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A GENERALIZATION OF DIRAC'S THEOREM FOR K(1,3)-FREE GRAPHS

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Abstract

It is known that if a 2-connected graph G of sufficiently large order n satisfies the property that the union of the neighborhoods of each pair of vertices has cardinality at least $\frac{n}{2}$, then G is hamiltonian. In this paper, we obtain a similar generalization of Dirac's Theorem for $K(1,3)$-free graphs. In particular, we show that if G is a 2-connected $K(1,3)$-free graph of order n with the cardinality of the union of the neighborhoods of each pair of vertices at least $\frac{(n+1)}{3}$, then G is hamiltonian. We also investigate several other related properties in $K(1,3)$-free graphs such as traceability, hamiltonian-connectedness, and pancyclicity.

1. Introduction

In [3] it was shown that if a 2-connected graph G of sufficiently large order n satisfies the property that the union of the neighborhoods of each pair of vertices has cardinality at least $\frac{n}{2}$, then G is hamiltonian. This is a generalization of the well-known theorem of Dirac [1], which states that if the neighborhood of each vertex of a graph of order n contains at least $\frac{n}{2}$ vertices, then the graph is hamiltonian. In [8], Matthews and Sumner proved that if G is a 2-connected $K(1,3)$-free graph of order n such that the neighborhood of each vertex contains at least $\frac{(n-2)}{3}$ vertices, then G is hamiltonian. We provide a natural generalization of this result for $K(1,3)$-free graphs. In particular, we determine a lower bound on the cardinality of the neighborhood union of arbitrary pairs of vertices that is sufficient to ensure that the graph is hamiltonian.

If u and v are arbitrary vertices in $V(G)$, for convenience we define the *generalized degree*

$$\deg\{u,v\} = |N(u) \cup N(v)|.$$

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We then denote by the generalized minimum degree $\delta_2(G)$, the minimum of $\deg\{u,v\}$, where this minimum is taken over all pairs of distinct vertices u, v in $V(G)$.

We prove the following theorem in Section 2.

THEOREM A. *If G is a 2-connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3},$$

then if n is sufficiently large, G is hamiltonian.

We also obtain analogous traceability results for $K(1,3)$-free graphs in Section 2. In Section 3 we utilize related conditions to explore stronger • properties such as hamiltonian-connectedness, pancyclicity, and a variation of panconnectedness.

2. Hamiltonian Cycles and Paths

Since we will assume generalized degree conditions that do not imply connectivity, we must impose some minimum connectivity conditions. Throughout this paper we will let $\beta(G)$ denote the vertex independence number of G.

In the proofs of the subsequent lemmas and theorems, we will make use of the following known results:

THEOREM B. [4] *If G is a $K(1,3)$-free graph of order n such that*

$$\beta(G) \geq 3$$

and

$$\delta_2(G) \geq r,$$

then $\beta(G) \leq s$, where s is the larger solution to $rs(s-1) = 2(n-s)(2s-3)$.

Note that when r is a positive fraction of n, $\beta(G)$ is bounded and independent of n. In particular, when $r = \frac{(n+1)}{3}$, then $\beta(G) \leq 11$.

THEOREM C. [3] *Let G be a graph of order n with $3 \leq s \leq \frac{n}{2}$ and $\delta_2(G) \geq s$.*
i.) If G is connected, then G contains a path of order at least $2s-1$.
ii.) If G is 2-connected, then G contains a cycle of order at least $2s-2$.

THEOREM D. [2] *If G is a graph of order $n \geq 9$ such that $\delta_2(G) \geq \frac{(n+5)}{2}$ and $\delta(G) \geq 3$, then G is hamiltonian-connected.*

For convenience we define for a path $P = x_1, \ldots, x_t$ in a graph G, x_i^- to be the predecessor of x_i and x_i^+ to be the successor of x_i along the path P. For standard notation not given here see [7]. Also note that in the proofs of the subsequent lemmas and theorems the fact that $\beta(G) \geq 3$ is an immediate consequence of the assumptions and techniques used.

LEMMA 1. *Let G be a connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3}.$$

In addition, if there exists a path P in G containing a vertex of degree less than or equal to nine, and $|P| > \frac{(2n+26)}{3}$, then for n sufficiently large, G contains a hamiltonian path.

PROOF. We may assume that $P = x_1, x_2, \ldots, x_t$ is a longest path in G of order greater than $\frac{(2n+26)}{3}$ containing a vertex of degree less than or equal to nine, but that P is not a hamiltonian path. Consider the graph $H = \langle V(G) - V(P) \rangle$. Suppose H consists of a single vertex v, and v is adjacent to some vertex x_i on P. (Clearly v is not adjacent to x_1 or x_t or a path longer than P is obtained.)

Without loss of generality, suppose v is adjacent to some other vertex x_j, for $i < j$, on P. Then $x_i^+ \notin N(x_j^+)$ or we obtain the path

$$x_1, \ldots, x_i, v, x_j, \ldots, x_i^+, x_j^+, \ldots, x_t$$

which is longer than P. Since $\beta(G) \leq 11$, this implies that v is adjacent to at most nine vertices on P. (Clearly x_1, the successor of any adjacency of v on P, and v are independent.) But P already contains a vertex of small degree by our assumption, whence $\delta_2(G)$ is not satisfied for n sufficiently large.

Thus, $|H| \geq 2$. Select two adjacent vertices $u, v \in V(H)$. (This is possible otherwise for n sufficiently large, the bound on $\delta_2(G)$ would be violated.) Now $\deg_P\{u,v\} \leq 9$ since $\beta(G) \leq 11$. So $|H| \geq \frac{(n+1)}{3} - 9 = \frac{(n-26)}{3}$. But since $|P| > \frac{(2n+26)}{3}$, we obtain our contradiction.

Therefore, P must be a hamiltonian path. ■

LEMMA 2. *Let G be a $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3}.$$

For n sufficiently large the following statements hold:
i.) If G is a nontraceable, connected graph with P a longest path in G, and $H = \langle V(G) - V(P) \rangle$, then one of four possibilities exists:

a.) $H \simeq K_1$

b.) $H \simeq H' \cup K_1$, where H' is a component of H.

c.) H is connected and contains one vertex v with $\deg(v) < 3$.

d.) H is connected with $\delta(H) \geq 3$.

ii.) If G is a nonhamiltonian, 2-connected graph with C a longest cycle in G, and $H = \langle V(G) - V(C) \rangle$ then one of the above four possibilities holds.

PROOF. We will prove (i.) and omit the proof of (ii.) since the method of proof for both is the same.

Suppose that P is a longest path in G with $H = \langle V(G) - V(P) \rangle$. Then from Theorem C we have that $|P| = t \geq \frac{(2n-1)}{3}$. Assume that $H \not\simeq K_1$.

If H consists of two or more components where more than one component consists of at least two vertices, then observe that there must exist a minimum component with at least two vertices but at most $\frac{(n+1)}{6}$ vertices. Call such a component H'. Suppose vertex $u \in V(H')$ is adjacent to x_i on P and vertex $v \in V(H')$ is adjacent to x_j on P, with $i < j$. Since H' is connected, $x_i^+ \notin N(x_j^+)$ or we obtain

$$x_1, \ldots, x_i, u, \ldots, v, x_j, \ldots, x_i^+, x_j^+, \ldots, x_t,$$

which is a path longer than P, a contradiction. Similarly, successors of neighbors of v on P are not adjacent and successors of neighbors of u are not adjacent. Since $\beta(G) \leq 11$ from Theorem B, this implies that u and v together are adjacent to at most nine vertices on P. (Clearly, x_1 is not adjacent to u, v and any successors of neighbors of u and v on P.) Thus for n sufficiently large, the condition $\delta_2(G)$ is violated.

Thus $H \simeq H' \cup K_1$ or H is connected. (If H consists of more than one component where each component contains exactly one vertex, then since $\beta(G) \leq 11$, clearly the generalized degree condition is not satisfied.) Note that if H is connected and contains a vertex v with $\deg_H(v) < 3$, then v is the only vertex with degree less than three in H. Otherwise, if there were another vertex $u \in V(H)$ with $\deg(u) < 3$, then $\deg(v) + \deg(u) \leq 13$ and the condition $\delta_2(G)$ is not satisfied. (Since H is connected, both u and v together can be adjacent to at most nine vertices on P, or $\beta(G) \leq 11$ is not satisfied.) Therefore, we have completed the proof of the lemma. ■

LEMMA 3. *Let G be a connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3}.$$

If P is a longest path in G, then for n sufficiently large, $|P| = n$ or $|P| = n-1$.

PROOF. Let $P = x_1, x_2, \ldots, x_t$ be a longest path in G. From Theorem C we have that $|P| = t \geq \frac{(2n-1)}{3}$. Assume that the subgraph H induced by $V(G) - V(P)$ is not isomorphic to K_1, and that G is not traceable. It follows that $|H| \leq \frac{(n+1)}{3}$, and H satisfies one of the three remaining conditions in Lemma 2. We proceed with arguments to contradict our assumption that $|P| \neq n$.

Consider a maximum component H' of H. Then observe that $|H'| \geq \frac{(n-26)}{3} \geq 9$ since otherwise any two vertices, say u, v of $V(H')$ would satisfy $\deg\{u, v\} \leq 17$, a contradiction to the generalized minimum degree condition for n sufficiently large.

Suppose that $H = H' \cup K_1$ or H consists of a single component containing a vertex of degree less than 3. (If H consists of a single component containing a vertex v of degree less than 3, then denote by H' the graph induced by $V(H) - \{v\}$.) Then $\delta(H') \geq 3$, and for n sufficiently large, $\delta_2(H') \geq \frac{(|H'|-5)}{2}$. Thus, by Theorem D, H' is hamiltonian-connected.

Suppose $|H'| = s$, where $9 \leq s < \frac{(n+1)}{3}$. Since G is $K(1,3)$-free, there must exist three vertices between any two consecutive neighbors on P of any pair of vertices $u, v \in V(H')$, otherwise we could form a path longer than P. Now $\delta_2(G) \geq \frac{(n+1)}{3}$, so there exists some pair of vertices in H' that must be adjacent to at least $\lceil\frac{(n+1)}{3}\rceil - s$ vertices on P. But since there exists at least three vertices between any two consecutive adjacencies on P of pairs of vertices in $V(H')$, we see that there are at least

$$T = 4\left[\left\lceil\frac{(n+1)}{3}\right\rceil - s\right] - 3$$

vertices covered between the first and last adjacency of the pair u, v in $V(H')$. We subtract three to account for the three vertices on the path to the right of the last adjacency of the pair u, v which are not included in the count for T.

Hence, using H and half of P, we obtain a path of order

$$\frac{T}{2} + \frac{t}{2} + s + 1.$$

By our assumption,

$$t \geq \frac{T}{2} + \frac{t}{2} + s + 1,$$

and since $n = t + s + 1$, this implies that $t \leq \frac{(2n-7)}{3}$, a contradiction.

In the case where H contains no vertices of degree less than 3, we must show that $|H| < \frac{(n+1)}{3}$. Assume to the contrary, that $|H| = \frac{(n+1)}{3}$. In other words, the order of the longest path P is $t = \frac{(2n-1)}{3}$. Since G is connected, there exists at least one adjacency from some vertex $u \in V(H)$ to some vertex x_i on P. As before, because G is $K(1,3)$-free, then $x_i^- x_i^+ \in E(G)$. Without loss of generality, assume that $i \geq \frac{t}{2}$. Then the path

$$x_1, \ldots, x_i^-, x_i^+, x_i, u, \ldots, V(H), \ldots$$

is a path of order at least $\frac{(n+1)}{3} + \frac{1}{2}\left(\frac{2n-1}{3}\right) + 1 = \frac{(4n+7)}{6} > \frac{(2n-1)}{3}$, a contradiction. Thus $|H| < \frac{(n+1)}{3}$. With this fact at hand we use the same argument for H as we did for H' to show that $t \leq \frac{(2n-1)}{3}$, also a contradiction.

Therefore, $|P| = n$. ■

With the help of the lemmas and the previously stated theorems we can now prove:

THEOREM 4. *If G is a connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3},$$

then for n sufficiently large G is traceable.

PROOF. Let P be a longest path in G. Then from Lemma 3, $|P| = n$ or $|P| = n-1$. Assume that G is not traceable. Then $P = y = x_1, x_2, \ldots, x_{n-1} = z$ and the graph $H = \langle V(G) - V(P)\rangle \simeq K_1$. Let the set R_s represent the region containing the first s vertices on P, and let the set R'_s represent the region containing the last s vertices of P.

Assume that $V(H) = \{x\}$ and let $x_i \in N(x)$ for some x_i on P. Of course, since G is $K(1,3)$-free, $x_i^- x_i^+ \in E(G)$. Observe that if $x_i, x_j \in N(x)$, for $i < j$, then $i - j \geq \frac{(n-17)}{3}$. Otherwise, the path

$$x_1, \ldots, x_i^-, x_i^+, x_i, x, x_j, x_j^-, x_j^+, \ldots, x_{n-1}$$

contains x as well as more than $\frac{(2n+26)}{3}$ vertices. Consequently, we obtain a contradiction using Lemma 1. Similarly, we see that $\frac{(n-20)}{3} \leq i, j \leq \frac{(2n+20)}{3}$ or we can force the appropriate path in Lemma 1. Thus, if $|N(x)| \geq 3$, then P contains at least

$$2\left(\frac{n-20}{3}\right) + 2\left(\frac{n-17}{3}\right)$$

vertices. For n sufficiently large, we reach a contradiction. Hence $|N(x)| = 1$ or $|N(x)| = 2$.

Case 1: Suppose $|N(x)| = 2$.

Let $N(x) = \{x_i, x_j\}$ with $i < j$. Then

$$\frac{(n-20)}{3} \leq i < i + \frac{(n-17)}{3} \leq j \leq \frac{(2n+20)}{3}.$$

Since there can be no path containing x with greater than $\frac{(2n+26)}{3}$ vertices, then x_i has no adjacencies in $R_{\frac{n}{3}-8}$ or $R'_{\frac{n}{3}-8}$. Furthermore, x_i has no adjacencies in the region R_j containing the last $\frac{n}{3} - 7$ vertices that precede x_j. Moreover, the sets $R_{\frac{n}{3}-8}$, $R'_{\frac{n}{3}-8}$, R_j, and $N(x_i)$ are disjoint. Since $\deg\{x, x_i\} \geq \frac{(n+1)}{3}$, then $|N_P(x_i)| \geq \frac{(n-5)}{3}$. For n sufficiently large, this implies that $|P| > n - 1$, which is a contradiction.

Case 2: Suppose that $|N(x)| = 1$.

Let $N(x) = \{x_i\}$. As in Case 1, $\frac{(n-20)}{3} \leq i \leq \frac{(2n+20)}{3}$, where x_i has no adjacencies in $R'_{\frac{n}{3}-8}$ and x_i has no neighbors in $R_{\frac{n}{3}-8}$. Likewise y has no adjacencies in $R'_{\frac{n}{3}-8}$, and z has no adjacencies in $R_{\frac{n}{3}-8}$.

If $x_i x_j, y x_k \in E(G)$ with $j < k < i$, then the path

$$x, x_i, x_j, \ldots, y, x_k, \ldots, x_i^-, x_i^+, \ldots, z$$

contains x and all of the vertices of P except those between x_j and x_k. Hence to avoid a path that gives a contradiction using Lemma 1, $k - j \geq \frac{(n-23)}{3}$. We can obtain a similar path if $i < j < k$. Symmetrically, we see that these statements are true for adjacencies of x_i and z as well. Whence, the adjacencies of y precede the adjacencies of x_i, which precede the adjacencies of z. (Note that there could exist one vertex y' with $y' \in |N(y) \cup N(x_i)|$, where y' is on the boundary between these two regions. Similarly, we might have $z' \in |N(z) \cup N(x_i)|$ on P.)

For any set S of vertices on P, we denote as S^+ the set of successors of vertices in S, and S^- is the set of predecessors of vertices in S. From the above arguments and the fact that P is a longest path, the sets $N(y)^-$, $N(z)^+$, and $N_P(x_i)$, are disjoint sets such that the first two contain at least $\frac{(n-2)}{3}$ vertices, and the last contains at least $\frac{(n-5)}{3}$ vertices. Thus we obtain a partition of either $P - \{x_i\}$ or $P - \{x_i, x_j\}$ for some x_j. Now each vertex $u \in N(y)^-$ can play the same role as y, since u is the end vertex of a path of order $n-1$, so we can assume that $N(u) \subseteq N(y)$. Analogously we can assume that, $N(v) \subseteq N(z)$ for any $v \in N(z)^+$.

Let

$$\omega \in V(G) - (N(y) \cup \{y\} \cup N(z) \cup \{z\} \cup \{x, x_i\}).$$

Then

$$(N(x) \cup N(\omega)) \cap (N(y)^- \cup N(z)^+) \cup \{x, \omega\} = \emptyset,$$

for otherwise there exists a path which leads to a contradiction by Lemma 1. This implies that $|N(x) \cup N(\omega)| \leq \frac{(n-2)}{3}$, which is a contradiction and completes the proof of this case and of the theorem. ■

The example below illustrates the sharpness of the result.

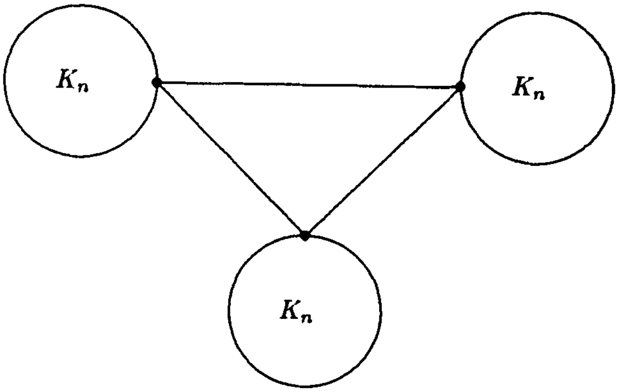

Fig. 1. A 1-connected $K(1,3)$-free graph that is not traceable

We now proceed with preliminary results that will be used to obtain sufficient conditions to get a hamiltonian cycle in a $K(1,3)$-free graph:

LEMMA 5. *Let G be a 2-connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3}.$$

If there exists a cycle C in G containing a vertex of degree less than or equal to ten, and $|C| \geq \lfloor\frac{(2n+6)}{3}\rfloor$, then for n sufficiently large G is hamiltonian.

PROOF. Suppose that G is not hamiltonian and choose a longest cycle C containing a vertex x where $\deg(x) \leq 10$ such that C has order at least $\lfloor\frac{(2n+6)}{3}\rfloor$. Let H be the subgraph induced by $V(G - C)$. Since x is a vertex of degree less than or equal to 10, H can contain no vertices of degree less than $\frac{(n+1)}{3} - 10$ or the generalized minimum degree condition is violated for n sufficiently large. In addition, $|H| \geq 9$. Hence from Theorem D, H is hamiltonian-connected.

Because $\delta_2(G) \geq \frac{(n+1)}{3}$, and $|C| \geq \lfloor\frac{(2n+6)}{3}\rfloor$, every pair of vertices in $V(H)$ has at least three adjacencies on C. From this fact and the knowledge that $|H| \geq 3$, we can easily obtain a 3-matching from H to C. (A 3-matching is simply a set of three pairwise independent edges from H to C.) With a 3-matching and the knowledge that H is hamiltonian-connected, we obtain a cycle containing x of order at least

$$\frac{2}{3}\left\lfloor\frac{(2n+6)}{3}\right\rfloor + 3 + \frac{(n-6)}{3} = \frac{(7n+21)}{9}.$$

Thus $|C| \geq \frac{(7n+21)}{9}$. From Theorem B, as before, we know that $\beta(G) \leq 11$. Thus, for every pair of vertices u, v in $V(H)$, $\deg_C\{u,v\} \leq 10$, and since the number of vertices off the cycle is at most $\frac{(n-30)}{3}$, we contradict the generalized minimum degree condition.

Therefore, G is hamiltonian. ∎

LEMMA 6. *Let G be a 2-connected, $K(1,3)$-free graph of order n such that $\delta_2(G) \geq \frac{(n+1)}{3}$ and C is a longest cycle in G. Then for n sufficiently large, $|C| = n$ or $|C| = n-1$.*

PROOF. Let $C = x_1, x_2, \ldots, x_t, x_1$ be a longest cycle in G such that $|C| \neq n-1$ and assume that G is not hamiltonian. Since $\delta_2(G) \geq \frac{(n+1)}{3}$, from Theorem C, we obtain a cycle of order at least $\frac{(2n-4)}{3}$. Denote as H, the subgraph induced by $V(G) - V(C)$.

Consider a maximum component F of H. Since G is 2-connected, there exist at least two vertices u, v in F such that $u \in N(x_i)$ and $v \in N(x_j)$ for some $x_i, x_j \in V(C)$, where $i \neq j$. (Note that from Lemma 2, the only other component of H other than F that could exist is a component isomorphic to K_1.) Since G is $K(1,3)$-free, certainly $x_i^- x_i^+$ and $x_j^- x_j^+ \in E(G)$.

Suppose that $F = H$ with $|H| = s$. As seen in Lemma 3, if there exists a vertex $\omega \in V(H)$ with $\deg_H(\omega) < 3$, we may simply consider the subgraph $H' = \langle V(H) - \{\omega\}\rangle$. Then for n sufficiently large, since $|H'| \geq 9$ we see that H' is hamiltonian-connected.

Assuming that this vertex of degree less than 3 exists in H, we will use the above observations and the edges ux_i and vx_j (select u, v here such that $u \neq \omega$ and $v \neq \omega$), to build a cycle of order at least

$$\left(\frac{n-s-2}{2}\right) + (s-1) + 4.$$

However, from our choice of the cycle C, we know that

$$\left(\frac{n-s-2}{2}\right) + (s-1) + 4 \leq n - s,$$

which implies that $s-1 \leq \frac{(n-7)}{3}$. Since $|H'| = s-1$ and $\delta_2(G) \geq \frac{(n+1)}{3}$, then nearly all pairs of vertices in H' have at least three adjacencies on C. (Pairs of vertices in $V(H')$ that include at least one neighbor of ω may not possess this property. Since $|H|$ is large enough, this fact does not affect the argument.) Hence we can obtain a 3-matching ux_i, vx_j, and zx_k, $i<j<k$, from H' to C. Without loss of generality, assume that the two largest of the three regions on C between x_i, x_j, and x_k are $R_1 = \{x_i^+, \ldots, x_j^-\}$ and $R_2 = \{x_j^+, \ldots, x_k^-\}$. Then the cycle

$$u, x_i, x_i^-, x_i^+, \ldots, x_j, \ldots, x_k^-, x_k^+, x_k, z, \ldots, V(H') \ldots, u$$

is a cycle of order at least

$$\frac{2}{3}(n-s-3)+(s-1)+5.$$

We add the five at the end to account for x_i, x_j, x_k, x_i^- and x_k^+.

Since C is a longest cycle then

$$\frac{2}{3}(n-s-3)+(s-1)+5 \leq n-s$$

which implies that $s \leq \frac{(n-6)}{4}$. This fact leads to a contradiction since it forces too many adjacencies on C from vertices in H', and the upper bound on $\beta(G)$ is not satisfied. Certainly if H is connected with no small degree vertex, we can use similar arguments to reach a contradiction.

If H is disconnected, and there exists a component isomorphic to K_1 in addition to the component F, then F cannot contain a low degree vertex or there would exist a pair of low degree vertices, which is a contradiction. Thus we are not forced to delete a vertex from the maximum component, and we can argue as we did previously to get the same upper bound for s and a contradiction.

Therefore, G is hamiltonian. ∎

By simply strengthening the connectivity condition in Theorem 4 we obtain a hamiltonian cycle in the graph G with the same lower bound on $\delta_2(G)$.

THEOREM 7. *If G is a 2-connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+1)}{3},$$

then for n sufficiently large G is hamiltonian.

PROOF. Assume that G is not hamiltonian. Let $C = x_1, x_2, \ldots, x_{n-1}, x_1$ be a cycle of order $n-1$ in G, assured by Lemma 6. Let x be the vertex not on C. Then since $\beta(G) \leq 11$ and G is 2-connected, $2 \leq \deg(x) \leq 10$. To resolve this case we will attempt to locate x on a cycle of order at least $\lfloor\frac{(2n+6)}{3}\rfloor$. Once this is accomplished we can use Lemma 5 to conclude that G is hamiltonian.

Suppose x has exactly two adjacencies x_i, x_j on C with $i<j$. Let Region 1 be denoted by $R_1 = \{x_i^{++}, \ldots, x_j^{--}\}$ and let Region 2 be denoted by $R_2 =$

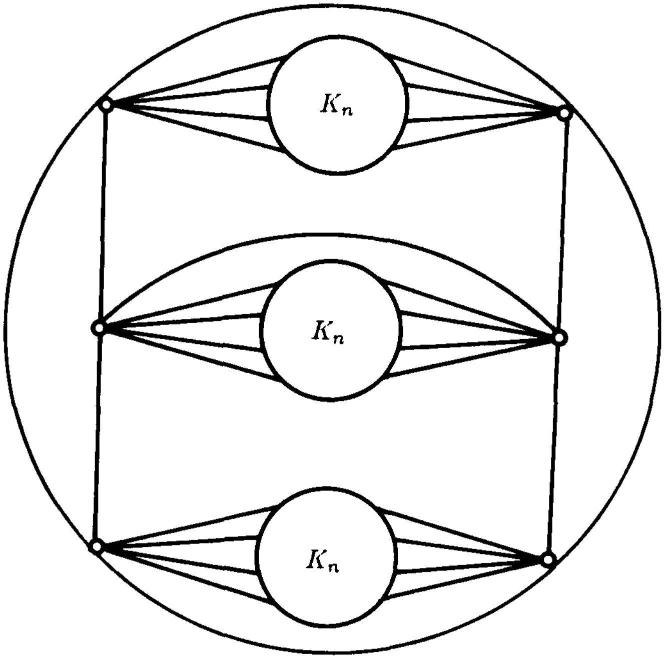

Fig. 2. A $K(1,3)$-free nonhamiltonian graph

$\{x_j^{++}, \dots, x_i^{--}\}$. Assume that x_k is the closest adjacency of x_j to x_i in Region 1 on C. Then if $S_1 = \{x_i^+, \dots, x_k^-\}$, we see that $|S_1| \geq \lceil \frac{n}{3} \rceil$ or

$$x_1, \dots, x_i^-, x_i^+, x_i, x, x_j, x_k, \dots, x_j^-, x_j^+, \dots, x_1$$

is a cycle of order at least $\lfloor \frac{(2n+6)}{3} \rfloor$ containing x. Also if x_l is the closest adjacency of x_j to x_i in Region 2 on C, then the set $S_2 = \{x_l^+, \dots, x_i^-\}$ is such that $|S_2| \geq \lceil \frac{n}{3} \rceil$.

From the hypothesis, $|N(x_i) \cup N(x)| \geq \lceil \frac{(n+1)}{3} \rceil$. Moreover, S_1, S_2 and $|N(x_i) \cup N(x)|$ are disjoint sets of vertices. Hence, there must be at least $\lceil \frac{(n+1)}{3} \rceil + \lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil$ vertices in the graph, which is too many. Thus we have reached our contradiction.

If x has degree ≥ 3, then we automatically get a cycle of length at least $\frac{(2n+10)}{3}$ containing x. From Lemma 5 we obtain the desired contradiction.

Therefore, G is hamiltonian. ■

This result is sharp as seen by the example in Figure 2. There $\delta_2(G) \geq \frac{n}{3}$, G is $K(1,3)$-free, and G is not hamiltonian.

3. Higher Hamiltonian Properties

In this section we supply generalized degree conditions that guarantee that a graph is hamiltonian-connected, pancyclic, and 8-panconnected. (We define a graph G of order n to be 8-panconnected if and only if for every pair of vertices $u, v \in V(G)$ there exists a uv-path of length l for each $8 \leq l \leq n-1$.)

LEMMA 8. *Let G be a connected $K(1,3)$-free graph of sufficiently large order n such that*

$$\delta_2(G) \geq \frac{(n+24)}{3}.$$

In addition, for vertices $x, y \in V(G)$, if there exists an xy-path P in G containing a vertex of degree less than or equal to eleven, and $|P| > \frac{(2n+9)}{3}$, then P contains all of the vertices of G.

PROOF. Suppose that P is a longest xy-path in G containing a vertex of degree less than or equal to eleven, but that P is not a hamiltonian xy-path. Consider the subgraph $H = \langle V(G) - V(P) \rangle$. If H consist of a single vertex v, then $\deg(v) \leq 11$, since from Theorem B we know that $\beta(G) \leq 11$. (The only situation where $\deg(v) = 11$ occurs is if v is adjacent to both x and y, otherwise $\deg(v) \leq 10$.) However, P already contains a vertex of small degree, whence $\delta_2(G)$ is not satisfied for n sufficiently large.

Thus $|H| \geq 2$. For adjacent vertices $u, v \in V(H)$, since $\beta(G) \leq 11$, $|N_P(u) \cup N_P(v)| \leq 11$. (Certainly we can find two adjacent vertices in H since otherwise for n sufficiently large the $\delta_2(G)$ condition would be violated.) Since $\delta_2(G) \geq \frac{(n+24)}{3}$, then $|H| \geq \frac{(n+24)}{3} - 11 = \frac{(n-9)}{3}$. But $|P| > \frac{(2n+9)}{3}$ which shows that the order of H is not large enough, and we obtain a contradiction to the generalized minimum degree condition.

Therefore, P must be a hamiltonian xy-path. ■

LEMMA 9. *Let G be a 3-connected $K(1,3)$-free graph of order n such that $\delta_2(G) \geq \frac{(n+24)}{3}$ and P is a longest xy-path in G for vertices $x, y \in V(G)$. Then for n sufficiently large, $|P| = n$ or $|P| = n-1$.*

PROOF. Let $P : x = x_1, x_2, \ldots, x_t = y$ be a longest xy-path in G, Assume $|P| \leq n-2$. We will proceed with arguments to reach a contradiction.

Consider a maximum component H' in $H = \langle\langle V(G) - V(P) \rangle$ where $|H| = s$. (As before, select H' such that it is a maximum connected subgraph of H with $\delta(H') \geq 3$. With arguments similar to those of Lemma 2, since $\beta \leq 11$, we know that $|H'| = |H|$ or $|H'| = |H| - 1$.) For n sufficiently large, $|H'| \geq 9$. Thus from Theorem D, we have that H' is hamiltonian-connected. Now since G is 3-connected we can locate a 3-matching from H' to P. With this information, we can form an xy-path of order at least

$$\frac{(n-s-2)}{2} + 3 + (s-1)$$

But

$$\frac{(n-s-1)}{2} + 2 + (s-1) \leq n - s$$

since we are assuming that P is a longest xy-path. This implies that $s \leq \frac{(n-1)}{3}$. Consequently, every pair of vertices must have at least nine adjacencies on P. With this many neighbors on P for every pair, using the above method, we can show that the path P is long enough to force $|H'| \leq \frac{(n-10)}{3}$. This would then contradict the fact that every pair of vertices has at most 11 adjacencies on the path P.

Therefore, $|P| = n$ and we have proven the lemma. ∎

We can now prove a theorem involving the hamiltonian-connected property and a generalized degree condition with the help of Lemmas 8 and 9 and Theorem D:

THEOREM 10. *If G is a 3-connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) \geq \frac{(n+24)}{3},$$

then for n sufficiently large G is hamiltonian-connected.

PROOF. Assume that G is not hamiltonian-connected. Then there are vertices $x, y \in V(G)$ such that there exists no xy-hamiltonian path. Let $P = x = x_1, x_2, \ldots, x_t = y$ be a longest xy-path in G. Then from the previous lemma, P has order $n-1$. If z is the vertex off the path P, then $3 \leq \deg(z) \leq 11$ since G is 3-connected and $\beta(G) \leq 11$. (In the special case where z is adjacent to both x and y, the $\deg_P(z)$ can be equal to eleven with $\beta(G) \leq 11$ still satisfied.)

Case 1: Suppose $4 \leq \deg(z) \leq 11$.

The worst case occurs when z is adjacent to both x and y since this situation leads to the shortest xy-path to be considered in the arguments below. Thus we will assume that there are at least four neighbors of z on P, namely x, y, x_i, and x_j for some $i < j$. As before, since G is $K(1,3)$-free, $x_i^- x_i^+$ and $x_j^- x_j^+ \in E(G)$. Since z is adjacent to both x and y and $4 \leq \deg(z) \leq 11$, then $\deg_P(x_i) \geq \frac{n+24}{3} - 12$.

We partition sections of the path P into three regions accordingly:

Let

$$R_1 = \{x^+, \ldots, x_i^{--}\}, R_2 = \{x_i^{++}, \ldots, x_j^{--}\}, \text{ and } R_3 = \{x_j^{++}, \ldots, y^-\}.$$

Now clearly $|R_3| = \frac{n-9}{3}$ (since $|H'| \geq 9$), otherwise we could form the xy-path

$$x, \ldots, x_j^-, x_j^+, x_j, z, y$$

of order greater than $n - \frac{(n-9)}{3} = \frac{(2n+9)}{3}$ containing z. By Lemma 8 we would reach a contradiction to the assumption that there is no xy-hamiltonian path. Similarly, $|R_1| = \frac{n-9}{3}$ and $|R_2| = \frac{n-9}{3}$.

Suppose that x_i has an adjacency ω_1 in R_1. Then

$$x, z, x_i, \omega_1, \ldots, x_i^-, x_i^+, \ldots, y$$

is an xy-path containing the small degree vertex z with order greater than or equal to

$$2\left(\frac{n-9}{3}\right)+10=\frac{(2n+12)}{3}.$$

Thus, by Lemma 8 this path is a hamiltonian xy-path, and we reach a contradiction. So we see that there exist no adjacencies of x_i into R_1. Similarly, if x_i has an adjacency ω_3 into R_3, then the xy-path

$$x,\ldots,x_i^-,x_i^+,\ldots,x_j,\ldots,\omega_3,x_i,z,y$$

is of order greater than or equal to $\frac{(2n+12)}{3}$ containing z. Once again, from Lemma 8 we obtain a hamiltonian xy-path. Lastly, for ω_2 in R_2 such that $\omega_2 \in N(x_i)$ the path

$$x,x_i^-,x_i^+,\ldots,\omega_2,x_i,z,x_j,x_j^-,x_j^+,\ldots,y$$

is an xy-path of order at least $\frac{(2n+12)}{3}$. Hence, x_i has no adjacencies into R_2.

Thus there exist three distinct regions of P each of order greater than or equal to $\frac{(n-9)}{3}$ where x_i has no adjacencies. But as mentioned previously, $\deg_P(x_i) \geq \frac{(n-12)}{3}$. Whence, for n sufficiently large we have forced more than n vertices in $V(G)$, and P must be a hamiltonian xy-path.

Case 2: Suppose $\deg(z)=3$.

As in Case 1, we will assume that $x,y \in N(z)$ since this is the worst situation possible. Suppose also that z is adjacent to some other vertex x_i on P.

Since $\deg(z)=3$, we have that each of $\deg_P(x)$, $\deg_P(y)$, and $\deg_P(x_i) \geq \frac{(n+24)}{3}-4=\frac{(n+12)}{3}$. As in Case 1, x_i has no adjacencies among the final $\frac{(n-4)}{3}$ and first $\frac{(n-4)}{3}$ vertices of P, (except for possibly x and y) otherwise we can create as before a path in each case containing z and of order greater than $\frac{(2n+9)}{3}$. Hence x_i has at least $\frac{(n+6)}{3}$ adjacencies (expect for possibly x and y) in the 'middle' region of P. Since these regions are disjoint, this means that there are at least $2\left(\frac{n-4}{3}\right)+\frac{(n+6)}{3}=n-\frac{2}{3}$ vertices on the path P. This is a contradiction, however, since this implies that $|P|=n$.

Therefore, P is a hamiltonian xy-path and we have proven this case and hence the theorem. ∎

Before we present a result involving a variation of the panconnected property, we must make some preliminary observations:

LEMMA 11. *If G is a $K(1,3)$-free 2-connected graph of order n such that*

$$\delta_2(G)>\frac{3n}{k},$$

$k \geq 4$, then the diameter of G, denoted diam(G), is less than $3\lceil\frac{k}{3}\rceil-1$.

PROOF. Suppose that $\operatorname{diam}(G) \geq 3\lceil\frac{k}{3}\rceil-1$. Then there exists a path P of order at least $3\lceil\frac{k}{3}\rceil$ such that the vertices on P have no additional adjacencies on P.

Select x to be an end vertex of this path P. For $1 \leq t \leq k$, let $N_t(x)$ denote the set of vertices at distance t from x. Since G is 2-connected, there exist at least two vertices in $N_1(x)$. Now $\delta_2(G) > \frac{3n}{k}$, so $|\{x\} \cup N_1(x) \cup N_2(x)| > \frac{3n}{k}$. Similarly, for $t = 3s$ $(1 \leq s \leq 3\lceil \frac{k}{3} \rceil - 1)$, $|N_{t-1}(x) \cup N_t(x) \cup N_{t+1}(x)| > \frac{3n}{k}$. Thus we have at least $3\lceil \frac{k}{3} \rceil$ pairs of vertices with disjoint neighborhoods, and each pair has greater than $\frac{3n}{k}$ vertices in its neighborhood. But this requires that there exist more than n vertices in $V(G)$. Therefore, we obtain a contradiction. ■

Note that the above argument implies that if $\delta_2(G) > \frac{n}{3}$, then $\mathrm{diam}(G) \leq 7$.

In the next lemma we use a result from [6] that provides the structure needed to construct paths of different lengths in the graph G.

THEOREM E. [6] *Let H ba a graph of order $n \geq 2$ with $\beta(G) \leq 2$, and let $G = K_1 + H$. Then either G is panconnected or $H = K_r \cup K_{p-r}$ for some $1 \leq r < p$.*

LEMMA 12. *If G is a 3-connected $K(1,3)$-free graph of order n such that*

$$\delta_2(G) > \frac{n}{3},$$

then for n sufficiently large, there exist xy-paths of length l, for each $8 \leq l \leq \frac{n}{6}$, and for all $x, y \in V(G)$.

PROOF. Since $\delta_2(G) > \frac{n}{3}$ there exists at most one vertex of degree $\leq \frac{n}{6}$. Consider two vertices $x, y \in V(G)$, and without loss of generality assume $\deg(x) > \frac{n}{6}$. Let H be the subgraph induced by $N(x) \cup x$. Then $\beta(H) \leq 2$ since G is $K(1,3)$-free. Hence H is panconnected or $H - x$ is the union of two complete graphs from Theorem E.

i.) Suppose H is panconnected. If $y \in N(x)$, clearly we get xy-paths of all lengths up to $\frac{n}{6} + 1$. If $y \notin N(x)$, since the diameter of G is less than or equal to seven (Lemma 11), we get all xy-paths of length l for each $8 \leq l \leq \frac{n}{6} + 1$.

ii.) Suppose $H - x$ is the union of two complete graphs. Let L be the largest complete component of $H - x$ so that $|L| > \frac{n}{12}$. Denote by S the smaller of the two components. Consider the shortest path P from y to x. Assume that P goes through L. Since L is complete and $\mathrm{diam}(G) \leq 7$, we obtain paths of each length from 8 to $\frac{n}{12}$.

Now consider the 2-connected graph $G - \{x\}$. We form a new graph G' with $V(G') = V(G - \{x\} + \{u, v\})$, where u is a new vertex adjacent to exactly two vertices in L, and v is a new vertex adjacent to y and a vertex in S. Since G' is 2-connected, there exist at least two internally disjoint uv-paths in G'. One path contains a path from y to L, and the other path contains a path from L to S. Select a pair of shortest disjoint uv-paths. Let P_1 be the path from y to L, and let P_2 represent the path from L to S. (Note that the end vertex of P_1 in L and the end vertex of P_2 in L are distinct.) Then $|P_1| \leq 8$ and $|P_2| \leq 8$. Otherwise, using arguments similar to those in Lemma 11, we can select alternating pairs of vertices on P_1, for example, such that these pairs have disjoint neighborhoods. Since these pairs must satisfy the generalized minimum degree condition, this forces too many vertices in our graph. Now for n sufficiently large, $16 < \frac{n}{12}$.

Using P_1 and P_2, and the vertices of both L and S, we can get all xy-paths of lengths from $\frac{n}{12}$ to $\frac{n}{6}$.

Assume the P goes through S. Then once again we are able to get all paths of lengths up to $\frac{n}{6}$ with the use of P_1 and P_2. ■

With these lemmas at hand, we provide a sufficient generalized degree condition for a graph G to be 8-panconnected. As a corollary, the graph G is pancyclic given the same generalized degree condition.

THEOREM 13. *Let G be a 3-connected $K(1,3)$-free graph of order n. Then there exists a constant c such that if*

$$\delta_2(G) \geq \frac{n}{3} + c,$$

then G is 8-panconnected.

PROOF. In this proof we show that $c = 128$ is sufficient (but not necessarily the best possible constant) to obtain the desired result. Assume that G is not 8-panconnected. Since n is large enough, from Lemma 12, we get all xy-paths of lengths $8, 9, \ldots, \frac{n}{6}$.

Select $t \geq \frac{n}{6}$ such that there exists a path $P_j(x,y)$, of length j, for all $8 \leq j \leq t$, but no $P_{t+1}(x,y)$. For $n < 193$ the theorem is trivially true, so for $n \geq 193$ we proceed by induction on n, the order of G. First notice that if G is a 3-connected $K(1,3)$-free graph of order 193 such that $\delta_2(G) \geq 193/3 + 128$, then G is complete, which implies that G is 8-panconnected. So assume the result for graphs of order k, for $193 \leq k \leq n-1$, that is, they are 8-panconnected. Using this hypothesis, we will build paths of the desired lengths.

Claim 1: We claim that $t > \frac{n}{3}$.

Suppose this is not the case. Let $P_t(x,y) : x = x_1, x_2, \ldots, x_t = y$ and consider $H = \langle V(G) - V(P_t(x,y))\rangle$. Then $\frac{2n}{3} \leq |H| \leq \frac{5n}{6}$, from our assumption and Lemma 12. Once again, from Theorem B we know that $\beta(G) \leq 11$. Consider a vertex v in H. If $v \in N(x_i)$ and $v \in N(x_j)$, for $x_i, x_j, i < j$, on $P_t(x,y)$, since G is $K(1,3)$-free, certainly $x_i^- x_i^+, x_j^- x_j^+ \in E(G)$. Also $x_i^- x_j^- \notin E(G)$ since otherwise the path

$$x = x_1, \ldots, x_i^-, x_j^-, \ldots, x_i^+, x_i, v, x_j, \ldots, x_t = y$$

is a $P_{t+1}(x,y)$, which is forbidden. Without loss of generality, assume that v is adjacent to x. (If v is adjacent to both x and y, the argument is the same.) Then $v \notin N(x_2) \cup N(x_3)$ since we would obtain a $P_{t+1}(x,y)$. But for any $x_i \in N(v), x_2 \notin N(x_i^+)$ or the path

$$x, v, \ldots, x_i, \ldots, x_2, x_i^+, \ldots, y$$

is a $P_{t+1}(x,y)$. Hence, since $\beta(G) \leq 11$, $\deg_{P_t}(v) \leq 11$.

We now investigate the connectivity of H by showing that H contains a large 3-connected subgraph K (one with at least $\frac{n}{3}$ vertices with $\delta_2(K) \geq \frac{n}{3}$). If H is

3-connected, we have located our 3-connected subgraph. So assume that H is not 3-connected.

Delete any vertex u of H such that $\deg_H(u) \leq 4$. (There is at most one such vertex.) If the resulting graph H' is 3-connected, we are done. If not, there is a cutset S, with $|S| \leq 2$, such that $H' - S$ is disconnected. The number of resulting components is two, for otherwise a $K(1,3)$ would occur. Let these two components be denoted H_1 and H_2.

Without loss of generality, consider $u, v \in H_1$. Since $\beta(G) \leq 11$, $|N_{G-H_1}(u) \cup N_{G-H_1}(v)| \leq 24$. (Each of u and v can have at most 11 adjacencies on P, and u and v might also be adjacent to two vertices in the cutset.) Thus, $\delta_2(H_1) \geq \frac{n}{3} + 104$ which implies that $|H_1| \geq \frac{n}{3} + 104$. Hence if H_1 is 3-connected, then H_1 is the desired subgraph. If H_1 is not 3-connected, then there exists some cutset S_1, with $|S_1| \leq 2$, such that $H_1 - S_1$ has two components H_{1_1} and H_{1_2}. Each of H_2, H_{1_1}, and H_{1_2} has at least $\frac{n}{3}$ vertices, which contradicts the order of G. Therefore, H must contain a 3-connected subgraph K with the needed properties.

Certainly K is $K(1,3)$-free and $\delta_2(K) \geq \frac{n}{3}$. Since $|K| = p \leq \frac{5n}{6}$, then $\delta_2(K) \geq \frac{p}{3} + 128$. From the induction hypothesis, K is 8-panconnected. Since G is 3-connected, we can locate disjoint paths $P_1 = x, \ldots, u$ and $P_2 = y, \ldots v$ from x to K and from y to K, respectively. From Lemma 11, the shortest paths from K to the path P have length less than or equal to 8. Thus we can get an xy-path

$$x, \ldots, u, \ldots, v, \ldots, y$$

of length at most 24. Now since $|K| \leq \frac{n}{3}$ and is 8-panconnected, we get all paths of lengths up to $\frac{n}{3} + 2$. Hence, we have shown that $t > \frac{n}{3}$ and have proven Claim 1.

Claim 2: We claim that $t \geq \frac{n}{2}$.

Using methods similar to those utilized in Claim 1 we can prove this statement. Once again, consider $G - P_t(x,y) = H$. By previous methods H contains a 3-connected subgraph K with the appropriate properties. By induction, K is 8-panconnected and we get all xy-path of desired lengths up to $\frac{n}{2}$.

Claim 3: Building further, we assert that $t \geq \frac{2n-1}{3}$. Again consider $H = G - P_t(x,y)$. Suppose that H is not 3-connected, and assume that when removing the cutset of H the result is at least two components each with at least two vertices. Since it has been established that $t \geq \frac{n}{2}$, then this implies that the smallest component has order less than equal to $\frac{n}{4}$. This fact contradicts the generalized degree condition since any pair of vertices in one component can have at most 22 adjacencies on the path. Thus H contains a 3-connected component K with $|K| \geq (n-t) - 1$. By induction, K is 8-panconnected. Since G is 3-connected, there exists a 3-matching from K to $P_t(x,y)$. Since $\operatorname{diam}(G) \leq 8$, we can get all paths of lengths t to $\frac{t+1}{2} + (n-t) - 1$. Thus, $\frac{t+1}{2} + (n-t) - 1 \leq t$ which implies that $t \geq \frac{2n-1}{3}$.

Now if $t \geq \frac{2n-1}{3}$, we immediately obtain that $t \geq n-1$, since when $t \geq \frac{2n-1}{3}$, every pair of vertices in $V(H) = V(G - P_t(x,y))$ must have 128 adjacencies on $P_t(x,y)$. This violates $\beta(G) \leq 11$. We see that $t = n$ from Theorem 10, which states that G is hamiltonian-connected.

Therefore, G is 8-panconnected. ∎

COROLLARY 14. *Let G be a 3-connected $K(1,3)$-free graph of order n. Then there exists a constant c such that if*

$$\delta_2(G) \geq \frac{n}{3} + c,$$

then G is pancyclic.

PROOF. Since G is 8-panconnected for $c = 128$, we get cycles of each length l for each $8 \leq l \leq n-1$. But since $\delta_2(G) \geq \frac{n}{3} + c$ and G is $K(1,3)$-free, we obtain cycles of lengths 3 to 7 by considering the neighborhoods of pairs of vertices.

Therefore, G is pancyclic. ∎

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