + 1$ and $\deg\{v, w\} \ge (n/2) + 1$, which contradicts Lemma 5. Therefore, we conclude that v_3 is not adjacent to u.

Next suppose that v_3 is adjacent to v. For every i, $3 \le i \le j - 2$, if $v_i \in N(v_2)$, then $v_{i+1} \notin N(v_2)$, for otherwise

$$x, v_{j-1}, \dots, v_{i+1}, v_2, v_i, \dots, v_3, v, s, u, w, t, \dots, y$$

is a hamiltonian $x - y$ path.

Recall, v_2 is not adjacent to v_{j-1} or u or w. For $i \ge k$, if $v_i, v_{i+1} \in N(v_2)$, then again a hamiltonian $x - y$ path results, namely

$$x, v_{j-1}, \dots, v_3, v, s, u, w, t, \dots, v_i, v_2, v_{i+1}, \dots, y .$$

For all i, if $v_i \in N(v)$, then both v_{i-1} and v_{i+1} are not in $N(v_2)$. This follows since for $i \ge k$, $v_{i-1} \in N(v_2)$ implies

$$x, w, u, s, \dots, v_2, v_{i-1}, \dots, t, v, v_i, \dots, y$$

is a hamiltonian $x - y$ path in G. While $v_{i+1} \in N(v_2)$ implies that

$$x, u, w, t, \dots, v_i, v, s, \dots, v_2, v_{i+1}, \dots, y$$

is a hamiltonian $x - y$ path in G. While for $3 \le i \le j$, if $v_{i-1} \in N(v_2)$, then

$$x, v_2, v_{i-1}, \dots, v_3, v, v_i, \dots, y$$

is a hamiltonian $x - y$ path in G; if $v_{i+1} \in N(v_2)$, then

$$x, v_{j-1}, \dots, v_{i+1}, v_2, v_3, \dots, v_i, v, s, u, w, t, \dots, y$$

is a hamiltonian $x - y$ path.

Hence, by these conditions and Lemma 2, $\deg\{v, v_2\} \le n/2$, contradicting Lemma 5.

Thus, we conclude that v_3 is not adjacent to v. Hence, from our earlier conclusions, $v_3 \notin N(u) \cup N(v)$. Thus, v_3 must be adjacent to w. Hence, $v_4 \notin N(v) \cup N(u)$. By Lemma 5, v_4 is adjacent to w or v_4 is a bonus vertex. Thus, we may assume that $v_3, v_4, \dots, v_r \in N(w)$ with $v_{r+1} \notin N(w)$. Obviously, $r \le j - 2$. Clearly, $v_{r+1} \notin N(u) \cup N(v) \cup N(w)$. Hence, v_{r+1} is a bonus vertex, contradicting Lemma 5. (Note that $v_3, v_4, \dots, v_r \in N(v) \cup N(u)$.) Hence, Lemma 12 is proved. ∎

Lemma 13. In G, let $Q : v_1, v_2, \dots, v_{n-1}$ be an $x - y$ path missing only vertex $v = v_0$. Let $N(x) = V(G) - x$ and $N(y) = V(G) - y$. Further, suppose that $\deg S_1 + \deg S_2 \geq n + 1$ for any pair of distinct 2-sets $S_1, S_2 \subset V(G)$. Then, G has a hamiltonian $x - y$ path.

Proof. Suppose this was not the case. Let $x' = v_2$ and $y' = v_{n-2}$. Let $v_j \in N(v)\,(2 < j < n - 2)$, which must exist as G is 3-connected (or else a hamiltonian $x - y$ path results). Then, v is not adjacent to x' or y' or a hamiltonian $x - y$ path is immediate. Note that x' is not adjacent to y' for otherwise

$$x, v, v_j, \ldots, x', y', \ldots, v_{j+1}, y$$

is a hamiltonian $x - y$ path in G.

Claim. For all $i \geq 3$, if $v_i \in N(v) \cup N(y')$, then $v_{i+1} \notin N(v) \cup N(x')$.

Proof of Claim. If $v_i \in N(v)$, the fact that $v_{i+1} \notin N(v)$ is obvious, while if $v_{i+1} \in N(x')$, then

$$x, v, v_i, \ldots, x', v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G.

If $v_i \in N(y')$ then if $v_{i+1} \in N(v)$ we see that

$$x, v, v_{i+1}, \ldots, y', v_i, \ldots, x', y$$

is a hamiltonian $x - y$ path in G. On the other hand, if $v_{i+1} \in N(x')$, then when $i < j$,

$$x, v, v_j, \ldots, v_{i+1}, x', \ldots, v_i, y', \ldots, v_{j+1}, y$$

is a hamiltonian $x - y$ path in G, while when $i > j$,

$$x, v, v_j, \ldots, x', v_{i+1}, \ldots, y', v_i, \ldots, v_{j+1}, y$$

is again a hamiltonian $x - y$ path. Note that $x' \in N(v_{j+1})$ implies that

$$x, v, v_j, \ldots, x', v_{j+1}, \ldots, y$$

is a hamiltonian $x - y$ path, proving our claim.

Hence, by the claim, $\deg\{x', v\} \leq n - 1 - \deg\{y', v\} + 1$. Thus, $\deg\{x', v\} + \deg\{y', v\} \leq n$, a contradiction. Thus, G must have an $x - y$ hamiltonian path. ∎

3. THE MAIN RESULT

We now turn our attention to the proof of the main result, Theorem C.

Proof of Theorem C. Suppose the result fails to hold. Let G be an edge-maximal counterexample of order n and suppose that G has no hamiltonian path between vertices x and y.

Claim 1. There is a vertex $v \notin N(x)$ such that $\deg v \geq (n + 1)/4$.

Proof of Claim 1. Suppose such a vertex v does not exist.

Claim 1.1. The graph G has at most two vertices not adjacent to x (or y).

Proof of Claim 1.1. Suppose there are three vertices not adjacent to x, say u, v, and w. Then $\deg u$, $\deg v$, and $\deg w$ are all less than $(n + 1)/4$. But then,

$$\deg\{u, v\} + \deg\{u, w\} < n + 1,$$

a contradiction.

Claim 1.2. $N(x) = V(G) - x$ (and $N(y) = V(G) - y$).

Proof of Claim 1.2. Suppose that $v \neq x$ and $v \notin N(x)$. By our choice of G, the graph $G + xv$ has a hamiltonian path from x to y and this path uses the edge xv. Hence, G has a path from v to y missing only x. By Claim 1.1, x has at most two nonneighbors. If x has only v as a nonneighbor, then by the 3-connectivity of G there exists a $v_i \in N(x)$ with $v_{i+1} \in N(v)$. But then a hamiltonian $x - y$ path is easily found.

Thus, x has two nonneighbors. Let the second nonneighbor be v_k, $k \geq 2$. If $k = 2$, then by the 3-connectivity and the above argument, a hamiltonian $x - y$ path can easily be found. If $k \geq 3$, then $v_k \notin N(v)$ or else

$$x, v_{k-1}, v_{k-2}, \ldots, v, v_k, v_{k+1}, \ldots, y$$

is a hamiltonian $x - y$ path.

Since G is 3-connected, v is adjacent to at least two vertices in $\{v_3, \ldots, y\}$. So v is adjacent to some v_j, $3 \leq j \leq n - 1$ and $j \neq k + 1$. Hence,

$$x, v_{j-1}, \ldots, v, v_j, \ldots, y$$

is a hamiltonian $x - y$ path, again producing a contradiction and completing the proof of Claim 1.2.

Therefore, we have that $N(x) = V(G) - x$ and similarly, $N(y) = V(G) - y$. Then, by Lemma 13, G has a hamiltonian $x - y$ path, a contradiction. Thus, we may assume that there exists a vertex $v \notin N(x)$ such that $\deg v \geq (n+1)/4$, proving Claim 1.

Claim 2. The graph G has a path $P: v = v_1, v_2, \ldots, v_{n-1} = y$ missing only x, where $x \notin N(v)$ and $\deg v \geq (n+1)/4$.

Proof of Claim 2. By Claim 1, we have $x \notin N(v)$, satisfying the desired degree condition. Now, by our choice of G as a maximal counterexample, $G + xv$ has a hamiltonian $x - y$ path. Thus, the desired path P clearly exists.

Now let $s = v_j$ be the neighbor of x on P closet to y where $s \neq y$. Let $x' = v_{j+1}$ (noting $x' = y$ is possible).

Claim 3. For $i \leq j$, if $v_i \in N(x) \cup N(x')$, then $v_{i+1} \notin N(v) \cup N(v_2)$. For $j + 2 \leq i \leq n - 1$, if $v_i \in N(x) \cup N(x')$, then $v_{i-1} \notin N(v) \cup N(v_2)$.

Proof of Claim 3. Suppose $i \leq j$. If $v_i \in N(x)$, then $v_{i+1} \in N(v)$ implies that

$$x, v_i, \ldots, v, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G; while $v_{i+1} \in N(v_2)$ implies that

$$x, v_i, \ldots, v_2, v_{i+1}, \ldots, y$$

is an $x - y$ path in G missing only v (with $\deg v \geq (n+1)/4$) and by Lemma 12, a hamiltonian $x - y$ path exists in G, again a contradiction.

Now suppose $v_i \in N(x')$. If $v_{i+1} \in N(v)$, then

$$x, s, \ldots, v_{i+1}, v, \ldots, v_i, x', \ldots, y$$

is a hamiltonian $x - y$ path in G; while if $v_{i+1} \in N(v_2)$, then

$$x, s, \ldots, v_{i+1}, v_2, \ldots, v_i, x', \ldots, y$$

is an $x - y$ path in G missing only v and again by Lemma 12 a hamiltonian $x - y$ path can be found, a contradiction.

For $j + 2 \leq i \leq n - 1$, if $v_i \in N(x')$ and $v_{i-1} \in N(v)$, then

$$x, s, \ldots, v, v_{i-1}, \ldots, x', v_i, \ldots, y$$

is a hamiltonian $x - y$ path; while $v_{i-1} \in N(v_2)$ implies that

$$x, s, \ldots, v_2, v_{i-1}, \ldots, x', v_i, \ldots, y$$

along with v and an application of Lemma 12 yields the desired hamiltonian path.

By a simple counting argument, we have that

$$\deg\{x, x'\} + \deg\{v, v_2\} \leq n + 1,$$

which implies that

$$\deg\{x, x'\} + \deg\{v, v_2\} = n + 1. \qquad (*)$$

Once again, we define $v_i (i < j)$ to be a bonus vertex (with respect to equation (*)) if $v_i \notin (N(x) \cup N(x'))^+ \cup (N(v) \cup N(v_2))$, while for $i \geq j + 2$, v_i is a bonus vertex if $v_i \notin (N(x) \cup N(x'))^- \cup (N(v) \cup N(v_2))$.

Claim 4. Suppose $k < j$, if $v_k \in N(x) \cup N(x')$ and $v_{k+1} \notin N(x) \cup N(x')$, then $v_{k+2} \in N(v) \cup N(v_2)$ and for any i, $\{v_i, v_{i+1}\} \not\subseteq N(v_{k+1})$.

Proof of Claim 4. Note that $v_{k+2} \in N(v) \cup N(v_2)$ or else it is a bonus vertex.

Now suppose $\{v_i, v_{i+1}\} \subset N(v_{k+1})$ and that $v_{k+2} \in N(v)$. Then for $i + 1 \leq k$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_{i+1}, v_{k+1}, v_i, \ldots, v_2, v, v_{k+2}, \ldots, y$$

is a hamiltonian $x - y$ path in G; while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{k+2}, v, v_2, \ldots, v_i, v_{k+1}, v_{i+1}, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path in G.

For $j - 1 \geq i \geq k + 2$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_2, v, v_{k+2}, \ldots, v_i, v_{k+1}, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path, while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{i+1}, v_{k+1}, v_i, \ldots, v_{k+2}, v, v_2, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path.

Finally, suppose $i \geq j + 2$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_2, v, v_{k+2}, \ldots, v_i, v_{k+1}, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G, while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{k+2}, v, v_2, \ldots, v_k, x', \ldots, v_i, v_{k+1}, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G. If instead, $v_{k+2} \in N(v_2)$, then all the same paths are produced, minus the vertex v. However, v and these paths satisfy Lemma 12, producing the desired contradiction.

Thus, in all cases a contradiction is reached and hence Claim 4 is proved.

Claim 5. No two vertices $v_k, v_l, k < l < j$ exist such that $v_k, v_l \in N(x) \cup N(x')$ and $v_{k+1}, v_{l+1} \notin N(x) \cup N(x')$.

Proof of Claim 5. Suppose the claim fails to hold. Then we may select k, l such that no vertex between v_k and v_l on P also has this property. Then by (*) and our choice of v_k and v_l, we may assume that

$$v_{k+2}, \ldots, v_p \in N(v) \cup N(v_2) \quad \text{and} \quad v_p, v_{p+1}, \ldots, v_l \in N(x) \cup N(x').$$

Note that $p = k + 2$ or $p = l$ are possible.

The following notation will now be useful. In denoting a path as

$$a, b, \ldots, (c), \ldots, d$$

the notation (c) means that vertex c may or may not be used in the path, thus really creating two paths as possibilities.

Claim 5.1. Note that $v_{k+1} \notin N(v_{l+1})$.

Proof of Claim 5.1. Suppose the claim fails to hold. If $x \in N(v_l)$, then

$$x, v_l, \ldots, v_{k+2}, v_2, (v), \ldots, v_{k+1}, v_{l+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12, while if $x' \in N(v_l)$, then

$$x, s, \ldots, v_{l+1}, v_{k+1}, \ldots, v_2, (v), v_{k+2}, \ldots, v_l, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. Thus, both cases lead to a contradiction.

Claim 5.2. Note that $v_{k+3}, \ldots, v_l \notin N(v_{k+1})$ and $v_{k+2}, \ldots, v_{l-1}, \notin N(v_{l+1})$.

Proof of Claim 5.2. This follows since if $v_i \in N(v_{k+1})$, $k + 3 \le i \le p + 1$, then if $x \in N(v_k)$

$$x, v_k, \ldots, v_2, (v), v_{i-1}, \ldots, v_{k+1}, v_i, \ldots, y$$

is a hamiltonian $x - y$ path or a path satisfying Lemma 12; while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_i, v_{k+1}, \ldots, v_{i-1}, (v), v_2, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. On the other hand, for $p + 1 \leq i \leq l$, if $x \in N(v_{i-1})$, then

$$x, v_{i-1}, \ldots, v_{k+2}, (v), v_2, \ldots, v_{k+1}, v_i, \ldots, y$$

is a hamiltonian $x - y$ path or a path satisfying Lemma 12; while if $x' \in N(v_{i-1})$, then

$$x, s, \ldots, v_i, v_{k+1}, \ldots, v_2, (v), v_{k+2}, \ldots, v_{i-1}, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. Thus, again we are lead to a contradiction.

If $v_{l+1} \in N(v_i)$, then for $k + 2 \leq i \leq p - 1$, if $x \in N(v_l)$, then

$$x, v_l, \ldots, v_{i+1}, (v), v_2, \ldots, v_i, v_{l+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12; while if $x' \in N(v_l)$, then

$$x, s, \ldots, v_{l+1}, v_i, \ldots, v_2, (v), v_{i+1}, \ldots, v_l, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. Also, for $p \leq i \leq l - 1$, if $x \in N(v_{i+1})$, then (recall by Claim 4, $v_{l+2} \in N(v) \cup N(v_2)$)

$$x, v_{i+1}, \ldots, v_{l+1}, v_i, \ldots, v_2, (v), v_{l+2}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12; while if $x' \in N(v_{i+1})$, then

$$x, s, \ldots, v_{l+2}, (v), v_2, \ldots, v_i, v_{l+1}, \ldots, v_{i+1}, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. Hence, the claim is proved.

Claim 5.3. If $v_i \in N(v_{l+1})$, then $v_{i+1} \notin N(v_{k+1})$ for $i + 1 \leq k$, $i \geq l + 2$.

Proof. To see this, note that for $i + 1 \leq k$ and $v_{i+1} \in N(v_{k+1})$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_{i+1}, v_{k+1}, \ldots, v_{l+1}, v_i, \ldots, v_2, (v), v_{l+2}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12; while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{l+2}, (v), v_2, \ldots, v_i, v_{l+1}, \ldots, v_{k+1}, v_{i+1}, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. Now when $i \geq l + 2$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_2, (v), v_{l+2}, \ldots, v_i, v_{l+1}, \ldots, v_{k+1}, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12; while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{i+1}, v_{k+1}, \ldots, v_{l+1}, v_i, \ldots, v_{l+2}, (v), v_2, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12 when $i + 1 \leq j$ and

$$x, s, \ldots, v_{l+2}, (v), v_2, \ldots, v_k, x', \ldots, v_i, v_{l+1}, \ldots, v_{k+1}, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path or a path satisfying Lemma 12 when $i \geq j + 2$, completing the proof of Claim 5.3.

Claim 5.4. If $v_i \in N(v_{l+1})$, then $v_{i-1} \notin N(v_{k+1})$ for $i \leq k$, $i + 1 \geq l + 2$.

Proof. Suppose $v_{i-1} \in N(v_{k+1})$. For $i \leq k$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_i, v_{l+1}, \ldots, v_{k+1}, v_{i-1}, \ldots, v_2, (v), v_{l+2}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12; while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{l+2}, (v), v_2, \ldots, v_{i-1}, v_{k+1}, \ldots, v_{l+1}, v_i, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12. Also, for $i - 1 \geq l + 2$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_2, (v), v_{l+2}, \ldots, v_{i-1}, v_{k+1}, \ldots, v_{l+1}, v_i, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12; while if $x' \in N(v_k)$, then

$$x, s, \ldots, v_i, v_{l+1}, \ldots, v_{k+1}, v_{i-1}, \ldots, v_{l+2}, (v), v_2, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12 when $i \leq j$; while for $i - 1 \geq j + 2$,

$$x, s, \ldots, v_{l+2}, (v), v_2, \ldots, v_k, x', \ldots, v_{i-1}, v_{k+1}, \ldots, v_{l+1}, v_i, \ldots, y$$

is a hamiltonian $x - y$ path producing the desired final contradiction.

In a manner similar to Lemma 2, we may now conclude that $|N(v_{k+1}) \cup N(v_{l+1})| \leq n/2$, a contradiction to equation (*). Therefore, there is at most one vertex v_k, $k < j$, with $v_k \in N(x) \cup N(x')$ and $v_{k+1} \notin N(x) \cup N(x')$.

Claim 6. No k exists such that $k < j$, $v_k \in N(x) \cup N(x')$ and $v_{k+1} \notin N(x) \cup N(x')$.

Proof of Claim 6. Suppose such a k exists. By Claim 5, it is unique. Note that if $v_i \in N(v_{k+1}) \cup N(x)$, then $v_{i-1}, v_{i+1} \notin N(v_{k+1})$. This follows since by Claim 4, $v_{i-1}, v_{i+1} \notin N(v_{k+1})$ if $v_i \in N(v_{k+1})$, so let $v_i \in N(x)$. If $v_{i-1} \in N(v_{k+1})$, then for $i \leq k$

$$x, v_i, \ldots, v_{k+1}, v_{i-1}, \ldots, v_2, (v), v_{k+2}, \ldots, y$$

is a hamiltonian $x - y$ path in G, or we obtain an $x - y$ path missing only v and apply Lemma 12 to produce the desired hamiltonian $x - y$ path, and hence a contradiction.

For $i \geq k + 4, v_{i-2} \notin N(x) \cup N(x') \cup N(v) \cup N(v_2)$, for otherwise if $v_{i-2} \in N(x)$, then

$$x, v_{i-2}, \ldots, v_{k+2}, (v), v_2, \ldots, v_{k+1}, v_{i-1}, \ldots, y$$

is a hamiltonian $x - y$ path in G or a path in which Lemma 12 applies; while if $v_{i-2} \in N(x')$, then

$$x, s, \ldots, v_{i-1}, v_{k+1}, \ldots, v_2, (v), v_{k+2}, \ldots, v_{i-2}, x', \ldots, y$$

is a hamiltonian $x - y$ path in G or a path satisfying Lemma 12, again a contradiction.

Now if $v_{i-2} \in N(v) \cup N(v_2)$, if $x \in N(v_k)$, then

$$x, v_k, \ldots, v_2, (v), v_{i-2}, \ldots, v_{k+1}, v_{i-1}, \ldots, y$$

is a hamiltonian $x - y$ path in G, or we again use Lemma 12 to obtain the desired contradiction. Otherwise, if $x' \in N(v_k)$, then

$$x, s, \ldots, v_{i-1}, v_{k+1}, \ldots, v_{i-2}, (v), v_2, \ldots, v_k, x', \ldots, y$$

is a hamiltonian $x - y$ path or again we use Lemma 12 to provide the desired contradiction.

Hence, by Claim 5, $v_p \notin N(x) \cup N(x')$ for every $p, k+2 \leq p \leq i-2$, and so v_{i-2} is a bonus vertex for (*), a contradiction.

Now let $v_{i+1} \in N(v_{k+1})$, and suppose $i+1 \leq k$. By Claim 5, $v_{i+1}, \ldots, v_k \in N(x) \cup N(x')$. This implies that $i+1 = k$, for otherwise $v_{i+2} \in N(x) \cup N(x')$, but $x \in N(v_{i+2})$ implies that

$$x, v_{i+2}, \ldots, v_{k+1}, v_{i+1}, \ldots, v_2, (v), v_{k+2}, \ldots, y$$

is a hamiltonian $x-y$ path or a path with which we again apply Lemma 12, providing a contradiction. Otherwise, if $x' \in N(v_{i+2})$, then

$$x, s, \ldots, v_{k+2}, (v), v_2, \ldots, v_{i+1}, v_{k+1}, \ldots, v_{i+2}, x', \ldots, y$$

produces the desired contradiction (or again use Lemma 12). Finally, $i \geq k+2$ implies

$$x, v_i, \ldots, v_{k+2}, (v), v_2, \ldots, v_{k+1}, v_{i+1}, \ldots, y$$

itself, or using Lemma 12 when $v_{k+2} \in N(v_2)$, gives the needed contradiction.

Now we let m_i be the number of vertices of P between the $(i-1)$-st and i-th neighbors of x. Let v_{k+1} be between the $(l-1)$st and lth neighbor of x, where m_1 equals the number of vertices between v_3 and the first neighbor of x on P (excluding neighbors of x). Note that $v_{k+1} \notin N(v) \cup N(v_2)$.

Therefore, in a manner similar to Lemma 2 (with some slight care), we may conclude that $\deg\{x, v_{k+1}\} \leq n/2$, a contradiction and Claim 6 follows.

Therefore, we may assume that $v_3, \ldots, v_l \in N(v) \cup N(v_2)$ and $v_l, \ldots, v_j \in N(x) \cup N(x')$.

Claim 7. No k, where $j+2 \leq k \leq n-3$, exists such that $v_{k+1} \in N(x) \cup N(x')$ but $v_k \notin N(x) \cup N(x')$.

Proof of Claim 7. Suppose such a vertex exists. Then $v_k \notin N(v) \cup N(v_2)$; otherwise

$$x, s, \ldots, v_2, (v), v_k, \ldots, x', v_{k+1}, \ldots, y$$

is a hamiltonian $x-y$ path (or we use Lemma 12), providing a contradiction. Hence, $v_k \neq v_{j+2}$ for otherwise x' is a bonus vertex for (*), and $v_{k-1} \in N(v) \cup N(v_2)$ for otherwise we may assume that $v_{k-1}, v_{k-2}, \ldots, v_r \in N(x) \cup N(x')$ (to avoid bonus vertices) and $v_{r-1} \notin N(x) \cup N(x')$ (possibly equal to x') would be a bonus vertex for (*).

We claim that if $v_i \in N(x) \cup N(v_k)$, then $v_{i-1}, v_{i+1} \notin N(v_k)$.

Suppose first v_k has consecutive neighbors along P, say v_i, v_{i+1}. For $i+1 \leq j$,

$$x, s, \ldots, v_{i+1}, v_k, v_i, \ldots, v_2, (v), v_{k-1}, \ldots, x', v_{k+1}, \ldots, y$$

is a hamiltonian $x - y$ path (or we use Lemma 12), a contradiction. For $j + 2 \leq i \leq k - 2$,

$$x, s, \ldots, v_2, (v), v_{k-1}, \ldots, v_{i+1}, v_k, v_i, \ldots, x', v_{k+1}, \ldots, y$$

is a hamiltonian $x - y$ path (or we use Lemma 12), providing the desired contradiction. For $i \geq k + 1$,

$$x, s, \ldots, v_2, (v), v_{k-1}, \ldots, x', v_{k+1}, \ldots, v_i, v_k, v_{i+1}, \ldots, y$$

is a hamiltonian $x - y$ path or we use Lemma 12 and obtain a contradiction.

If $v_i \in N(x)$, then $i \leq j$ or $i = n - 1$. For $i \leq j$, if $v_{i-1} \in N(v_k)$, then

$$x, v_i, \ldots, v_{k-1}, (v), v_2, \ldots, v_{i-1}, v_k, \ldots, y$$

produces a hamiltonian $x - y$ path (or an application of Lemma 12); while if $v_{i+1} \in N(v_k)$, then

$$x, v_i, \ldots, v_2, (v), v_{k-1}, \ldots, v_{i+1}, v_k, \ldots, y$$

is a hamiltonian $x - y$ path (or we use Lemma 12), producing a contradiction. Hence, by Lemma 5, $\deg\{x, v_k\} \leq (n - 2)/2 + 1 = n/2$, a contradiction when combined with (*). Hence, Claim 7 follows.

Thus, by Claim 7 we may assume that $v_{j+2}, \ldots, v_p \in N(x) \cup N(x')$ and $v_p, \ldots, y \in N(v) \cup N(v_2)$. Clearly, $x', v_{j+2}, \ldots, v_{p-1} \notin N(v) \cup N(v_2)$, otherwise a hamiltonian $x - y$ path exists (or we find one via an application of Lemma 12). When $x' \neq y$, by direct counting (see Figure 2) we get

$$\deg\{v, v_2\} + \deg\{x, x'\} = (l + (n - p)) + (j - l + 1 + p - (j + 1)) = n$$

a contradiction.

Thus, $x' = y$, which implies that $x' \notin N(v) \cup N(v_2)$.

By 3-connectivity, some v_k, $2 < k < l$, must be adjacent to some v_q, $q > l$. If $v_q = x'$, then

$$x, s, \ldots, v_{k+1}, v_2, (v), \ldots, v_k, y$$

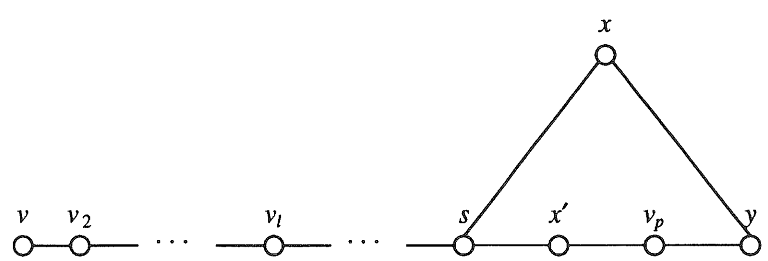

FIGURE 2

is a hamiltonian $x - y$ path (or a path on which we apply Lemma 12). Thus, let $q \leq j$. If $v_{q-1} \in N(x)$, then

$$x, v_{q-1}, \ldots, v_{k+1}, v_2, (v), \ldots, v_k, v_q, \ldots, y$$

is a hamiltonian $x - y$ path (or one on which Lemma 12 applies). If $v_{q-1} \in N(x')$, then

$$x, s, \ldots, v_q, v_k, \ldots, v_2, (v), v_{k+1}, \ldots, v_{q-1}, y$$

is a hamiltonian $x - y$ path (or one is produced via Lemma 12). This provides the final contradiction, completing the proof. ∎

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