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SOME RAMSEY TYPE RESULTS ON TREES VERSUS COMPLETE GRAPHS

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ABSTRACT

For an arbitrary tree T of order m and an arbitrary positive integer n, Chvátal proved that the ramsey number $r(T, K_n) = 1 + (m-1)(n-1)$, i.e., for any coloring of the edges of $K_{1 + (m-1)(n-1)}$ with two colors, there exists a monochromatic tree T or a monochromatic K_n. Chvátal's theorem is extended by showing that, in certain cases, the result still follows if $K_{1 + (m-1)(n-1)}$ is replaced by an appropriate proper spanning subgraph of $K_{1 + (m-1)(n-1)}$.

For graphs G_1 and G_2, the ramsey number $r(G_1, G_2)$ is the least positive integer p such that if every edge of the complete graph K_p is arbitrarily colored red or blue, then there exists either a red G_1 (a subgraph isomorphic to G_1 all of whose edges are colored red) or a blue G_2. Equivalently, $r(G_1, G_2)$ is the least positive integer p such that if $K_p = R \oplus B$ is an arbitrary factorization of K_p (i.e., R and B have order p and $E(R) \cup E(B)$ is a partition of $E(K_p)$), then G_1 is a subgraph of R (in symbols $G_1 \subset R$) or G_2 is a subgraph of B.

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$K_{1+2n} - e \to (T_3, K_{n+1})$. Let $K_{1+2n} - e = R \oplus B$ be an arbitrary factorization and assume that $T_3 \not\subset R$. Hence there exist n pairwise nonadjacent edges $e_i = u_i v_i$, $1 \le i \le n$, such that

$$K_{1+2n} - e - \{e_1, e_2, \ldots, e_n\} \subset B.$$

If w is the remaining vertex of $K_{1+2n} - e$, then one of the induced subgraphs $\langle\{w, u_1, u_2, \ldots, u_n\}\rangle$ and $\langle\{w, v_1, v_2, \ldots, v_n\}\rangle$ is isomorphic to K_{n+1} so that $K_{n+1} \subset B$.

Assume now that

$$K_{1+(m-1)n} - e \to (T_m, K_{n+1})$$

for a fixed but arbitrary $m \ge 3$. We show that $K_{1+mn} - e \to (T_{m+1}, K_{n+1})$. Let T be an arbitrary tree of order $m + 1$; let v be an end-vertex of T and let u be the vertex of T adjacent with v. Suppose there exists a factorization $K_{1+mn} - e = R \oplus B$, where $T \not\subset R$ and $K_{n+1} \not\subset B$. Let w_1 and w_2 be the vertices of $K_{1+mn} - e$ such that $w_1 w_2$ is not an edge of $K_{1+mn} - e$. Let S be a set of n vertices of $K_{1+mn} - e$ such that $\{w_1, w_2\} \subseteq S$. Then $K_{1+mn} - S$ is a complete graph H of order $1 + (m-1)n$. By Theorem A, $r(T - v, K_{n+1}) = 1 + (m-1)n$ so that $T - v \subset R \cap H$ or $K_{n+1} \subset B \cap H$. Since $K_{n+1} \not\subset B$, we conclude that $T - v \subset R \cap H$. Furthermore, neither w_1 nor w_2 is a vertex of $T - v$. Now

$$K_{1+mn} - e - V(T - v) \cong K_{1+m(n-1)} - e.$$

By the inductive hypothesis, $K_{1+m(n-1)} - e \to (T, K_n)$ so that

$$T \subset R \cap [K_{1+mn} - e - V(T - v)]$$

or

$$K_n \subset B \cap [K_{1+mn} - e - V(T - v)].$$

Since $T \not\subset R$, then exists $F \cong K_n$ such that $F \subset B \cap [K_{1+mn} - e - V(T - v)]$.

Consider the vertex u of $T - v$. By construction, $u \ne w_1$ and $u \ne w_2$. If any edge at least $1 + (m - 2)(n - 1)$ edges of R joining $V(H')$ and $V(G) - V(H')$. Therefore, there is a vertex of H' that is incident with at least

$$\frac{1 + (m - 2)(n - 1)}{n - 1} > m - 2$$

edges of R, implying that $K(1, m - 1) \subset R$, a contradiction. ∎

We now investigate the possibility of a result similar to Theorem 1 where two adjacent edges are deleted from $K_{1+(m-1)(n-1)}$. We begin with a lemma.

In the next two results, $K_p - e - f$ $(p \geq 3)$ denotes the graph obtained by the removal of two arbitrary adjacent edges from K_p. Further, P_4 denotes the path of order 4.

Lemma 1. For $n \geq 2$,

$$K_{3n-2} - e - f \rightarrow (P_4, K_n).$$

Proof. We proceed by induction on n, the result following immediately for $n = 2$.

Assume that $K_{3n-2} - e - f \rightarrow (P_4, K_n)$ for a fixed but arbitrary $n \geq 2$. Let $G = K_{3n+1} - e - f$; we show that $G \rightarrow (P_4, K_{n+1})$. Let $x, y, z \in V(G)$ such that $xy, yz \notin E(G)$. Suppose that $G = R \oplus B$, where $P_4 \not\subset R$. We prove that $K_{n+1} \subset B$. Let H be a component of R having smallest order. We consider five cases.

Case 1. Suppose H has order 3. Then $G - V(H)$ is isomorphic to K_{3n-2}, $K_{3n-2} - e$ or $K_{3n-2} - e - f$. By applying Theorem A, Theorem 1 or the inductive hypothesis, respectively, we conclude that $G - V(H) \rightarrow (P_4, K_n)$. Thus, $P_4 \subset R \cap (G - V(H))$ or $K_n \subset B \cap (G - V(H))$. Since $P_4 \not\subset R$, it follows that there exists a graph F isomorphic to K_n such that

x, y and z. These $3n - 3$ vertices induce a subgraph K_{3n-3} in B. Because $n \geq 2$, $K_{n+1} \subset B$.

Case 5. Suppose none of Cases 1 - 4 holds, i.e., R has two or more components and each has order at least 4. This implies that $n \geq 3$. Let $R_1, R_2, \ldots, R_k$ $(k \geq 2)$ be the components of R. Necessarily $R_i \cong K(1, n_i)$ for $i = 1, 2, \ldots k$ and $n_i \geq 3$. Then R has at least $(3n + 1) - k - 3 = 3n - 2 - k$ end-vertices that are none of the vertices x, y and z. Thus, these $3n - 2 - k$ vertices induce $K_{3n-2-k} \subset B$.

Since each component R_i has order at least 4, it follows that $k \leq (3n + 1)/4$. Hence,

$$3n - 2 - k \geq 3n - 2 - \frac{3n + 1}{4} = \frac{9n - 9}{4} \geq n + 1$$

since $n \geq 3$. Therefore, $K_{n+1} \subset B$. ∎

We are now prepared to present an analogue to Theorem 1 in which two adjacent edges are deleted from $K_{1+(m-1)(n-1)}$.

Theorem 3. For $m \geq 4$ and $n \geq 2$,

$$K_{1+(m-1)(n-1)} - e - f \rightarrow (T_m, K_n).$$

Proof. We proceed by induction on $m (\geq 4)$. For $m = 4$, there are two trees T_4, namely P_4 and $K(1, 3)$. The result for P_4 follows by Lemma 1 while the result for $K(1, 3)$ follows by Theorem 2.

Assume that $K_{1+(m-1)(n-1)} - e - f \rightarrow (T_m, K_n)$ for a fixed $m \geq 4$ and all $n \geq 2$. We show that $K_{1+m(n-1)} - e - f \rightarrow (T_{m+1}, K_n)$ for all $n \geq 2$ by induction on n. Since $m \geq 4$, it follows immediately that $K_{m+1} - e - f \rightarrow (T_{m+1}, K_2)$. Assume that

$$K_{1+m(n-1)} - e - f \rightarrow (T_{m+1}, K_n)$$

for some $n \geq 2$. Let T be an arbitrary tree of order $m + 1$. We show that $K_{1+mn} - e - f \rightarrow (T, K_{n+1})$.

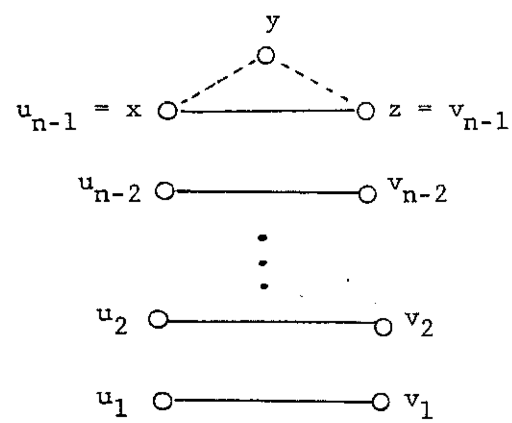

Figure 1

REFERENCES

[1] G. Chartrand, R. J. Gould and A. D. Polimeni, On ramsey numbers of forests versus graphs with large clique number. Submitted for publication.

[2] V. Chvátal, Tree-complete graph ramsey numbers. J. Graph Theory 1(1977) 93.