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Neighborhood Unions and the Cycle Cover Number of a Graph

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ABSTRACT

For several years, the study of neighborhood unions of graphs has given rise to important structural consequences of graphs. In particular, neighborhood conditions that give rise to hamiltonian cycles have been considered in depth. In this paper we generalize these approaches to give a bound on the smallest number of cycles in G containing all the vertices of G.

We show that if for all $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 2n/5 + 1$, then $V(G)$ is coverable by at most two cycles. Several related results and extensions to t cycles are also given.

INTRODUCTION

There has been considerable interest in recent years in the idea of determining structural features of graphs that have a given "neighborhood union" condition. To be more specific, let G be a graph, for $x \in V(G)$, the *neighborhood* of x, denoted $N(x)$, is the set of vertices adjacent to x. If S is a subset of $V(G)$, then the *degree* of S is

$$\deg(S) = \bigcup_{x \in S} N(x).$$

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If S_2 is an arbitrary subset of two vertices of $V(G)$, then δ_2 is the minimum degree over all such two element subsets. Several "Dirac"-like results pertaining to this minimum degree and cycles in graphs have been given in [2, 6, 7]. For example, in [6] the following result is given:

Theorem A. Let G be a 2-connected graph of order n, if $\delta_2 \geq n/2$, then G is hamiltonian for sufficiently large n.

This result is generalized in [2], where the following is given.

Theorem B. If G is a 2-connected graph of order n, then G contains a cycle of length at least $2\delta_2 - 2$.

In this paper, we continue this line of study and give several results about the minimum number of cycles necessary to cover all the vertices of G.

Many of the papers pertaining to neighborhood unions have concentrated on various hamiltonian related properties. In this paper we give the following result:

Theorem 2. Let G be a 2-connected graph of order $n, n \geq 10$. If $\delta_2 \geq 2n/5 + 1$, then there is a 2-cycle cover of G. Furthermore, one of the cycles can be chosen to be a longest cycle in G.

Clearly, by considering $K_{p,p-1}$, the condition is not sufficient to imply a hamiltonian cycle; thus to be able to cover the vertices with two cycles is, in some sense, the next best possibility. The idea for this research was stimulated by results in [5]; in particular, it is shown that if G is a graph of order n with minimum degree at least $n/3$, then the vertices of G are covered by two cycles. In [4], this result is generalized, and it is shown that if G is a graph of order n and the sum of the degrees of any three independent vertices is at least n, then the vertex set of G can be covered by at most two cycles, edges, or vertices. In the later paper the question of extending this to neighborhood unions is posed. We address that question, and generalize to t-cycle covers.

For completeness we give the following definitions. A cycle C in a graph G is a *dominating cycle* if $V(G) - V(C)$ is an independent subset of vertices. A set of cycles, $C_1, C_2, \ldots, C_t$ is a *t-cycle cover* if each vertex in G is contained in at least one of these cycles. If this is the smallest such set of cycles, then we say that the *cycle cover number* of G is t.

MAIN RESULTS

Theorem 1. If G is a 2-connected graph of order n, with $\delta_2 \geq (n + 5)/3$, then G contains a longest cycle, which is a dominating cycle.

Proof. Over all cycles of maximum length in G, choose C to be one in which $G - C$ has as many components as possible. If all components of $G - C$ are trivial, then C is a dominating cycle, so let T be a nontrivial component of $G - C$. By Theorem B, it follows that

$$|V(C)| \geq 2\left(\frac{n+5}{3}\right) - 2 = \frac{2n+4}{3},$$

and thus $|V(T)| \leq (n-4)/3$. Since $|V(T)| \leq (n-4)/3$, and $\delta_2 \geq (n+5)/3$ it follows that T contains most at one vertex adjacent to fewer than two vertices of C.

We wish to find a path P from one vertex of C to another vertex of C, such that the interior vertices of this path forms a longest path of T. Choose a longest path in T. Suppose one of the endvertices, say u, has no adjacencies in C. It is the only such vertex in T, and since $\delta(G) \geq 2$, u must have an adjacency on P. The predecessor on P of this new adjacency to u is the endvertex of a maximum length path in T, and this path has both endvertices adjacent to at least two vertices on C.

Let P be such a longest path in T with endvertices u and v. Of course $(N(u) \cup N(v)) \cap T$ is contained in P and there are two distinct vertices on C, one adjacent to u and one adjacent to v (these vertices on C could be adjacent to both u and v, but we have shown the existence of a distinct adjacency for each of u and v on C). Let $|P| = k \geq 2$. We will proceed by finding a lower bound for the number of nonadjacencies of u and v in G.

Let $S = \{x_1, x_2, \ldots, x_s\} = (N(u) \cup N(v)) \cap C$. Note, $|N(u) \cup N(v)| \leq k + s$. For convenience, we will refer to an open segment (x_{i-1}, x_i) as a gap, and it contains all the vertices strictly between x_{i-1} and x_i, none of which are adjacent to u nor v. Also, the set S induces s such gaps. Since u and v both have neighbors on C, there are at least two places on the cycle where x_{i-1} is a neighbor of u and x_i is a neighbor of v or vice versa. Since C was chosen to be a longest cycle in G, these two gaps must have at least k vertices each, and as noted above, all of which are nonadjacent to u and v. We now will show that for each of the other $s - 2$ gaps there are at least 2 distinct non-neighbors of u and v.

If the gap contains at least two vertices, then they must all be non-neighbors of u and v. So we need only consider gaps containing a single vertex. Let x_{i-1}, y, x_i be such a gap. Clearly, both x_{i-1} and x_i are adjacent to just one of u or v, say u, for otherwise a longer cycle would result. If y is not adjacent to a vertex off C, then $C' = (C - y) \cup \{u\}$ is a cycle having the same length as C with more components in $G - C'$ than in $G - C$. Hence, y is adjacent to y' off C, and it follows that y' is not in T since a longer cycle would clearly exist. Consequently, neither u nor v is adjacent to y'. Now for each such y, a distinct y' results, for otherwise a longer cycle would exist including all of the vertices of C, y', and at least one vertex of T. Therefore, the number of vertices nonadjacent to u and v is at least

$2k + 2(s - 2)$, which implies

$$2k + 2s - 4 + k + s \leq n,$$

since $|S \cup T| \geq k + s$. This gives

$$|N(u) \cup N(v)| \leq k + s \leq \frac{n + 4}{3},$$

a contradiction. ∎

This result is best possible. Consider the graph obtained by identifying two vertices from three distinct copies of $K_{(n+4)/3}$. This graph is 2-connected of order n with $\delta_2 = (n + 4)/3$, but contains no dominating cycle.

It is well known that a dominating cycle in G corresponds to a hamiltonian cycle in the line graph; thus we get the following:

Corollary. If G is a 2-connected graph of order n, with $\delta_2 \geq (n + 5)/3$, then $L(G)$ contains a hamiltonian cycle.

We will use Theorem 1 to help find a connection between neighborhood unions and the cycle cover number.

Theorem 2. Let G be a 2-connected graph of order $n, n \geq 10$. If $\delta_2 \geq (2n/5) + 1$, then there is a 2-cycle cover of G. Furthermore, one of the cycles can be chosen to be a longest cycle in G.

Proof. Choose C to be a longest cycle in G, say of length m, such that C is a dominating cycle. Of course the existence of such a cycle is guaranteed by Theorem 1. Let $V(C) = \{x_1, x_2, \ldots, x_m\}$ with this "orientation." Also, recall from Theorem B,

$$m \geq 2\left(\frac{2n}{5} + 1\right) - 2 = \frac{4n}{5}.$$

We will proceed by showing that there is a cycle in G containing all the vertices of $G - C$, which we will call W. This will give the desired result.

We introduce some convenient notation. If $w \in W$ and $x_i, x_j \in N(w)$, then we refer to the segment $((x_i, x_j), (x_i, x_j]$, or $[x_i, x_j))[x_i, x_j]$ as an *(open, half open) arc subtended by w.* The *length* of such an arc is the number of vertices included in the segment.

Let $w_0 \in W$, be a vertex of minimum degree t. By the δ_2 requirement in the hypothesis of the theorem, this implies that the remaining vertices in W have degree at least $\max\{t, (2n/5) + 1 - t\}$. Let $[x_{i_0}, x_{i_1})$ be one of the shortest half open arcs subtended by w_0. It follows that the length of this

arc is at most m/t. Furthermore, since C is a longest cycle in G, for any vertex $w \in W - \{w_0\}$

$$|N(w) \cap [x_{i_0}, x_{i_1})| \leq \frac{|[x_{i_0}, x_{i_1})| + 1}{2} \leq \frac{m + t}{2t}. \tag{1}$$

Find the largest k so that there is a subset of k vertices $\{w_1, w_2, \ldots, w_k\}$ in $W - \{w_0\}$, which subtend nonintersecting "right" half open arcs in $C - [x_{i_0}, x_{i_1})$. Over all such k element subsets, choose one where the sum of the lengths of these subtended arcs is as small as possible. Without loss of generality, we may assume that the vertex w_j subtends the arc $[x_{i_{2j}}, x_{i_{2j+1}})$ and that given the orientation of C, we have

$$x_1 \leq x_{i_0} < x_{i_1} \leq x_{i_2} < x_{i_3} \leq \cdots \leq x_{i_{2k}} < x_{i_{2k+1}} \leq x_1 .$$

We observe that for any $w \in W - \{w_0, w_1, w_2, \ldots, w_k\}$ and any $\alpha = 1, 2, \ldots, k$;

$$|N(w) \cap [x_{i_{2\alpha-1}}, x_{i_{2\alpha}})| \leq 1 \tag{2}$$

and

$$|N(w) \cap [x_{i_{2\alpha}}, x_{i_{2\alpha+1}})| \leq 1 , \tag{3}$$

follows by the maximality of k and the minimality of the length of the sum of these subtended arcs, respectively. Note (2) and (3) also hold when the right half open interval is replaced with the left half open interval. For a vertex $w \in W - \{w_0, w_1, w_2, \ldots, w_k\}$, we will refer to a pair of consecutive segments $[x_{i_{2\alpha-1}}, x_{i_{2\alpha}})$ and $[x_{i_{2\alpha}}, x_{i_{2\alpha+1}})$ as *w-good* if equality holds for both (2) and (3) and these intervals. Let $\ell = n - m - k - 1$, that is

$$\ell = |W - \{w_0, w_1, w_2, \ldots, w_k\}| .$$

Label the vertices of $W - \{w_0, w_1, w_2, \ldots, w_k\}$ with $u_1, u_2, \ldots, u_\ell$. We will show for each u_j that there are at least ℓ_{uj}-good pairs of consecutive segments. Before proving this claim, we will demonstrate how this will in fact complete the proof. Since each of the ℓ vertices have ℓ distinct good pairs, we can match each vertex with a distinct pair of good consecutive segments.

At this point it is clear there is a cycle K containing all of $\{w_0, w_1, w_2, \ldots, w_k\}$;

$$K = x_{i_0}, w_0, x_{i_1}, \ldots, x_{i_2}, w_1, x_{i_3}, x_{i_4}, w_2, x_{i_5}, \ldots, x_{i_{2k}}, w_k, x_{i_{2k+1}}, \ldots, x_{i_0} .$$

We need to only be able to alter this cycle and add the remaining ℓ vertices of W. To accomplish this we merely proceed along K and add the u_i's as

they occur. The adjacencies of u_i could either be the included endvertex or an internal vertex for each of the consecutive pair of segments matched with u_i. Thus, there are four possibilities for the adjacencies of the u_i-good pair. Figure 1 shows how each case is handled. Hence, if a matching of the u_i's with the possible "good" pairs of consecutive segments exist, then a cycle containing all of the vertices of W must be in G. The claim assures such a matching.

Proof of Claim. Suppose to the contrary that there is a vertex $u \in W - \{w_0, w_1, w_2, \ldots, w_k\}$ with fewer than ℓu - good pairs of consecutive segments. Recall that $[x_{i_0}, x_{i_1})$ is the only segment in which u could have more than one adjacency. Thus, since u could be adjacent to at most $(k + 1) + (\ell - 1)$ of the $2k + 2$ segments that C is partitioned into, it follows that

$$\deg(u) \leq (k + 1) + (\ell - 2) + \frac{m + t}{2t}.$$

Thus,

$$\deg(u) \leq k + \ell + \frac{n - (k + 1 + \ell) + t}{2t} - 1, \tag{4}$$

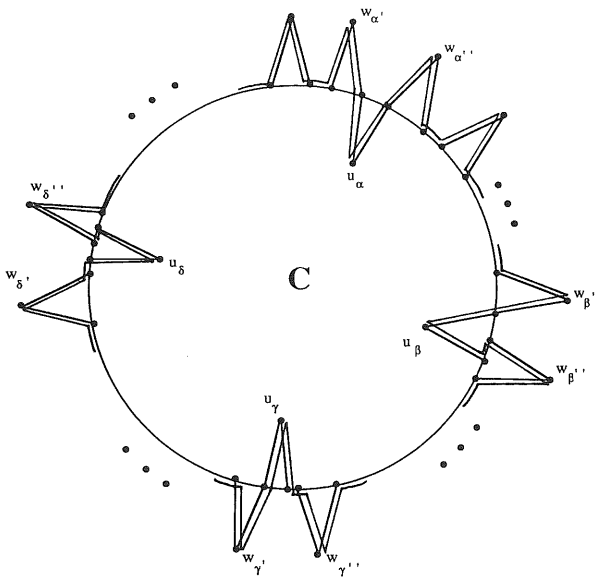

FIGURE 1. Resolving the possibilities for u-good pairs.

since $m = n - (k + 1 + \ell)$. We also know that

$$\deg(u) \geq \max\left\{t, \frac{2n}{5} + 1 - t\right\}.$$

Case 1. Suppose $\max\{t, (2n/5) + 1 - t\} = (2n/5) + 1 - t$. Note that this implies that $t \leq (2n + 5)/10$. We get

$$k + \ell + \frac{n - (k + 1 + \ell) + t}{2t} - 1 \geq \frac{2n}{5} + 1 - t.$$

$$\frac{2t - 1}{2t}(k + \ell) \geq \frac{2n}{5} - t - \frac{n - 1 + t}{2t} + 2.$$

$$(k + \ell) \geq \frac{1}{2t - 1}\left(\frac{4n}{5}t - 2t^2 - n + 3t + 1\right).$$

But since $k + \ell + 1 \leq n/5$, this gives

$$\frac{n}{5} - 1 \geq \frac{1}{2t - 1}\left(\frac{4n}{5}t - 2t^2 - n + 3t + 1\right).$$

Consequently, we get

$$f(t) = 2t^2 - \left(\frac{2n}{5} + 5\right)t + \frac{4n}{5} \geq 0.$$

We observe that $f(2) = -2 < 0$ and $f((2n + 5)/10) = -2 < 0$, and since $2 \leq t \leq (2n + 5)/10$, this inequality is satisfied for no possible values of t, and hence this case is impossible.

Case 2. Suppose $\max\{t, (2n/5) + 1 - t\} = t$. Note that this implies that $t > (2n + 5)/10$, which gives that $(m + t)/2t < 3$ and thus

$$\frac{n}{5} < \deg(u) \leq (k + 1) + \ell,$$

which immediately gives the contradiction

$$k + \ell > \frac{n}{5} - 1.$$

Exhausting these two cases, the claim is proved, and as previously remarked, a cycle containing all the vertices of W is achieved. Thus, a cycle cover of all of G with 2-cycles is attained. ∎

We will now give an indication of the usefulness of the technique given in the previous theorem.

Lemma 3. If G is a graph of order n with maximal dominating cycle C, so the $G - C$ has a set B containing b vertices each of degree at least b, then G contains a cycle containing all the vertices of B.

Proof. As in the theorem, label the vertices of C and find the largest k so that there is a subset of B with k vertices, $W = \{w_1, w_2, \ldots, w_k\}$, which subtend nonintersecting "right" half open arcs in C. Over all such k element subsets, choose one where the sum of the lengths of these subtended arcs is as small as possible. Without loss of generality, we may assume that the vertex w_j subtends the arc $[x_{i_{2j-1}}, x_{i_{2j}})$ and that given the orientation of C, we have

$$x_1 \leq x_{i_1} < x_{i_2} \leq x_{i_3} < \cdots \leq x_{i_{2k-1}} < x_{i_{2k}} \leq x_1 .$$

Let $\ell = b - k$. If for each vertex u of $B - W$, u has ℓ u-good pairs, then the desired cycle would exist. If there existed a vertex u with fewer than ℓ u-good pairs, then it would follow that its degree would be at most $k + \ell - 1$. But since each of these vertices is required to have degree b, and $b = k + \ell$, a contradiction would arise, and the lemma follows. ∎

Theorem 4. If G is 2-connected of order n with $\delta_2 \geq (n + 5)/3$, then G has cycle cover number at most 3.

Proof. By Theorem 1 and Theorem B, it follows that G contains a dominating cycle C of length at least $2((n + 5)/3) - 2$. We now will apply Lemma 3 twice. Let x be vertex of $G - C$ of smallest degree, say d. Choose any $d - 1$ other vertices in $G - C$ to form set D_1. Let $D_2 = G - C - D_1$. The set D_1 is a subset of d vertices all of degree at least d, and D_2 is a subset of at most $((n - 4)/3) - d$ vertices, each of degree at least $((n + 5)/3) - d$. By Lemma 3, there is a cycle containing all the vertices in D_1 and a cycle containing all the vertices of D_2; thus with C, it follows that G contains a 3-cycle cover. ∎

The key to this result is the existence of the dominating cycle. If we assume that a dominating cycle exists, then the result can be extended to cycle cover number t.

Theorem 5. If G is a 2-connected graph of order n, with dominating cycle C, and having

$$\delta_2 \geq \frac{2n}{t + 1},$$

$t \geq 6$, then G has cycle cover number at most t.

Proof. Let C be a maximum length dominating cycle. Since there are vertices of degree at least $n/(t+1)$, it follows that C has length at least $2n/(t+1)$. As in the proof of Theorem 4, continued applications of the lemma gives the desired cycles. ∎

Unfortunately, this result is far from best possible. Even for $t = 3$, Theorem 4 is a vast improvement over Theorem 5. The principle reason for this is that the dominating cycle is assured to be a cycle of maximum length in Theorems 2 and 4, while that is not the case for Theorem 5. It would be of interest to find the "best" possible value for δ_2. Also, the argument depends heavily on the degree of the vertices. It might help if the neighborhood union property could be used more effectively to give a smaller cycle cover number. Finally, it is possible that some structure other than a dominating cycle might be useful in bounding this parameter.

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