52

# Generalized Degree Sums and Hamiltonian Graphs

Avrim Blum*
MIT Lab for Computer Science
545 Technology Sq., Cambridge, MA, U.S.A. 02139
e-mail: avrim@theory.lcs.mit.edu

Ronald J. Gould†
Department of Mathematics and Computer Science
Emory University, Atlanta, GA, U.S.A. 30322

**Abstract**

In this paper we consider a generalized form of a degree sum condition studied by Chen. Given any set $S = \{v_1, \ldots, v_t\}$ of $t$ independent vertices in a graph, we bound from below the following sum:

$$t\deg(S) + \deg(v_1) + \deg(v_2) + \ldots + \deg(v_t).$$

Bounding from below the sum of the degrees of the vertices of $S$ plus the degree of the set $S$ itself, is shown to provide a sufficient condition for a graph to be hamiltonian.

**Dedicated to Roger Entringer on the occasion of his 60th birthday**

## 1 Introduction

Over the past few years a form of generalized degree condition for sets of vertices—where the sets satisfy various conditions—has been used to further the study of a variety of graph properties. In [FGJS1] and [F], hamiltonian properties were studied using sets of independent vertices of various

*Supported in part by an NSF Graduate Fellowship.
†Partially supported under ONR contract number N00014-91-J-1085.

sizes, while in [FGJS2], matchings and extremal path and cycle lengths were studied for the same types of sets. In [FGS] and [FGL], these generalized degree conditions were used to study connectivity. In [FGJL], a Turan-type extremal result was obtained. A survey of recent results using generalized degrees can be found in [L].

To be specific, we define the *neighborhood* $N(S)$ of a set of vertices $S$ to be the union of the neighborhoods of the vertices in $S$. We then define $\deg(S)$, the *degree* of $S$, to be the cardinality of its neighborhood. That is, $\deg(S) = |\bigcup_{v \in S} N(v)|$.

This definition of degree provides a straightforward generalization of the idea of the degree of a vertex and as can be seen from the references above, has been a useful tool in studying several types of problems. As an example of this new approach, the following result was proved in [FGJS1].

**Theorem A ([FGJS1])** *Let $G$ be a 2-connected graph of order $n \geq 3$. If for each set $S = \{x, y\}$ of two independent vertices in $G$,*

$$\deg(S) \geq (2n - 1)/3,$$

*then $G$ is hamiltonian.*

Chen [C] provided a variation on this approach when he combined standard degrees with this new generalized degree for pairs of independent vertices.

**Theorem B (Chen [C])** *If $G$ is a graph of order $n \geq 3$ such that for each set $S = \{x, y\}$ of two independent vertices in $G$,*

$$2 \deg(S) + \deg(x) + \deg(y) \geq 2n - 1,$$

*then $G$ is hamiltonian.*

The purpose of this paper is to further explore such generalized degree sums and their implications for hamiltonian graphs. In particular, we wish to consider a generalization of Chen's result to $t$ independent vertices instead of two vertices. We prove here the following theorem.

**Theorem 1** *If $G$ is a $t$-connected graph $(t \geq 2)$ of order $n > 6t^5$ such that for each set $S = \{v_1, \ldots, v_t\}$ of $t$ independent vertices in $G$,*

$$t \deg(S) + \deg(v_1) + \ldots + \deg(v_t) \geq tn + 3t^2,$$

*then $G$ is hamiltonian.*

In the following discussion we will suppose that the graph $G$ on $n$ vertices is *not* hamiltonian, that it *is* $t$-connected and that for all sets of $t$ mutually non-adjacent vertices, the sum of the degrees of the vertices plus $t$ times the degree of the set exceeds $tn + 3t^2$, and then derive a contradiction. The analysis assumes that $n$ is much larger than $t$; in particular $n > 6t^5$ suffices.

# 2 Preliminaries

For standard terms and notation not found here see [G].

Given a $t$-connected graph $G = (V, E)$ on $n$ vertices, we will denote an edge from vertex $a$ to vertex $b$ by the concatenation "$ab$" of the vertex labels.

- For $S \subseteq V$, let $(S)$ be the sum of the degrees of all the vertices in $S$. That is, $(S) = \sum_{v \in S} \deg(v)$.

- We say that a set of vertices $S$ is a *bad t-set* if it is an independent set of size $t$ and $(S) + t\deg(S) < tn + 3t^2$.

- Since we are assuming that $G$ is not hamiltonian, let $C$ be the largest cycle in the graph $G$ and let $n(C) = |V(C)|$. Note that $n(C) < n$. Label the vertices in $C$: $v_0, v_1, \ldots, v_{n(C)-1}$ so that vertex $v_i$ is adjacent to vertex $v_{(i+1)(\text{mod } n(C))}$ on the cycle. We will say that vertex $v_{i+1}$ is the *successor* of vertex $v_i$ on the cycle and $v_i$ is the *predecessor* of $v_{i+1}$.

  For convenience, if $v = v_j$ is a vertex on the cycle, define "$v + i$" to equal $v_{(j+i)(\text{mod } n(C))}$.

- Let $G - C$ denote the subgraph induced by the vertices of $V - V(C)$.

- Given two vertices $v$ and $w$ on cycle $C$, let $[v, w]$ represent the sequence of vertices $v, v+1, \ldots, w$. Similarly, let $[v, w)$, $(v, w]$, and $(v, w)$ represent the sequence without $w$, $v$, or both, respectively. If $v = w$ then $[v, w] = \{v\}$ and the other three intervals are empty.

- A path between two vertices $u, v$ is *C-disjoint* if it consists (except for possibly its endpoints) of vertices not on cycle $C$. We will write such a path as "$u \rightarrow v$."

- A set of vertices $S$ is *$C$-isolated* if there exists no $C$-disjoint path between two vertices in $S$. That is, $S$ is independent and furthermore, no two members of $S$ are in the same connected component of the graph $G - (V(C) - S)$.
- A set $S$ of vertices on cycle $C$ has the *non-crossing property* if for any two vertices $u, v \in S$, and for any vertex $a$ in $(u, v)$, there does not exist both an edge from $u$ to $a + 1$ and an edge from $v$ to $a$.

## 2.1 The basic plan

The basic idea for the proof of the main theorem is as follows. We will present a sequence of lemmas that bound $(S) + t\deg(S)$ for any $C$-isolated set $S$ with the non-crossing property. We will then consider several cases for the structure of graph $G$, for each showing how to find large such sets $S$. For the main case (type II below), this will involve a two-stage process.

More specifically, we consider three different cases for $G$ depending on the structure of the vertices and connected components of the subgraph $G - C$. For the first case, we say $G$ is type I if some vertex in $V - C$ has more than $\frac{n}{3t}$ neighbors in $C$; label some such vertex $z$. For the second case, we say $G$ is type II if $G$ is not type I and there exists some connected component $G_z$ of $G - C$ that either has less than $\frac{n}{t^4}$ vertices, or else is not hamiltonian-connected. If $G$ is type II, let $z$ be the vertex in $G_z$ with the fewest neighbors in $G_z$.

For $G$ of type I or II, since $G$ is $t$-connected we know there exist at least $t$ vertex-disjoint paths from $z$ to the cycle $C$. A maximal collection of these paths intersect $C$ at vertices we will label as $x_1, \ldots, x_T$ $(T \geq t)$. Notice that the neighbors of $z$ on $C$, if any, are contained in the set of $x_i$, so for $G$ of type I, we have $T \geq \frac{n}{3t}$.

For the final case, $G$ is type III if it is not types I or II; so all connected components of $G - C$ both have greater than $\frac{n}{t^4}$ vertices and are hamiltonian-connected. If $G$ is type III, let $G_z$ be some connected component of $G - C$. Since $G$ is $t$-connected and $\frac{n}{t^4} > t$ (by our earlier condition on $n$), we know there exist $t$ distinct vertices $z_1, \ldots, z_t$ in $G_z$ with distinct neighbors $x_1, \ldots, x_t$ respectively on the cycle $C$. Let $T = t$.

Now, for $G$ of any type, let $y_1, \ldots, y_T$ be the predecessors of $x_1, \ldots, x_T$ respectively and let $Y$ be the set of all $y_i$. For $G$ of types II and III, let $n_z = |G_z|$.

What we will show is that if $G$ is type I, then some subset of $Y$ is a bad $t$-set. Here we use the fact that set $Y$ is large (size at least $n/3t$ by the definition of type I graphs). If $G$ is type II, then either some subset of $Y \cup \{z\}$ is a bad $t$-set or else some subset of a different set $B_{\text{solo}}(Y)$, which is defined later, is a bad $t$-set. The reason for the conditions defining type II is they imply that $z$ has "few enough" neighbors in $G_z$: note that by Ore's theorem [O], if $G_z$ is not hamiltonian-connected, then $z$ has at most $|G_z|/2$ neighbors in $G_z$. If $G$ is type III, we will use the fact that $G_z$ is hamiltonian-connected to prove stronger facts about the set $Y$ and show that some subset of $Y \cup \{v\}$ for some $v \in G_z$ must be a bad $t$-set.

## 3 Some properties of the set "$Y$"

We now describe some useful properties held by the set $Y$ defined in Section 2.1.

**Proposition 1** *If $G$ is type I or II, then $Y \cup \{z\}$ is a $C$-isolated set.*

**Proof:** First, there can be no $C$-disjoint path between $z$ and any $y_i$. If such a path $y_i \rightarrow z$ existed, let $v$ be the vertex at which this path first intersects the path $x_i \rightarrow z$ (if the paths only touch at $z$ then $v = z$). A cycle larger than $C$ would then be:

$$(v \rightarrow x_i, x_i + 1, \ldots, y_i \rightarrow v).$$

Now, consider two vertices $y_i$ and $y_j$ on the cycle. Any path $y_i \rightarrow y_j$ cannot intersect paths $x_i \rightarrow z$ or $x_j \rightarrow z$ because this would imply the existence of a path $y_i \rightarrow z$. So, any path $y_i \rightarrow y_j$ results in the cycle:

$$(z \rightarrow x_i, x_i + 1, \ldots, y_j \rightarrow y_i, y_i - 1, \ldots, x_j \rightarrow z)$$

which is also larger than $C$ (see figure 1). □

**Proposition 2** *If $G$ is type III, then for all $v \in G_z$, the set $Y \cup \{v\}$ is $C$-isolated.*

**Proof:** The proof is essentially the same as that for Proposition 1. If there exists a path $y_i \rightarrow v$, then since there also exists a path $z_i \rightarrow v$

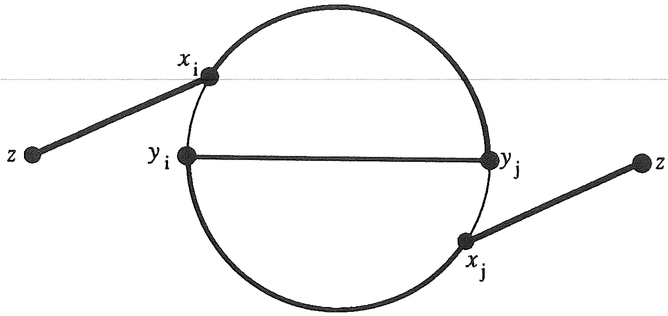


**Figure 1.**

because $G_z$ is connected, there must be a path $y_i \to z_i$. So, we get the cycle

$$(z_i, x_i, x_i + 1, \ldots, y_i \to z_i).$$

If there is no such path but there exists a path $y_i \to y_j$ then we have the cycle

$$(z_i, x_i, x_i + 1, \ldots, y_j \to y_i, y_i - 1, \ldots, x_j, z_j \to z_i).$$

□

**Proposition 3** *The set $Y$ has the non-crossing property.*

**Proof:** Suppose $y_i, y_j$ violate this property. So, there exist edges $y_i(a+1)$ and $y_j a$ for some $a \in [y_i, y_j]$, which cannot intersect by Propositions 1 and 2. We know that $a \neq y_i$ and $a + 1 \neq y_j$ since $Y$ is an independent set. Thus, if $G$ is type I or II we have the cycle larger than $C$ of:

$$(z, x_i, x_i + 1, \ldots, a, y_j, y_j - 1, \ldots, a + 1, y_i, y_i - 1, \ldots, x_j, z).$$

If $G$ is type III, we have the cycle:

$$(z_i, x_i, x_i + 1, \ldots, a, y_j, y_j - 1, \ldots, a + 1, y_i, y_i - 1, \ldots, x_j, z_j \to z_i).$$

(See figure 2.) □

Notice that if $S$ is a $C$-isolated set with the non-crossing property, then so is any subset of $S$.

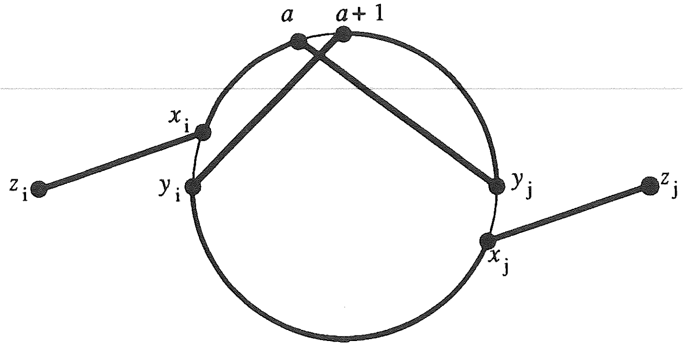


Figure 2.

# 4 Degree bounds for $C$-isolated sets with the non-crossing property

Let $S$ be a $C$-isolated set of $p$ vertices on cycle $C$ with the non-crossing property. We will now prove some facts about how the edges from the set $S$ may be distributed to other vertices on the cycle. In order to do this, we will need some additional definitions.

- Let $M(S) = \{v \in V(C) \mid |N(v) \cap S| \geq 2\}$. We call these vertices "multiples" since they are incident to at least two edges from set $S$.

In the following definitions, if $v \in V(C)$, let $v_-$ and $v_+$ denote the vertices *in* $N(S)$ such that the interval $[v_-, v_+]$ is the smallest interval containing $v$.

- Let $B(S) = \{v \in V(C) - N(S) \mid$ for some $s, s' \in S, \exists$ edges $sv_-$ and $s'v_+$ with $v \in [s', s]\}$.

  Note that $S \subseteq B(S)$, since for $v \in S$, we know $v \notin N(S)$ and we may let $s = s' = v$. For $v$ *not* in $S$, the edges $sv_-$ and $s'v_+$ in the definition must "cross" in that vertices $v_+, s, s', v_-$ appear in order around $C$. Set $B(S)$ is a subset of the "blanks"—the vertices not in $N(S)$—but does not necessarily contain the entire set.

- Let $B_{\text{rightmost}}(S) = \{v \in B(S) \mid v + 1 \notin B(S)\}$. That is, for $v \in B_{\text{rightmost}}(S)$, $v_+ = v + 1$.

- Let $B_{\text{solo}}(S) = \{v \in B(S) \mid v - 1 \notin B(S) \text{ and } v + 1 \notin B(S)\}$.
- Let $b(S) = |B(S)|$.
- Let $r(S) = |B_{\text{rightmost}}(S)|$. This counts the number of non-empty segments or *regions* along the cycle between vertices in $B(S)$.

In this section, since we mainly consider just one $C$-isolated set $S$, we will drop the arguments and just write $B$, $B_{\text{solo}}$, $B_{\text{rightmost}}$, $b$, $r$, and $M$. We will specify the argument only if we wish to discuss these values with respect to some other such set (eg. some subset of $S$).

An important part of our main argument rests on showing that $B_{\text{solo}}$ is large, because in certain cases we will be able to show that $B_{\text{solo}}$ is $C$-isolated with the non-crossing property as well. Lemma 1 below allows us to put a lower bound on the size of $B_{\text{solo}}$ based on $b$ and $r$, numbers that will be easier to deal with.

**Lemma 1** $|B_{\text{solo}}| \geq 2r - b$.

**Proof:** Say $B_{\text{left-or-right}} = \{v \in B(S) \mid v - 1 \in N(S) \text{ or } v + 1 \in N(S)\}$. Then, $2r = |B_{\text{left-or-right}}| + |B_{\text{solo}}|$. That is, for each $v \in B_{\text{rightmost}}$ there exists a corresponding $v'$ in the same block of vertices from $B$ such that $v' - 1 \in N(S)$, where $v' = v$ exactly when $v \in B_{\text{solo}}$. Clearly, $|B_{\text{left-or-right}}| \leq b$, so:

$$|B_{\text{solo}}| = 2r - |B_{\text{left-or-right}}| \geq 2r - b.$$

□

We will now look at how the edges from $S$ may fall into a region between two "consecutive" vertices of $B$. Choose $b_1, b_2 \in B$ such that $(b_1, b_2)$ is non-empty and there is no $v \in B$ inside $(b_1, b_2)$; so, $b_1 \in B_{\text{rightmost}}$ and $b_2$ would be in a correspondingly defined $B_{leftmost}$. For convenience, label the vertices in $S$ : $s_1, s_2, \ldots, s_p$ clockwise around the cycle starting from $b_2$. If $b_2 \in S$ then $s_1 = b_2$. Also, notice there is no vertex of $S$ in $(b_1, b_2)$.

**Lemma 2** *For any $v, w \in [b_1, b_2]$, if there exist edges $s_i v$ and $s_j w$ for $i < j$, then $[w, v] \subset [b_1, b_2]$. (That is, either $v = w$ or else vertices $b_1$, $w$, $v$, $b_2$ appear in order around the cycle.)*

**Proof:** We must show that we cannot have $[v, w] \subset [b_1, b_2]$ and $v \neq w$. We will use induction on the distance along the cycle from $v$ to $w$. First, if $w = v + 1$, then $s_i v$ and $s_j w$ violate the non-crossing property. So, suppose we have shown that $w$ cannot equal $v + 1, \ldots, v + d$ for $d \geq 1$ and let us show that $w$ cannot equal $v + d + 1$.

Suppose $w = v + d + 1$. If all vertices in the range $(v, w)$ are not in $N(S)$, then they would all be in $B$ since they reside in the interval $[s_j, s_i]$, and this would contradict our definition of $b_1$ and $b_2$. So, there must be an edge $us_k$ for some $s_k \in S$ and $u \in (v, w)$. If $k > i$, then the edges $s_k u$ and $s_i v$ violate the induction condition because $i < k, v \neq u, [v, u] \subset [b_1, b_2]$ and $u = v + d'$ for some $d' \leq d$. Similarly, if $k < j$ then the edges $s_k u$ and $s_j w$ violate the condition. (In the range $i < k < j$, both pairs violate the condition.) □

**Corollary 1** *The vertex $s_k \in S$ shares at most one neighbor inside $[b_1, b_2]$ with any of $s_1, \ldots, s_{k-1}$. That is, $|N(s_k) \cap N(\{s_1, \ldots, s_{k-1}\}) \cap [b_1, b_2]| \leq 1$.*

**Proof:** Suppose that $s_k$ shares some neighbor $v$ with $s_i$ and a neighbor $w$ with $s_j$ for $i \leq j < k$ and $v, w \in [b_1, b_2]$. By Lemma 2, since $i < k$ and there are edges $s_i v$ and $s_k w$, we must have $[w, v] \subset [b_1, b_2]$. But also by Lemma 2, since $j < k$ and there are edges $s_j w$ and $s_k v$, we must have $[v, w] \subset [b_1, b_2]$. Therefore, $v = w$. □

**Lemma 3** *The number of edges between vertices in $S$ and vertices in the region $[b_1, b_2]$ is at most $|(b_1, b_2)| + p - 1$. (Recall, $p = |S|$.)*

**Proof:** Say that each $s_i$ has $n_i$ edges into $[b_1, b_2]$. By Corollary 1, $s_2$ may have at most one neighbor in the segment in common with $s_1$, which means the union of the neighborhoods of $s_1$ and $s_2$ inside $[b_1, b_2]$ has size at least $n_1 + n_2 - 1$. Inductively, suppose that $[N(s_1) \cup \ldots \cup N(s_k)] \cap [b_1, b_2]$ has size at least $n_1 + (n_2 - 1) + \ldots + (n_k - 1)$. According to Corollary 1, vertex $s_{k+1}$ has at most one neighbor in that set, which implies that the size of $[N(s_1) \cup \ldots \cup N(s_{k+1})] \cap [b_1, b_2]$ is at least $n_1 + (n_2 - 1) + \ldots + (n_{k+1} - 1)$. The size of $N(S) \cap [b_1, b_2]$ is at most $|(b_1, b_2)|$ since vertices $b_1, b_2$ are not in $N(S)$; so, we have: $|(b_1, b_2)| \geq n_1 + (n_2 - 1) + \ldots + (n_p - 1)$. Thus, $n_1 + \ldots + n_p$, the total number of edges from $S$ into $[b_1, b_2]$, is at most $|(b_1, b_2)| + p - 1$. □

**Corollary 2** *If* $n_0 = |(V - V(C)) - N(S)|$, *then* $(S) \leq (n - n_0 - b) + r(p-1)$.

**Proof:** Each region $(b_i, b_{i+1})$ of $l_i > 0$ consecutive vertices from $V(C) - B(S)$ has at most $l_i + p - 1$ edges entering it from $S$. There are at most $r$ such regions on the cycle, and the sum of the $l_i$ is at most $n(C) - b$. So, the number of edges between vertices in $S$ and vertices in $C$ is at most $(n(C) - b) + r(p-1)$. Since $S$ is $C$-isolated, no vertex of $G - C$ can be the neighbor of more than one element of $S$; that is, the number of edges between $S$ and the vertices off $C$ is at most $(n - n(C) - n_0)$. So, the total sum of degrees of vertices in $S$ is at most $n(C) - b + r(p-1) + (n - n(C) - n_0) = (n - n_0 - b) + r(p-1)$. □

**Corollary 3** *If* $n_0 = |(V - C) - N(S)|$, *then for each* $q \leq p$, *there exists a set* $S' \subseteq S$ *of size* $q$ *such that* $(S') + t \deg(S') < (t + \frac{q}{p})(n - n_0) + qr - tb$.

**Proof:** Let $S'$ be the set of $t$ vertices of least degree in $S$. So, $(S') \leq (q/p)(S)$. Also, clearly $\deg(S') \leq n - n_0 - b$. Plugging in the bound of Corollary 2 yields: $(S') + t\deg(S') \leq \frac{q}{p}[n - n_0 - b + r(p-1)] + t(n - n_0 - b) < (t + \frac{q}{p})(n - n_0) + qr - tb$. □

**Lemma 4** $|M| \leq (p-1)r$, *and for any* $S' \subseteq S$ *of size* $q$, $|M(S')| \leq (q-1)r$.

**Proof:** First, notice that since $S$ is $C$-isolated, all vertices of $M$ lie on the cycle $C$. By Corollary 1, each $s_k \in S$ shares at most one neighbor in $[b_1, b_2]$ with any of $s_1, \ldots, s_{k-1}$. Thus, if we "charge" each $v \in M(S)$ in $[b_1, b_2]$ to its neighbor in $S$ of highest index, we will charge each vertex in $S - \{s_1\}$ at most once and give no charge to $s_1$, implying a total charge of at most $(p-1)$. Similarly, if we charge each $v \in M(S') \cap [b_1, b_2]$ to its neighbor in $S'$ of highest index, we will achieve a total charge of at most $q - 1$. Since there are $r$ nonempty regions $(b_i, b_{i+1})$, we get $|M(S)| \leq r(p-1)$ and $|M(S')| \leq r(q-1)$. □

**Lemma 5** *For each* $q < p$, *there exists some set* $S' \subseteq S$ *of size* $q$ *such that* $(S') + t\deg(S') \leq \frac{q}{p}[(S) + t(\deg(S) - |M|)] + t|M|$.

**Proof:** For each vertex $s \in S$, calculate $\deg(s) + t|N(s) - M|$, and let $S'$ be the $q$ vertices in $S$ of lowest such value. Since over all vertices $s \in S$, the sets $[N(s) - M]$ are disjoint by definition of $M$, we know that

$$\begin{aligned}\sum_{s \in S} [\deg(s) + t|N(s) - M|] &= (S) + t|N(S) - M| \\ &= (S) + t(\deg(S) - |M|). \\ &\quad \text{(since } M \subseteq N(S)\text{)}\end{aligned}$$

By definition of the set $S'$, it must be that

$$\begin{aligned}\sum_{s \in S'} [\deg(s) + t|N(s) - M|] &\leq \frac{q}{p}[(S) + t(\deg(S) - |M|)], \text{ so} \\ (S') + t(\deg(S') - |M|) &\leq \frac{q}{p}[(S) + t(\deg(S) - |M|)] \\ (S') + t\deg(S') &\leq \frac{q}{p}[(S) + t(\deg(S) - |M|)] + t|M|.\end{aligned}$$

□

# 5 The main result

We now apply the lemmas of the previous sections to the set $Y$ to prove the following theorem.

**Theorem 2** *There exists a bad $t$-set in $G$.*

The proof involves considering the three types of graphs defined in Section 2, and showing that the theorem holds true in each case. For this section, unless otherwise specified, the terms $B$, $B_{\text{solo}}$, $B_{\text{rightmost}}$, $b$, $r$, and $M$ are implicitly defined with respect to $Y$.

## 5.1 $G$ is type I

We first consider the simplest case that $G$ is type I. In this case $Y$ has $T$ elements and $T \geq \frac{n}{3t}$. Applying Corollary 3 with $p = T$ and $q = t$ yields a set $Y' \subseteq Y$ of size $t$ with

$$(Y') + t\deg(Y') < (t + 3t^2/n)n + t(r - b) \leq tn + 3t^2. \quad \text{(since } b \geq r\text{)}$$

So, $Y'$ is a bad $t$-set.

## 5.2 $G$ is type II

If $G$ is type II, then we will show that either some subset of $Y \cup \{z\}$ or some subset of $B_{\text{solo}}(Y) \cup \{z\}$ is a bad $t$-set. To do this, we need the following theorem that we will prove in the appendix.

**Theorem 3** *If $G$ is type II, then $B_{\text{solo}}(Y)$ is a $C$-isolated set with the non-crossing property and $B_{\text{solo}}(Y) \cup \{z\}$ is $C$-isolated.*

The plan for finding a bad $t$-set will be as follows. We show that either a subset of $Y \cup \{z\}$ is a bad $t$-set (applying Corollary 2 and Lemmas 4 and 5) or else that $B_{\text{solo}}(Y)$ is large. In the latter case, using Corollary 3, we show that some subset of $B_{\text{solo}}(Y) \cup \{z\}$ is a bad $t$-set.

**Lemma 6** *If $G$ is type II then either there is a bad $t$-set $\tilde{Y} \subseteq Y \cup \{z\}$ or else $(t-1)|B_{\text{solo}}| > n - \frac{(t-1)n}{T} - (T + \frac{n}{t^4})(t+1)$.*

**Proof:** Let $Y'$ be the $t$ vertices of $Y$ of least degree. So, $(Y') \leq \frac{t}{T}(Y)$, which by Corollary 2 implies that $(Y') \leq \frac{t}{T}[n - n_z - b + r(T-1)] < \frac{t}{T}(n - n_z - b) + rt$. We can clearly upper bound $\deg(Y')$ by $n - n_z - b$ and using Lemma 4 we know $|M(Y')| \leq r(t-1)$. We now apply Lemma 5 with $q = t-1$ and $p = t$. Thus, we find some set $Y'' \subset Y'$ of $t-1$ elements such that:

$$\begin{aligned}
(Y'') + t\deg(Y'') &\leq \tfrac{t-1}{t}(Y') + (t-1)\deg(Y') + |M(Y')| \\
&\leq \tfrac{t-1}{t}[\tfrac{t}{T}(n - n_z - b) + rt] + (t-1)(n - n_z - b) \\
&\qquad + r(t-1) \\
&= \tfrac{t-1}{T}(n - n_z - b) + (t-1)(n - n_z - b) \\
&\qquad + 2r(t-1) \\
&< \tfrac{(t-1)n}{T} + (t-1)(n - n_z - b) + 2r(t-1) \\
&\leq \tfrac{(t-1)n}{T} + (t-1)(n - n_z) + (t-1)|B_{\text{solo}}|. \\
&\qquad \text{(using Lemma 1)}
\end{aligned}$$

Now, let $\tilde{Y} = Y' \cup \{z\}$. We know $z$ has $T$ neighbors on $C$ and at most $\frac{n}{t^4} + \frac{n_z}{2}$ neighbors off of $C$ by the definition of type II graphs. So, either $\tilde{Y}$ is a bad $t$-set or else we have

$$\tfrac{(t-1)n}{T} + (t-1)(n - n_z) + (t-1)|B_{\text{solo}}| + (t+1)(T + \tfrac{n}{t^4} + \tfrac{n_z}{2}) > tn,$$

which implies that

$$\begin{aligned}(t-1)|B_{\text{solo}}| &> n - \tfrac{(t-1)n}{T} - (T+\tfrac{n}{t^4})(t+1) - n_z(\tfrac{3-t}{2}) \\ &\geq n - \tfrac{(t-1)n}{T} - (T+\tfrac{n}{t^4})(t+1). \quad (\text{since } \tfrac{3-t}{2} \leq 0)\end{aligned}$$

□

**Theorem 4** *If $G$ is type II then there exists a bad $t$-set.*

**Proof:** Suppose $T > 6t$. Then, $\frac{(t-1)n}{T} < \frac{n}{6}$. So, by Lemma 6, either there is a bad $t$-set or else $(t-1)|B_{\text{solo}}| > \frac{5}{6}n - (T + \frac{n}{t^4})(t+1)$. Now, since $G$ is type II, we know $T \leq \frac{n}{3t}$, so we have

$$\tfrac{5}{6}n - (T+\tfrac{n}{t^4})(t+1) \geq n(\tfrac{5}{6} - \tfrac{1}{3} - \tfrac{1}{3t} - \tfrac{t+1}{t^4}) > n/3 \quad \text{for } t \geq 3.$$

Thus, $|B_{\text{solo}}| > \frac{n}{3t}$. Now, using Theorem 3 and Corollary 3 (with $q = t$) we get that some set $S' \subseteq B_{\text{solo}}$ of $t$ vertices satisfies $(S') + t\deg(S') < [t + t/(n/3t)]n + tr(B_{\text{solo}}) - tb(B_{\text{solo}})$ which is at most $tn + 3t^2$. So, $S'$ is a bad $t$-set.

On the other hand, suppose $T \leq 6t$. Since $T \geq t$, we have $\frac{(t-1)n}{T} \leq n - n/t$ and since $T \leq 6t$, we have $(T + \frac{n}{t^4})(t+1) \leq n/t^2$ for $n$ large enough. So, by Lemma 6, either there is a bad $t$-set or else $(t-1)|B_{\text{solo}}| > n/t - n/t^2$, which implies that $|B_{\text{solo}}| > n/t^2$. We again use Theorem 3 and Corollary 3 (this time with $q = t-1$) to find some set $S' \subseteq B_{\text{solo}}$ of $t-1$ vertices that satisfies:

$$\begin{aligned}(S') + t\deg(S') &\leq (t + \tfrac{t-1}{n/t^2})(n - n_z) + (t-1)r(B_{\text{solo}}) - tb(B_{\text{solo}}) \\ &< t(n-n_z) + t^3 - b(B_{\text{solo}}). (\text{since } b(B_{\text{solo}}) \\ &\geq r(B_{\text{solo}}))\end{aligned}$$

Now consider $S' \cup \{z\}$. Vertex $z$ has at most $6t$ neighbors on $C$ and at most $\frac{n}{t^4} + \frac{n_z}{2}$ neighbors off of $C$. Thus,

$$\begin{aligned}(S'\cup\{z\}) + t\deg(S'\cup\{z\}) &\leq t(n-n_z) + t^3 - b(B_{\text{solo}}) + 6t(t+1) \\ &\quad + (t+1)\tfrac{n_z}{2} + (t+1)\tfrac{n}{t^4} \\ &\leq tn + t^3 + 6t(t+1) + (t+1)\tfrac{n}{t^4} - b(B_{\text{solo}}) \\ &\leq tn. (\text{ since } b(B_{\text{solo}}) \geq |B_{\text{solo}}| > n/t^2, \\ &\quad \text{and } n \text{ is large})\end{aligned}$$

So, $S' \cup \{z\}$ is a bad $t$-set. □

## 5.3 $G$ is type III

Next, we consider the case that the graph $G$ is type III. For this case we will need to prove two additional lemmas and then we will use Corollary 2 and Lemmas 4 and 5 to show the existence of a bad $t$-set. For vertices $u, v \in G_z$, let "$u \Rightarrow v$" denote the path from $u$ to $v$ that is hamiltonian in the subgraph $G_z$ (which exists since $G_z$ is hamiltonian-connected).

**Lemma 7** *If $G$ is type III then $r \leq t + \frac{b-t}{n_z}$.*

**Proof:** Consider again the set $B_{\text{rightmost}} = \{v \in B \mid v+1 \notin B\}$. We know the $t$ vertices $y_1, \ldots, y_t$ lie in $B_{\text{rightmost}}$ since $y_i + 1 \in N(Y)$ for all $i$. To prove the lemma, we show that of the $b-t$ vertices of $B$ left over, at most $1/n_z$ of them may also be in $B_{\text{rightmost}}$.

Suppose $v \in B_{\text{rightmost}} - Y$, so $v+1 \notin B$. It is enough to show that the vertices $v-1, v-2, \ldots, v-(n_z-1)$ all lie in $B$, because then for each $v \in B_{\text{rightmost}} - Y$ we have $n_z - 1$ other vertices in $B$ that are not in $B_{\text{rightmost}}$. Thus, assume to the contrary that the vertex $v-k$ for $0 < k < n_z$ is not in $B$ and let $k$ be the least value such that this holds. By definition of $B$, we know $v-k \in N(y_i)$ and $v+1 \in N(y_j)$ such that $(v-k, v+1) \subseteq B \cap (y_j, y_i)$. The cycle:

$$(z_j, x_j, x_j+1, \ldots, v-k, y_i, y_i-1, \ldots, v+1, y_j, y_j-1, \ldots, x_i, z_i \Rightarrow z_j)$$

has length $|C| + n_z - |(v-k, v+1)| = |C| + n_z - k$ which is greater than $|C|$, a contradiction to $C$ being the longest cycle. □

**Lemma 8** *If $G$ is type III then some $u \in G_z$ has less than $n(C)/n_z$ neighbors on $C$.*

**Proof:** Suppose vertex $v \in C$ is a neighbor of more than two vertices of $G_z$. Then the vertices $v+1, \ldots, v+n_z$ cannot be neighbors of *any* vertex of $G_z$. Otherwise, if $z' \in G_z$ were a neighbor of $w \in [v+1, v+n_z]$ and $z'' \in G_z$ a different neighbor of $v$ (which exists since $v$ has at least two neighbors in $G_z$), then we get the cycle longer than $C$ of:

$$(v, z'' \Rightarrow z', w, w+1, \ldots, v).$$

So, for each vertex $v$ in $C$ that is a neighbor of at least two (and at most $n_z$) vertices of $G_z$, there are $n_z$ others: $v+1, \ldots, v+n_z$, that are neighbors of *no* vertices of $G_z$. So, if there are $k$ vertices $v \in C$ with more than one neighbor in $G_z$, the number of edges between $C$ and $G_z$ is at most $k \cdot n_z + (n(C) - kn_z) \cdot 1 = n(C)$ total edges. Thus, some vertex $u \in G_z$ has at most $n(C)/n_z$ neighbors in $C$. □

**Lemma 9** *If G is type III then there exists a bad t-set.*

**Proof:** We know that none of the vertices in $G_z$ are in the neighborhood of the set $Y$ (by the $C$-isolatedness of $Y \cup \{z_i\}$). So, if $n_z \geq n/(t+1)$ then Corollary 3 with $q = p = t$ implies that $(Y) + t\deg(Y) < (t+1)(n - n_z) \leq tn$, so $Y$ is a bad $t$-set. If $n_z < n/(t+1)$ then we will use Lemma 5.

Lemma 5 shows that some set $Y' \subset Y$ of $t-1$ elements satisfies $(Y') + t\deg(Y') \leq \frac{t-1}{t}[(Y) + t\deg(Y)] + |M(Y)|$. Applying Corollary 2 and Lemma 4 yields:

$$\begin{aligned}
(Y') + t\deg(Y') &\leq \tfrac{t-1}{t}[n - n_z - b + r(t-1) + t(n - n_z - b)] \\
&\qquad +(t-1)r \\
&< (t - 1/t)(n - n_z) + t(2r - b) \\
&\leq (t - 1/t)(n - n_z) + t(2t - \tfrac{2(b-t)}{n_z} - b) \\
&\qquad (\text{by Lemma 7}) \\
&= (t - 1/t)(n - n_z) + t[t + (t-b)(1 - \tfrac{2}{n_z})] \\
&\leq (t - 1/t)(n - n_z) + t^2 \\
&\qquad (\text{since } b \geq t \text{ and } n_z \geq 2)
\end{aligned}$$

Now consider the vertex $u \in G_z$ with fewest neighbors in $C$. By Lemma 8, it has at most $n(C)/n_z$ neighbors on $C$ and we know it must have less than $n_z$ neighbors not on $C$. So (using $n(C) \leq n - n_z$),

$$\begin{aligned}
(Y' \cup \{u\}) + t\deg(Y' \cup \{v\}) &< (t - 1/t)(n - n_z) + t^2 \\
&\qquad +(t+1)(n/n_z + n_z - 1) \\
&\leq tn - n/t + t^2 + n_z\tfrac{t+1}{t} \\
&\qquad +(t+1)n/n_z - t - 1.
\end{aligned}$$

In the allowed range $\frac{n}{t^4} \leq n_z \leq \frac{n}{t+1}$, (and with $n > 6t^5$), the value $n_z\frac{t+1}{t} + (t+1)n/n_z$ is maximized at $n_z = \frac{n}{t+1}$. So, using this value of $n_z$

we get:

$$\begin{aligned}(Y'\cup\{u\})+t\deg(Y'\cup\{v\}) &< tn-n/t+t^2+n/t+(t+1)^2-t-1\\ &= tn+2t^2+t\\ &\leq tn+3t^2.\end{aligned}$$

Thus, the set $Y'\cup\{u\}$ is a bad $t$-set. □

In conclusion then, no matter which case holds—$G$ is type I, II, or III—we have produced a bad $t$-set. Therefore, our assumption that $G$ is non-hamiltonian must be false, and we have proved our main result, Theorem 1.

## 6 Conclusion

In conclusion, we have shown that Chen's bound for pairs of independent vertices can be generalized to sets of $t$ ($t \geq 2$) independent vertices. However, our general argument is not sharp enough to determine the best possible bound for each value of $t$. We conjecture the following:

**Conjecture 1** *If $G$ is a $t$-connected graph of order $n$, and if for each set $S = \{x_1, \ldots, x_t\}$ of $t$ independent vertices,*

$$t\deg(S) + \deg(x_1) + \ldots + \deg(x_t) \geq tn - (t-1),$$

*then $G$ is hamiltonian.*

An example to show that this conjectured bound is sharp is the following:

$$G = (t+1)K_{\frac{n-t}{t+1}} + K_t.$$

That is, $t+1$ copies of $K_{\frac{n-t}{t+1}}$ joined to a $K_t$.

## References


[C] G. Chen One Sufficient Condition for Hamiltonian Graphs. *J. Graph Theory*, **14** (1990), 723-730.

[FGJL] R. J. Faudree, R. J. Gould, M. S. Jacobson, and L. Lesniak. On a Neighborhood Condition Implying the Existence of Disjoint Complete Graphs. *European Journal of Combinatorics*, **10**(1989), 427-433.

[FGJS1] R. J. Faudree, R. J. Gould, M. S. Jacobson, and R. H. Schelp. Neighborhood Unions and Hamiltonian Properties in Graphs. *J. Combin. Theory B*, **46**(1989), 1-20.

[FGJS2] R. J. Faudree, R. J. Gould, M. S. Jacobson, and R. H. Schelp. Extremal Problems Involving Neighborhood Unions. *J. Graph Theory*, **11**(1987), 555-564.

[FGL] R. J. Faudree, R. J. Gould, and L. Lesniak. Generalized Degrees and Menger Path Systems, *Discrete Applied Math.* **37/38**, (1992), 179–191.

[FGS] R. J. Faudree, R. J. Gould, and R. H. Schelp. Menger Path Systems, *J. Combin. Math. and Combin. Computing*, **6**(1989), 9-21.

[F] P. Fraisse. A New Sufficient Condition for Hamiltonian Graphs. *J. Graph Theory*, **10**(1986), 405-410.

[G] R. J. Gould. *Graph Theory*, Benjamin/Cummings Publishing Co., Menlo Park, CA, 1988.

[L] L. Lesniak. Neighborhood Unions and Graphical Properties. *Proceeding of the Sixth Intern. Conf. on the Theory and Applications of Graphs, Kalamazoo*, Wiley Interscience, 1991. Edited by Alavi, Chartrand, Oellerman, and Schwenk.

[O] O. Ore. A Note on Hamiltonian Circuits. *American Math. Monthly*, 67(1960), 55.

# 7 Appendix

We now must prove Theorem 3, that when $G$ is type II, $B_{\text{solo}}(Y)$ is a $C$-isolated set with the non-crossing property and $B_{\text{solo}}(Y) \cup \{z\}$ is $C$-isolated. While we believe there should exist an elegant proof of this, our current proof is simply a decomposition into a series of cases, each one checked separately.

Unless otherwise specified, $B_{\text{solo}}$ refers to $B_{\text{solo}}(Y)$. For convenience, we define the following additional notation.

- For $v \in B_{\text{solo}} - Y$, let $y_+(v)$ and $y_-(v)$ be vertices of $Y$ such that $[y_+(v), y_-(v)]$ is the smallest interval containing $v$ and there exist edges $(v+1)y_+(v)$ and $(v-1)y_-(v)$. That is, these are the vertices $s$ and $s'$ respectively in the definition of set $B$.

**Corollary 4** *If $v \in B_{\text{solo}} - Y$, then $y_+(v) \in [y_-(v), v]$.*

**Proof:** By definition of $B_{\text{solo}}$, we have $v \in [y_+(v), y_-(v)]$, so $y_+(v)$, $v$, and $y_-(v)$ appear in order around the cycle. □

**Lemma 10** *$B_{\text{solo}} \cup \{z\}$ is a $C$-isolated set.*

**Proof:** First, there can be no $C$-disjoint path between any $v \in B_{\text{solo}} - Y$ and $z$, because this would imply a $C$-disjoint path between $v$ and $x_j$ for $y_j = y_-(v)$, and the argument in the proof of Proposition 1 holds.

Now, let $v$ and $w$ be two vertices of $B_{\text{solo}}$ and suppose for contradiction that there exists a $C$-disjoint path $v \rightarrow w$. So, certainly not both $v$ and $w$ are in $Y$.

If one of the two vertices (say vertex $v$) belongs to $Y$, then we just use the fact that either $y_-(w) \in [w, v]$ or else $y_+(w) \in [v, w]$. Let $y_j$ be $y_-(w)$ or $y_+(w)$ respectively. Then, vertices $v$ and $y_j$ violate the non-crossing property if we extend the definition to include paths of vertices not on $C$ (in particular the path $v \rightarrow w$) as well as edges; the proof of Proposition 3 holds for this extended definition.

We now consider the situation where $v, w \in B_{\text{solo}} - Y$. First, suppose that $y_-(v) \in [v, w]$, and let $y_i = y_-(v)$. We now have three cases:

1. $y_-(w) \in [w, v]$. Let $y_j = y_-(w)$. We then have the cycle larger than $C$ of:

$$\begin{aligned}(z &\rightarrow x_j, x_j + 1, \ldots, v - 1, y_i, y_i - 1, \ldots, \\ v &\rightarrow w, \ldots, y_j - 1, y_j, w - 1, \ldots, x_i + 1, x_i \rightarrow z).\end{aligned}$$

   (See figure 3.)

2. $y_-(w) \in [v, y_i]$. Let $y_j = y_-(w)$. We then have the cycle:

$$\begin{aligned}(z &\rightarrow x_i, x_i + 1, \ldots, w - 1, y_j, y_j - 1, \ldots, \\ v &\rightarrow w, w + 1, \ldots, v - 1, y_i, y_i - 1, \ldots, x_j \rightarrow z).\end{aligned}$$

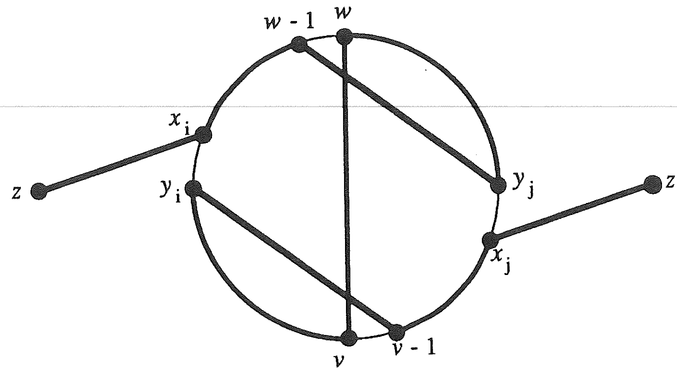


**Figure 3.**

3. $y_-(w) \in [y_i, w]$. Corollary 4 tells us that $y_+(w)$ is also in $[y_i, w]$. Let $y_j = y_+(w)$. So, we get the cycle:

$$\begin{array}{lll}(z & \to & x_i, x_i+1, \ldots, y_j, w+1, w+2, \ldots, v-1, y_i, y_i-1, \ldots, \\ v & \to & w, w-1, \ldots, x_j \to z).\end{array}$$

These three cases apply when $y_-(v) \in [v, w]$, or (by switching the labels of $v$ and $w$), to the case where $y_-(w) \in [w, v]$. The only additional case to consider is:

4. $y_-(v) \in [w, v]$ *and* $y_-(w) \in [v, w]$. In this case (letting $y_i = y_-(v)$ and $y_j = y_-(w)$) we have:

$$\begin{array}{lll}(z & \to & x_j, x_j+1, \ldots, w-1, y_j, y_j-1, \ldots, \\ v & \to & w, w+1, \ldots, y_i, v-1, \ldots, x_i \to z).\end{array}$$

□

**Lemma 11** $B_{\text{solo}}$ *satisfies the non-crossing property.*

**Proof:** Again, let $v$ and $w$ be two vertices of $B_{\text{solo}}$, and let us suppose for contradiction that there are edges from $v$ to $a$ and $w$ to $b$ where $b = a + 1$ if $a \in [w, v]$ and $b = a - 1$ if $a \in [v, w]$.

We may reduce the number of cases to consider with the following two observations.

**Observation 1.** Suppose we we would have a cycle $(z, \ldots, v \rightarrow w, \ldots, z)$ longer than $C$ if $v$ and $w$ were in fact connected by a $C$-disjoint path. Then if $a$ and $b$ are adjacent *in order* from left to right in this cycle, we get the longer path of either: $(z, \ldots, a, v, \ldots, b, w, \ldots, z)$ or $(z, \ldots, v, a, \ldots, w, b, \ldots, z)$, depending on the position of $a$ and $b$.

**Observation 2.** Suppose that were $v$ and $w$ connected by a $C$-disjoint path, we would then have two disjoint cycles together containing all the vertices of $V(C) \cup \{z\}$, with $v$ and $w$ in one cycle. Then, if $a$ and $b$ appear adjacent in the *other* cycle, we can connect the two cycles together into a single cycle containing $V(C) \cup \{z\}$.

Since $Y$ has the non-crossing property, we may assume not both of $v$ and $w$ belong to $Y$, and let us first consider the case where in fact neither $v$ nor $w$ belongs to $Y$.

- First, case 1 in the proof of Lemma 10 is completely covered by observation 1.
- For case 2 in the proof of Lemma 10, the only difficulty is if $a, b \in [y_-(v), w-1]$. We now look at the position of $y_+(v)$. If $y_+(v) \in [y_-(w), w]$, then observation 2 holds. For example, if $y_i = y_+(v)$, $y_j = y_-(w)$, and $y_i \in [y_j, b]$, we get the cycle:

  $$(z \rightarrow x_j, \ldots, y_i, v{+}1, \ldots, y_j, w{-}1 \ldots, a, v, \ldots, w, b, \ldots, x_i \rightarrow z).$$

  On the other hand, if $y_+(v) \in [w, v]$, then observation 1 holds. (If we had a path $v \rightarrow w$, we would get the cycle: $(z \rightarrow x_i, \ldots, v, w, \ldots, y_i, v+1, \ldots, y_j, w-1, \ldots, x_j \rightarrow z)$.)
- For case 3 in the proof of Lemma 10, observation 1 applies unless $a, b \in [y_-(v), y_+(w)]$. We now look at $y_j = y_-(w)$. If $y_j \in [w, v]$, then we have case 1, with the labels on $v$ and $w$ reversed. If $y_j \in [y_-(v), w]$, we use observation 2.
- Case 4 of the proof of Lemma 10 is again completely covered by observation 1.

The final situation to consider is where one of $v, w$ belongs to set $Y$, and is handled analogously. □

Thus, we have proved Theorem 3.