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On the Ascending Subgraph Decomposition Problem

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Abstract. The following has become known as the Ascending Subgraph Decomposition Problem: Given a graph G of size $\binom{n+1}{2}$, does there exists a sequence of nonempty edge disjoint subgraphs $G_1, G_2, \ldots, G_n$ such that for each i, $1 \leq i \leq n-1$, the graph G_i is isomorphic to a proper subgraph of G_{i+1}? In this paper we obtain a lower bound on the length of such a sequence of subgraphs and by modifying the problem slightly we also obtain an upper bound on the number of graphs necessary in a decomposition in which G_i is isomorphic to a subgraph (possibly the graph itself) of G_{i+1}.

1. Introduction

For standard terms and notation not found here see [G]. Decomposition problems, that is, problems dealing with partitioning the edge set of a graph into subgraphs with specified properties, have been widely studied. A common theme has been to partition the edge set so that the induced subgraphs are isomorphic. However, in [**ABC**], a problem that is in some sense the opposite of this type of question was considered.

A graph G of order p and size q, with $\binom{n+1}{2} \leq q < \binom{n+2}{2}$, is said to have an *Ascending Subgraph Decomposition* (or an ASD) if the edge set of G can be decomposed into nonempty subgraphs $G_1, G_2, \ldots, G_n$ such that G_i is isomorphic to a proper subgraph of G_{i+1}, $1 \leq i \leq n-1$. Note that if $q = \binom{n+1}{2}$, then this implies that $|E(G_i)| = i$, $i \leq n$. Further, it was shown that no matter what the value of q, if G has an ASD, then it has an ASD in which G_i has size i, $i < n$. We call the graphs $G_1, G_2, \ldots, G_n$ the *ascending sequence* of subgraphs.

In [**ABC**], it was conjectured that every graph has an ascending subgraph decomposition. Several papers have provided classes of graphs which satisfy the ASD property. The following are examples of such results.

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Theorem A. [F]. *If G is a graph with $\binom{n+1}{2}$ edges and $\Delta(G) \leq \frac{n}{2}$, then G has an ascending subgraph decomposition in which each graph in the ascending sequence is a matching.*

This result was extended in [FGJL], where the graphs used in the decomposition were the union of short paths.

Theorem B. [FGJL]. *If G is a graph with $\Delta(G) \leq (2-\sqrt{2})n$, then G has an ascending subgraph decomposition.*

Another large special class is also known to have ASD's

Theorem C. [FG]. *Every forest F has an ascending subgraph decomposition.*

In this paper we shall investigate the maximum length of an ascending sequence that one can insure exists in an arbitrary graph. In order to do this we construct ascending sequences $G_1, G_2, \ldots, G_s$ and examine the ratio $\frac{s}{n}$ (which we call the *sequence ratio*). We define $sr(G) = \max \frac{s}{n}$, where this maximum is taken over all ascending sequences $G_1, \ldots, G_s$ of subgraphs in G. Clearly, if an ASD can be found, then $sr(G) = 1$. However, if no ASD exists, then $sr(G) < 1$.

We also consider the following variation of the original problem: Given a graph G with size q, we say the sequence of subgraphs $G_1, G_2, \ldots, G_k$ is a *weak* ASD if the edge set of G can be decomposed into these k subgraphs such that G_i is isomorphic to a subgraph (possibly the graph itself) of G_{i+1}, for $1 \leq i \leq k-1$ and each G_i has size at most i. Clearly, a weak ASD might fail to maintain the growth in size between consecutive subgraphs, hence, there may be many more than n terms in a weak ascending sequence. We investigate weak ASD's for arbitrary graphs and find a bound on the number of graphs necessary for a weak decomposition to exist. In this case we call $\frac{k}{n}$ the *weak ratio* and denote by $wr(G) = \min \frac{k}{n}$, where this minimum is taken over all weak ascending sequences of subgraphs $G_1, \ldots, G_k$ of G. In a sense, this parameter can be thought of as an upper bound on the number of graphs necessary to obtain a sequence of subgraphs that maintains the decomposition property (but not necessarily the regular growth in size).

We note that both the ratio and the weak ratio are well defined, since the graph itself provides an ascending sequence and the sequence of subgraphs obtained from each of the individual edges of the graph provides a weak ascending sequence. Clearly, our objective is to maximize the sequence ratio and minimize the weak ratio. A restatement of the Ascending Subgraph Decomposition Conjecture in these terms would be: For every graph G, of size $\binom{n+1}{2}$, $sr(G) = wr(G) = 1$.

The following notation will be useful. The complete bipartite graph $K_{1,m}$ is called an m-star and also denoted S_m.

2. Lower Bounds

We know from Fu's Theorem (Theorem A) that if $\Delta(G) \leq \frac{n}{2}$, then G not only has an ASD, but in fact, each graph in the sequence is a matching. This result and several others led to the following extension of the ASD conjecture. This conjecture also remains open.

Conjecture. *(Faudree, Gyarfas, Schelp [FGS]) Let G be a graph with* $\binom{n+1}{2}$ *edges, then G has an ASD in which each member of the ascending sequence is a star forest.*

We begin our investigation by examining the length of an ascending sequence that we can construct maintaining the star forest property, that is, each graph in the sequence will induce a star forest. Suppose that G has size $\binom{n+1}{2}$. Further, suppose that $\Delta(G) > \frac{n}{2}$, (or else we would apply Fu's Theorem to find an ASD in which each graph is a matching, hence a star forest). Suppose that we can decompose a portion of $E(G)$ into stars

$$H_1 = S_{(\beta-\varepsilon)n+1},\ H_2 = S_{(\beta-\varepsilon)n+2}, \ldots,\ H_{\varepsilon n} = S_{\beta n}$$

where $\beta n \geq |E(H_i)| > \frac{n}{2}$ for each i. We continue this star decomposition as long as possible. The graph $H = G - \cup_{i=1}^{\varepsilon n} E(H_i)$ has $\Delta(G) < (\beta - \varepsilon)n$ and size

$$\binom{n+1}{2} - \sum_{i=1}^{\varepsilon n} |E(H_i)| = \binom{n+1}{2} - [(\beta-\varepsilon)n\varepsilon n + \binom{\varepsilon n+1}{2}],$$

and hence,

$$|E(H)| \geq \frac{n^2}{2} - (\beta n)(\varepsilon n) + \frac{(\varepsilon n)^2}{2}.$$

We now remove edges from the stars H_i, $1 \leq i \leq \varepsilon n$, to form smaller stars

$$S_1 . S_2, \ldots, S_{\varepsilon n},$$

We temporarily hold these removed edges in reserve. Our goal is to find $(\beta - \varepsilon)n$ disjoint matchings of size βn in H, the graph that remained after the initial stars were removed. This will be possible provided

$$(\frac{1}{2} - \varepsilon\beta + \frac{\varepsilon^2}{2})n^2 \geq \beta n(\beta - \varepsilon)n,$$

or equivalently, provided

$$\beta^2 - \frac{1}{2} \leq \varepsilon^2.$$

However, this inequality is clearly true when $\beta \leq \frac{\sqrt{2}}{2}$. Thus, we form these matchings of size βn and call them $M_1, M_2, \ldots, M_{(\beta-\varepsilon)n}$.

In this case we now form an ascending sequence of subgraphs as follows: For each star S_i, $1 \leq i \leq \varepsilon n$, form the union $S_i \cup M_i$. Since M_i is a matching and the size of S_i is i $(1 \leq i \leq \varepsilon n)$, at most i edges of M_i $(1 \leq i \leq \varepsilon n)$ are incident with vertices of S_i. Thus, for each subgraph $S_i \cup M_i$ $(1 \leq i \leq \varepsilon n)$, it is possible to remove εn edges from M_i (call these removed edges M_i') and insure that

$$V(S_i) \cap V(M_i - M_i') = \emptyset.$$

The subgraphs thus constructed will be called

$$G_{(\beta-\varepsilon)n+1}, G_{(\beta-\varepsilon)n+2}, \ldots, G_{\beta n}$$

and are an ascending sequence of subgraphs with sizes $(\beta-\varepsilon)n+1, (\beta-\varepsilon)n+2, \ldots, \beta n$ respectively.

To complete the construction of our ascending sequence, we let $G_i(\varepsilon n + 1 \leq i \leq (\beta - \varepsilon)n)$ consist of i independent edges from the matching M_i. Since each M_i has size $(\beta - \varepsilon)n$, this is clearly possible. Finally, the subgraphs G_j, $1 \leq j \leq \varepsilon n$, will consist of j independent edges removed from the matching M_j. Since εn edges were removed from each of those εn matchings, this is also possible.

The sequence $G_1, G_2, \ldots, G_{\beta n}$ thus constructed is an ascending sequence of subgraphs with $sr(G) = \frac{\sqrt{2}}{2}$. We summarize this in the following result. We note that these graphs are always star forests, which lends support to the Faudree, Gyárfás, Schelp Conjecture.

Theorem 1. *If G is a graph of size $\binom{n+1}{2}$, then G has an ascending sequence of subgraphs $G_1, \ldots, G_s$ with $sr(G) \geq \frac{\sqrt{2}}{2}$.*

Proof: If $\Delta(G) \leq \frac{n}{2}$, then we apply Fu's Theorem to obtain a sequence with ratio 1. Otherwise, the above construction provides a sequence of subgraphs achieving the lower bound. ∎

3. Upper Bounds

We now turn our attention to weak ascending sequences. We will find the following notation useful. Given a positive integer m, let $\overline{m}$ denote $\lceil \frac{m}{2} \rceil$ and let $\underline{m}$ denote $\lfloor \frac{m}{2} \rfloor$.

Lemma 1. *Let G_1 and G_2 be subgraphs of a graph G satisfying the properties that $G_1 = K_{1,n}$ with center at vertex x and $G_2 = mK_2$ and the vertex $x \notin V(G_2)$. If m is even or $m \leq \frac{n}{2} - 1$ then there exists subgraphs H_1 and H_2 in G with the following properties:*

1. $E(H_1) \cap E(H_2) = \emptyset$.
2. $H_1 \cup H_2 = G_1 \cup G_2$.
3. $H_1 = K_{1,\overline{n}} \cup \overline{m}K_2$ *and* $H_2 = K_{1,\underline{n}} \cup \underline{m}K_2$ *(where both unions are disjoint).*

Proof: Let $H = G_1 \cup G_2$. For convenience we consider the edges of G_1 and G_2 as partitioned into three sets as follows.

The edges of G_2 incident with two edges of G_1 will be denoted $A_2 = \{e_1, e_2, \ldots, e_a\}$. The edges of G_2 incident with one edge of G_1 will be denoted $B_2 = \{f_1, f_2, \ldots, f_b\}$. The edges of G_2 incident with no edges of G_1 will be denoted $C_2 = \{g_1, g_2, \ldots, g_c\}$. Hence, $E(G_2) = A_2 \cup B_2 \cup C_2$. Now let $A_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{2a}\}$ be the set of edges of G_1 incident with edges of A_2. Also let $B_1 = \{\beta_1, \beta_2, \ldots, \beta_b\}$ be the edges of G_1 incident with one edge of G_2. Finally, let $C_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$ be the edges of G_1 incident with no edges of G_2. Without loss of generality, suppose that e_i is incident with α_{2i-1} and α_{2i} and f_j is incident with β_j.

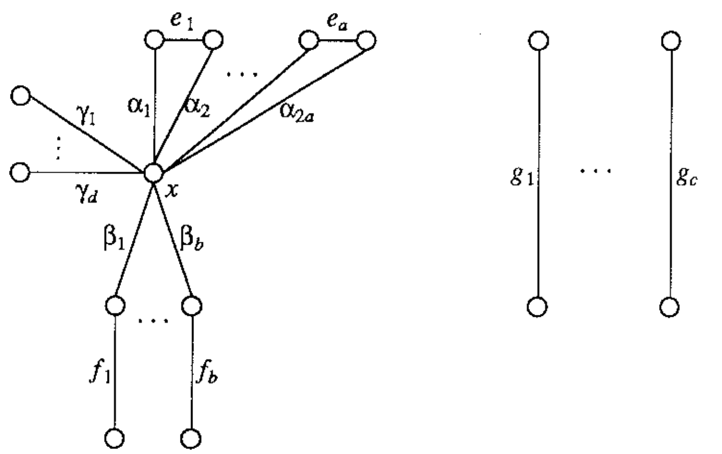

Figure 1. The graph $H = G_1 \cup G_2$.

Case 1. *Suppose m is even and a is even.*

Subcase i. *Suppose b is odd or b is even and d is even.*

Here we form H_1 and H_2 as subgraphs of H as follows:

$$H_1 = S^1 \cup M^2 \text{ and } H_2 = S^2 \cup M^1,$$

where $S^1 = K_{1,a+\overline{b}+\underline{d}}$ and $S^2 = K_{1,a+\underline{b}+\overline{d}}$ and $M^1 = (\overline{a} + \overline{b} + \underline{c})K_2$ and $M^2 = (\underline{a} + \underline{b} + \overline{c})K_2$. Further,

$$E(H_1) = \{e_1, e_2, \ldots, e_{\underline{a}}, f_1, f_2, \ldots, f_{\underline{b}}, g_1, g_2, \ldots, g_{\overline{c}}, \alpha_{a+1}, \ldots, \alpha_{2a}, \beta_{\underline{b}+1}, \ldots, \beta_b, \gamma_1, \ldots, \gamma_{\underline{d}}\}$$

and

$$E(H_2) = \{e_{\underline{a}+1}, \ldots, e_a, f_{\underline{b}+1}, \ldots, f_b, g_{\overline{c}+1}, \ldots, g_c, \alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_{\underline{b}}, \gamma_{\underline{d}+1}, \ldots, \gamma_d\}.$$

Since a is even and $m = a + b + c$ is also even, we see that b and c must have the same parity. Then, in fact,

$$\overline{m} = \overline{a} + \overline{b} + \underline{c} = \underline{a} + \underline{b} + \overline{c} = \underline{m}.$$

Next note that

$$\overline{n} = a + \overline{b} + \underline{d} \geq a + \underline{b} + \overline{d} = \underline{n}$$

provided b is odd or both b is even and d is even. The graphs H_1 and H_2 thus satisfy all the properties 1-3 and by the way the star edges were partitioned, these are disjoint unions. This completes the proof in this subcase.

Subcase ii. *Suppose b is even and d is odd.*

Again, $H_1 = S^1 \cup M^2$ and $H_2 = S^2 \cup M^1$. We proceed as in the previous case except that

$$S^1 = K_{1,a+\overline{b}+\overline{d}} \text{ and } S^2 = K_{1,a+\underline{b}+\underline{d}}$$

while

$$M_1 = (\overline{a} + \overline{b} + \underline{c})K_2 \text{ and } M^2 = (\underline{a} + \underline{b} + \overline{c})K_2.$$

Here we see that

$$\overline{n} = a + \overline{b} + \overline{d} > a + \underline{b} + \underline{d} = \underline{n}$$

while, since m, a and b are even, clearly c is also even and hence,

$$\overline{m} = \overline{a} + \overline{b} + \underline{c} = \underline{a} + \underline{b} + \overline{c} = \underline{m}.$$

Further,

$$E(H_1) = \{e_1, e_2, \ldots, e_{\underline{a}}, f_1, f_2, \ldots, f_{\underline{b}}, g_1, g_2, \ldots, g_{\overline{c}}, \alpha_{a+1}, \ldots, \alpha_{2a}, \beta_{\underline{b}+1}, \ldots, \beta_b, \gamma_1, \ldots, \gamma_{\overline{d}}\}$$

and

$$E(H_2) = \{e_{\underline{a}+1}, \ldots, e_a, f_{\underline{b}+1}, \ldots, f_b, g_{\overline{c}+1}, \ldots, g_c, \alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_{\underline{b}}, \gamma_{\overline{d}+1}, \ldots, \gamma_d\}.$$

Once again, this partition of the edges is disjoint and completes this subcase.

Case 2. *If m is even and a is odd.*

Note that a similar construction applies in this case.

Case 3. *Suppose m is odd and $m \leq \frac{n}{2} - 1$.*

We note that this implies that $c \geq 2$. Then H_1 and H_2 can again be formed in a manner analogous to the first case. ∎

We now present an upper bound on $wr(G)$, the ratio of a weak ascending sequence of subgraphs in G.

Theorem2. *Suppose that G has $\binom{n+1}{2}$ edges, then $wr(G) \leq \frac{9}{8}\sqrt{2}$.*

Proof: Let G have the indicated number of edges. Also, let $S^1, S^2, \ldots, S^{\varepsilon n}$ be a sequence of stars from G, each containing at least $\frac{\sqrt{2}}{2}n$ edges selected as follows: Choose a vertex of maximum degree Δ in G. If $\Delta < \frac{\sqrt{2}}{2}n$, then we are done (i.e., no stars are selected). Otherwise,

$$\Delta = (\frac{\sqrt{2}}{2}n)t + r \text{ where } 0 \leq r < \frac{\sqrt{2}}{2}n.$$

Let $\frac{\sqrt{2}}{2}n = z$ and we select the stars as

$$S^1 = S_z = \cdots = S^{t-1} = S_z, S^t = S_{z+r}.$$

We continue removing large stars and partitioning them in this manner until it is no longer possible, that is, until $\Delta < \frac{\sqrt{2}}{2}n$. At this time we will have an ascending sequence of stars (by relabeling if necessary) $S^1, \ldots, S^{\varepsilon n}$, each of size at least $\frac{\sqrt{2}}{2}n$.

From the remaining edges, we form matchings. Since there are at most $(\frac{1}{2} - \frac{\sqrt{2}}{2}\varepsilon)n$ edges remaining, and the maximum degree is less than $\frac{\sqrt{2}}{2}n$, we can form $\frac{\sqrt{2}}{2}n$ matchings with at most $(\frac{\sqrt{2}}{2} - \varepsilon)n$ edges each. Since $\varepsilon \leq \frac{\sqrt{2}}{2}n$, we select εn matchings of even order (removing an edge if necessary).

Now with each star S^i and corresponding matching M^i we apply Lemma 1, to obtain two subgraphs partitioned as the disjoint union of a star of size at least $\frac{\sqrt{2}}{4}n$ and a matching of size $\frac{1}{2}(\frac{\sqrt{2}}{2} - \varepsilon)n$. Do this for each i, $1 \leq i \leq \varepsilon n$, and then arrange the resulting graphs in descending order. This can be accomplished by Lemma 1.

Each of the remaining $(\frac{\sqrt{2}}{2} - \varepsilon)n$ matchings M can now be paritioned into two matchings with $\left\lceil \frac{|M|}{2} \right\rceil$ and $\left\lfloor \frac{|M|}{2} \right\rfloor$ edges respectively. These matchings can easily be arranged into the next $2(\frac{\sqrt{2}}{2} - \varepsilon)n$ graphs in the descending sequence.

Thus, we now have

$$2\varepsilon n + 2(\frac{\sqrt{2}}{2} - \varepsilon)n = \sqrt{2}n$$

graphs in this descending sequence. To complete our task we need to modify the initial $\frac{\sqrt{2}}{8}n$ graphs in our present sequence (each is a matching of size $\frac{1}{2}(\frac{\sqrt{2}}{2} - \varepsilon n)$ and thus does not yet satisfy our size restriction). This is done by removing one edge from the top matching (i.e. matching $\frac{\sqrt{2}}{8}n$) and making this edge the first graph of our ascending sequence. Then remove two edges from the next smallest matching and make this the second graph of the ascending sequence. Continue in this fashion for the $\frac{\sqrt{2}}{8}n$ matchings and we obtain an ascending sequence of subgraphs with $\frac{9}{8}\sqrt{2}\,n$ graphs, now satisfying the size restrictions. There are still possibly $\frac{n}{2}$ edges unaccounted for. These may all be placed in the top graph which has at most $\frac{\sqrt{2}}{2}n$ edges. Thus, we have constructed the desired weak ascending sequence. ∎

Again it is worth noting that these graphs are star forests except for the largest graph.

4 Conclusions

We have succeeded in bounding the ascending subgraph sequence length from below and the weak ascending sequence length from above. This does not answer the original conjecture, however, it does lends some support to its possible truth. Extending either of these bounds would be interesting. Can one improve the bounds given in Theorems 1 and 2 by restricting consideration to special classes of graphs, in a manner similar to [F] and [**FGJL**]?

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