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Generalized degrees and Menger path systems

R.J. Faudree*

Memphis State University, Memphis, TN, USA

R.J. Gould*

Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

L.M. Lesniak

Drew University, Madison, NJ, USA

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Abstract

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For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G, there are m internally disjoint paths of length at most d. For a positive integer t, a graph G satisfies the minimum generalized degree condition $\delta_t(G) \geq s$ if the cardinality of the union of the neighborhoods of each set of t vertices of G is at least s. Generalized degree conditions that insure that $P_{d,m}(G)$ is satisfied are investigated. For example, if for fixed positive integers $t \geq 5$, $d \geq 5t^2$, and $m \geq 2$, an m-connected graph G of order n satisfies the generalized degree condition $\delta_t(G) \geq (t/(t+1))(5n/(d+2)) + (m-1)d + 3t^2$, then for n sufficiently large G has property $P_{d,m}(G)$. Also, if the order of magnitude of $\delta_t(G)$ is decreased, then $P_{d,m}(G)$ will not hold; so the result is sharp in terms of order of magnitude of $\delta_t(G)$.

1. Introduction

Consider a graph G that models a computer network with each vertex representing a processor and each edge representing a two-way communication link. To insure that the network is fault-tolerant with respect to processor failures, it is necessary that the number of internally disjoint paths between each pair of vertices

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of G exceed the number of possible failures. Connectivity is clearly the crucial graph concept. However, the length of time for the information to arrive is also important, so it is desirable that the internally disjoint paths be short. This requires that between each pair of vertices of the graph G there is a specified number of internally disjoint paths, with a bound on the length of each.

For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G there are at least m internally disjoint paths, each of length at most d. The graph G representing a computer network prone to processor failures should satisfy $P_{d,m}(G)$ for appropriate values of d and m. Menger's classic result [8] on connectivity solves the problem of the existence of a system of such paths, if there is no concern for the length of the paths in the system. Although Menger's theorem gives no information about the length of the paths, the "length problem" has been studied. For example, in [1] Bond and Peyrat studied the effect of adding or deleting edges on the diameter of a network, and Chung and Garey considered diameter bounds in [3]. Menger type results for paths of bounded length were proved by Lovasz, Neumann-Lara and Plummer in [7] and by Pyber and Tuza in [11], and Mengerian theorems for "long paths" (i.e., at least a given length) were given by Montejano and Neumann-Lara in [9] and by Hager in [6]. In [10] property $P_{d,m}(G)$ and its application to computer networks and distributed processing was introduced. Extremal results for $P_{d,m}(G)$ were investigated in [5]. These results were extended in [4] where various combinations of connectivity, minimum degree, degree properties and neighborhood conditions implying $P_{d,m}(G)$ were studied.

The *neighborhood* of a vertex v of a graph G is the collection of vertices adjacent to v, and will be denoted by $N_G(v)$. More generally, if $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{v \in S} N_G(v)$. For any set T of vertices of G the neighborhood of v in T, that is $N_G(v) \cap T$, will be denoted $N_T(v)$ and if $S \subseteq V(G)$, we define $N_T(S) = \bigcup_{v \in S} N_T(v)$. For a fixed positive integer t and a graph G, we write $\delta_t(G) \geq s$ if the cardinality of the union of the neighborhoods of each set of t vertices of G is at least s. Here we investigate conditions based on this generalized minimum degree that imply $P_{d,m}(G)$.

2. Results

Notation and standard definitions in this paper will generally follow that found in [2]. Any special notation will be described as needed. For vertices x and y of a graph G let $P_{d,m}(x,y)$ denote the property that there are m internally disjoint $x-y$ paths in G, each of length at most d. A collection of such paths is called a *Menger path system* for x and y. Our first result, a technical lemma, will be used in the proofs of Theorems 1, 2 and 3.

Lemma. *Let $d, m \geq 2$ and t be fixed positive integers and $k > 1$ a fixed real number. Furthermore, let G be an m-connected graph of order n with $\delta_t(G) \geq n/k$. If there* *exists a pair x, y of vertices of G such that G that does not satisfy $P_{d,m}(x,y)$, but $G+uv$ does satisfy $P_{d,m}(x,y)$ for each pair u, v of nonadjacent vertices, then for n sufficiently large, G satisfies $P_{\max\{d+1,\lfloor 2k\rfloor+t\},m}(x,y)$.*

Proof. Suppose that G contains a pair u, v of nonadjacent vertices such that $|N_G(u)\cap N_G(v)|>md$. By hypothesis, $G+uv$ contains m internally disjoint $x-y$ paths $P_1,P_2,\ldots,P_m$, each of length at most d, where, without loss of generality, we may assume $uv\in E(P_1)$. Since $|N_G(u)\cap N_G(v)|>md$, there is a vertex z of G such that $uz,zv\in E(G)$ and $z\notin V(P_i)$, $i=1,2,\ldots,m$. Clearly, then, G contains the desired system of $x-y$ paths. Thus, we may assume that for every pair u, v of nonadjacent vertices of G, $|N_G(u)\cap N_G(v)|\le md$.

Since $\delta_t(G)\ge n/k$, there are at most $t-1$ vertices of G of degree less than $n/(kt)$. Let $A=\{v\in V(G)\mid \deg_G v\ge n/(kt)\}$. Construct a sequence $v_1,v_2,\ldots,v_l$ of vertices of G as follows. Let v_1 be a vertex of A with minimum degree in G and let $A_1=N_A(v_1)\cup\{v_1\}$. Let v_2 be a vertex of $A-A_1$ with minimum degree in G and let $A_2=N_A(v_2)\cup\{v_2\}$. In general, let v_i be a vertex of $A-\bigcup_{j=1}^{i-1}A_j$ with minimum degree in G and let $A_i=N_A(v_i)\cup\{v_i\}$ and $A=\bigcup_{j=1}^{l}A_j$. Then for $i\ne j$, the vertices v_i and v_j are nonadjacent and, consequently, $|A_i\cap A_j|\le md$. Furthermore, since $\deg v\ge n/(kt)$ for each $v\in A$ and n is sufficiently large, we have that $l\le kt+1$.

For $i=1,2,\ldots,l$, let $B_i=A_i-\bigcup_{j\ne i}A_j$. Let $u,v\in B_i$ for some $1\le i\le l$ and suppose $uv\notin E(G)$. Since $uv\notin E(G)$, we have that $|N_G(u)\cap N_G(v)|\le md$. Thus one of u and v, say u, has at most $(|B_i|+md)/2$ adjacencies in B_i. Furthermore, u has at most lmd adjacencies in $A-B_i$. Thus for n sufficiently large,

$$\deg_G u\le(|B_i|+md)/2+lmd+t-1<\deg_G v_i,$$

contradicting the choice of v_i. Thus, $\langle B_i\rangle_G$ is complete for $i=1,2,\ldots,l$.

Suppose there is a vertex $w\in A-\bigcup_{i=1}^{l}B_i$. Since $\deg_G w\ge n/(kt)$, and n is large, for some $1\le i\le l$ we have $|N_G(w)\cap B_i|>md$. Consequently, w is adjacent to each vertex of B_i. Thus G contains disjoint sets $C_1,C_2,\ldots,C_l$ of vertices such that $|\bigcup_{i=1}^{l}C_i|\ge n-t+1$ and $\langle C_i\rangle_G$ is complete for $i=1,2,\ldots,l$.

If for some i, $|C_i|<n/k-(mdl)t$, then, since $\delta_t\ge n/k$, each vertex of C_i, with at most $t-1$ exceptions, will be adjacent to at least md vertices of some C_j. However, if a vertex is adjacent to md vertices of C_j, it is adjacent to all vertices of C_j. Thus, for some $j\ne i$, each of the vertices of C_j is adjacent to each of the vertices of C_i. Hence, for n sufficiently large, we can assume that $|C_i|\ge n/k-(mdl)t$ for $i=1,2,\ldots,l$, and $l\le k$.

Let $P_1,P_2,\ldots,P_m$ be a collection of m internally disjoint $x-y$ paths, the sum of whose lengths is minimum. Then each path P_i contains at most two vertices of C_i for $i=1,2,\ldots,l$ and $l\le k$. Thus each P_i has length at most $2\lfloor k\rfloor+t$. This completes the proof of the lemma. □

Our first main result gives a sufficient condition for $P_{d,m}(G)$ based on $\delta_t(G)$ in the case that $t\ge 5$ and $d\ge 5t^2$.

Theorem 1. *Let $t \geq 5$, $d \geq 5t^2$, and $m \geq 2$ be fixed integers. If G is an m-connected graph of order n with*

$$\delta_t(G) \geq \left(\frac{t}{t+1}\right)\left(\frac{5n}{d+2}\right) + (m-1)d + 3t^2,$$

then for n sufficiently large, $P_{d,m}(G)$ is satisfied.

Proof. Assume, to the contrary, that G is an m-connected graph with

$$\delta_t(G) \geq \left(\frac{t}{t+1}\right)\left(\frac{5n}{d+2}\right) + (m-1)d + 3t^2$$

that does not satisfy $P_{d,m}(G)$ but that $G+uv$ does satisfy $P_{d,m}(G)$ for each pair u, v of nonadjacent vertices of G. Since G does not satisfy $P_{d,m}(G)$ there are vertices x and y of G for which G does not satisfy $P_{d,m}(x,y)$. By the lemma, G contains a collection of m internally disjoint $x-y$ paths, each of length at most $\max\{d+1, \lfloor 2((t+1)/t)((d+2)/5)\rfloor + t\} \leq d+1$. Among all such collections let $P_1, P_2, \ldots, P_m$ be one, the sum of whose lengths is minimum.

Assume now that P_1 has length $d+1$, say P_1: $x=x_1, x_2, \ldots, x_{d+2}=y$ and let $M = V(G) - \bigcup_{j=2}^{m} V(P_j) \cup \{x,y\}$. Observe that if $v \in M$, then v can be adjacent to at most three vertices of P_1, and for any i, vx_i and vx_{i+3} are not simultaneously in $E(G)$. Let $d_M(u,v)$ denote the distance between u and v in the graph induced by the vertices in M. For $i=1,2,\ldots,d+2$ define N_i as follows:

$$N_1 = \{x_1\};$$

$$N_i = \{v \in M \mid d_M(x,v) = i-1\} \quad \text{for } 2 \leq i \leq d+1;$$

$$N_{d+2} = M - \bigcup_{i=1}^{d+1} N_i.$$

For each i, $x_i \in N_i$, and the N_i form a partition of M. Note that if $S \subseteq N_i$, then $N_G(S) \subseteq \bigcup_{j=2}^{m} V(P_j) \cup N_{i-1} \cup N_i \cup N_{i+1}$. Let $\bar{N}(S)$ denote $N_M(S) \subseteq N_{i-1} \cup N_i \cup N_{i+1}$. So if $|S| \geq t$, then $|\bar{N}(S)| \geq (t/(t+1))(5n/(d+2)) + 3t^2 > (t/(t+1))(5n/(d+2))$. Define a *block* of G to be a sequence $N_i, N_{i+1}, \ldots, N_j$ that satisfies the following conditions:

(1) no two consecutive terms N_l and N_{l+1}, with $i \leq l \leq j-1$, have $|N_l| < t$ and $|N_{l+1}| < t$;

(2) if $i \neq 1$, then $|N_{i-1}| < t$ and $|N_i| < t$; and

(3) if $j \neq d+2$, then $|N_j| < t$ and $|N_{j+1}| < t$.

The *length* of the block $N_i, N_{i+1}, \ldots, N_j$ is defined to be $j-i+1$; the *left endpoint* is x_i and the *right endpoint* is x_j.

Observe first that G contains at most $t-1$ blocks of length 1; otherwise, suppose $N_{i_1}, N_{i_2}, \ldots, N_{i_t}$ were t blocks of length 1. Then if $S = \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$, we have $|N_G(S)| < 3t^2 + (m-1)d$, contradicting $\delta_t(G) \geq (t/(t+1))(5n/(d+2)) + (m-1)d + 3t^2$. Assume, then, that G has $t-l$ blocks of length 1 where $l \geq 1$.

We first show that since $\delta_t(G) \geq (t/(t+1))(5n/(d+2)) + (m-1)d + 3t^2$, then we also have that $\delta_t(G) \geq (l/(l+1))(5n/(d+2-t+l)) + (m-1)d + 3t^2$. Since $l \leq t$, it suffices to show that for $1 \leq l \leq t-1$ we have

$$\frac{l+1}{(l+2)(d+2-t+l+1)} \geq \frac{l}{(l+1)(d+2-t+l)}.$$

However, this inequality is equivalent to the inequality

$$d \geq l^2 + l + t - 2,$$

which is certainly true since $l \leq t$.

First consider the case $l \geq 5$. We show that every block $N_i, N_{i+1}, \ldots, N_j$ of G of length s, where $s \not\equiv 1 \bmod 4$, has $|N_i \cup N_{i+1} \cup \cdots \cup N_j| > sn/(d+2+l-t)$.

If N_i, N_{i+1} is a block of length 2, then necessarily $i+1 = d+2$ and $|N_{d+2}| \geq t$. Let $S \subseteq N_{d+2}$, where $|S| = t$. Then $\bar{N}(S) \subseteq N_{d+1} \cup N_{d+2}$ and so

$$|N_{d+1} \cup N_{d+2}| \geq |\bar{N}(S)| > \left(\frac{l}{l+1}\right)\left(\frac{5n}{d+2+l-t}\right) > \frac{2n}{d+2+l-t},$$

since $l \geq 5$.

If N_i, N_{i+1}, N_{i+2} is a block of length 3, then $|N_{i+1}| \geq t$ and so, as above, we have

$$|N_i \cup N_{i+1} \cup N_{i+2}| > \left(\frac{l}{l+1}\right)\left(\frac{5n}{d+2+l-t}\right) > \frac{3n}{d+2+l-t}.$$

If $N_i, N_{i+1}, N_{i+2}, N_{i+3}$ is a block of length 4, then $|N_{i+1}| \geq t$ and $|N_{i+2}| \geq t$ and again we have

$$|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3}| > \frac{4n}{d+2+l-t}.$$

If $N_i, N_{i+1}, \ldots, N_j$ is a block of length $s > 4$ with $s \equiv 0 \bmod 4$, then we obtain the desired result by considering consecutive groups of four N_p. By allowing a group of three N_p at the beginning and/or end of a block we obtain the result for $s \equiv 2 \bmod 4$ and $s \equiv 3 \bmod 4$.

If G contains no blocks of length $s \equiv 1 \bmod 4$ with $s \geq 5$, then

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > (d+2-(t-l))\left(\frac{n}{d+2+l-t}\right) = n,$$

and we arrive at a contradiction.

Thus we may assume that G contains blocks of length $s \equiv 1 \bmod 4$ with $s \geq 5$. Let $N_i \cup N_{i+1} \cup \cdots \cup N_j$ be such a block of length s. By considering only $N_i, \ldots, N_{j-2}$ and consecutive groups of four N_p and a final group of three, we can choose t-sets appropriately within these N_p to show that

$$|N_i \cup N_{i+1} \cup \cdots \cup N_{j-2}| > \frac{sn}{d+2+l-t} - \left(\frac{1}{l+1}\right)\left(\frac{5n}{d+2+l-t}\right).$$

Suppose first that G contains more than l blocks of length $s\equiv 1 \bmod 4$ with $s\geq 5$. Select l such blocks with right endpoints $x_{j_1}, x_{j_2}, \ldots, x_{j_l}$. Then these l vertices, together with $t-l$ vertices selected one from each of the blocks of length 1, form a set S that must satisfy $|\bar{N}(S)|\geq (l/(l+1))(5n/(d+2+l-t))+3t^2$. Note that there are at most $3t^2$ vertices in $\bar{N}(S)$ that are not in one of the l blocks of length $s\equiv 1 \bmod 4$ with $s\geq 5$. Thus at least one block $N_i \cup N_{i+1} \cup \cdots \cup N_j$ has $|N_{j-1}\cup N_j|\geq (1/(l+1))(5n/(d+2+l-t))$, and so

$$|N_i \cup N_{i+1} \cup \cdots \cup N_j| \geq \frac{sn}{d+2+l-t}.$$

Continue in this fashion to obtain blocks $N_i, N_{i+1}, \ldots, N_j$ of length $s\equiv 1 \bmod 4$ with $s\geq 5$ and such that $|N_i \cup N_{i+1} \cup \cdots \cup N_j| \geq sn/(d+2-t+l)$ until exactly l blocks of length $s\equiv 1 \bmod 4$ with $s\geq 5$ remain. Then the same argument shows that if the sum of the length of these blocks is s^*, then their union contains at least $s^*n/(d+2-t+l)$ vertices. Thus, again we have that

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| \geq (d+2-(t-l))\left(\frac{n}{d+2-t+l}\right) = n,$$

which is a contradiction.

Thus we may assume that if $l\geq 5$ and G has $t-l$ blocks of length 1, then G has fewer than l blocks of length $s\equiv 1 \bmod 4$ with $s\geq 5$. We wish to show that in this case, too, we have

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| \geq (d+2-(t-l))\left(\frac{n}{d+2-t+l}\right) = n.$$

Certainly we have that

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > n - \left(\frac{l}{l+1}\right)\left(\frac{5n}{d+2-t+l}\right)$$

since G has fewer than l blocks of length $s\equiv 1 \bmod 4$ with $s\geq 5$.

For each block $N_i, N_{i+1}, N_{i+2}, N_{i+3}$ of length 4, we observe that

$$\begin{aligned}|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3}| &\geq \left(\frac{l}{l+1}\right)\left(\frac{5n}{d+2+l-t}\right)\\ &= \frac{4n}{d+2+l-t} + \left(\frac{l-4}{l+1}\right)\left(\frac{n}{d+2-t+l}\right).\end{aligned}$$

Since $l\geq 5$, it follows that

$$|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3}| \geq \frac{4n}{d+2+l-t} + \left(\frac{1}{l+1}\right)\left(\frac{n}{d+2-t+l}\right).$$

Certainly, for a block of length $s=2$ or $s=3$ we also have that the union of the sets in the blocks contains at least

$$\frac{sn}{d+2+l-t}+\left(\frac{1}{l+1}\right)\left(\frac{n}{d+2+l-t}\right)$$

vertices. Similarly, if $N_i, N_{i+1}, \ldots, N_j$ is a block of length $s=6$, 7 or 8, then by looking at two groups of three or four consecutive N_p we see that $|N_i \cup N_{i+1} \cup \cdots \cup N_j|$ is at least

$$\frac{sn}{d+2+l-t}+2\left(\frac{1}{l+1}\right)\left(\frac{n}{d+2+l-t}\right)$$

and, in general, if $N_i, N_{i+1}, \ldots, N_j$ is a block of length $s \equiv 0$, 2 or 3 mod 4, then

$$|N_i \cup N_{i+1} \cup \cdots \cup N_j| > \frac{sn}{d+2+l-t}+\left(\left\lceil \frac{s}{4} \right\rceil\right)\left(\frac{1}{l+1}\right)\left(\frac{n}{d+2+l-t}\right).$$

Furthermore, if $N_i, N_{i+1}, \ldots, N_j$ is a block of length $s \equiv 1$ mod 4 with $s \geq 5$, then

$$|N_i \cup N_{i+1} \cup \cdots \cup N_j| \geq \frac{sn}{d+2+l-t}-\left(\frac{1}{l+1}\right)\left(\frac{5n}{d+2+l-t}\right) +\left(\left\lfloor \frac{s}{4} \right\rfloor -1\right)\left(\frac{1}{l+1}\right)\left(\frac{n}{d+2+l-t}\right).$$

This, however, implies that

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > n,$$

since $t \geq 5$ and $d \geq 5t^2$.

Thus, it must be the case that G has $t-l$ blocks of length 1, where now $1 \leq l \leq 4$. We wish to show that in this case, too, we have

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > (d+2-(t-l))\left(\frac{n}{d+2-t+l}\right)=n.$$

For each block $N_i, N_{i+1}, N_{i+2}, N_{i+3}$ of length 4, we observe by hypothesis that

$$|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3}| > \left(\frac{t}{t+1}\right)\left(\frac{5n}{d+2}\right).$$

However, $5t/((t+1)(d+2)) > 4/(d+2+l-t)$, since $t \geq 5$ and $d \geq 5t^2$. Thus $|N_i \cup N_{i+1} \cup \cdots \cup N_j| \geq 4n/(d+2+l-t)+\varepsilon n$ for some $\varepsilon > 0$. Similarly, for a block of length $s=2$ or $s=3$ we also have that the union of the sets in the block contains at least $sn/(d+2+l-t)+\varepsilon n$. Similarly, if $N_i, N_{i+1}, \ldots, N_j$ is a block of length $s=6$, 7 or 8, then by looking at two groups of three or four consecutive N_p we see that $|N_i \cup N_{i+1} \cup \cdots \cup N_j|$ is at least $sn/(d+2+l-t)+2\varepsilon n$ and, in general, if N_i, $N_{i+1}, \ldots, N_j$ is a block of length $s \equiv 0$, 2 or 3 mod 4, then

$$|N_i \cup N_{i+1} \cup \cdots \cup N_j| > \frac{sn}{d+2+l-t}+\left\lceil \frac{s}{4} \right\rceil \varepsilon n.$$

Furthermore, suppose among its blocks, G contains two blocks $N_{i_1}, N_{i_1+1}, \ldots, N_{j_1}$ and $N_{i_2}, N_{i_2+1}, \ldots, N_{j_2}$ of length 5. We claim that if $j_2 = d+2$ and $|N_{d+2}| \geq t$, then the corresponding block of length 5 has at least $5n/(d+2+l-t)+\varepsilon n$ vertices. To see this let S be the t-set consisting of $x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}$ and $t-4 \leq t-l$ vertices chosen one per block of length 1. Since this set S has at most $(m-1)d$ adjacencies on the other paths P_i for $i \neq 1$, $|\bar{N}(S)| \geq (t/(t+1))(5n/(d+2))+3t^2$. Note also that there are at most $3t^2$ vertices in $\bar{N}(S)$ that are not in one of the two blocks of length 5. It follows that one of $N_{i_1+1}, N_{j_1-1}, N_{i_2+1}, N_{j_2-1}$ contains at least $(1/4)(t/(t+1))(5n/(d+2))$ vertices so that one of the two blocks of length 5 contains at least

$$\left(\frac{t}{t+1}\right)\left(\frac{5n}{d+2}\right)+\left(\frac{1}{4}\right)\left(\frac{t}{t+1}\right)\left(\frac{5n}{d+2}\right)$$

vertices, that is, at least

$$\frac{25tn}{4(t+1)(d+2)}$$

vertices. But since $5t/(4(t+1)(d+2)) > 1/(d+2+l-t)$ for $d \geq 5t^2$, we have that one of the blocks of length 5 contains at least $5n/(d+2+l-t)+\varepsilon n$ vertices.

More generally, all but at most one block of length $s \equiv 1 \bmod 4$ with $s \geq 5$ contain at least

$$\frac{sn}{d+2+l-t}+\left\lfloor\frac{s}{4}\right\rfloor \varepsilon n$$

vertices and the remaining block, if it exists, contains at least

$$\frac{(s-5)n}{d+2+l-t}+\left\lfloor\frac{s-5}{4}\right\rfloor \varepsilon n+\left(\frac{t}{t+1}\right)\left(\frac{5n}{d+2}\right)$$

vertices. This, however, implies (by adding the cardinalities of the sets in the blocks) that

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > n,$$

a contradiction, and the proof of the theorem is complete. □

"Wheel type" graphs give important information on the extremal properties related to $P_{d,m}(G)$. We start with the wheel graph $W_b = K_1 + C_b$ that has b spokes and b vertices on the rim. Replace each vertex of W_b with some complete graph, and make each vertex of the corresponding complete graph adjacent to the vertices in the neighborhood of the replaced vertex. The graphs obtained by this expansion of vertices of a wheel form a family of "generalized wheels". More precisely, order the vertices of W_b starting with the center and followed by the vertices on the rim in a natural order around the cycle. For positive integers $p(i)$ $(0 \leq i \leq b)$, the generalized wheel obtained from W_b by replacing the ith vertex with a complete graph $K_{p(i)}$ will be denoted by $W(p(0), p(1), \ldots, p(b))$.

In many of the cases of interest to us, most of the $p(i)$ in the generalized wheel will be the same or will follow some pattern, so we will adopt the more compact notation of representing the sequence $(p(j),\ldots,p(k))$ by $(k-j+1;p)$ when $p=p(j)=\cdots=p(k)$. Thus, $W(1,r;1)=W_r$ and $W(m-2,n-m+2;1)=K_{m-2}+C_{n-m+2}$. Also, if the pattern $(p(1),p(2),\ldots,p(r))$ is repeated s times, we will represent this by $(s;(p(1),p(2),\ldots,p(r)))$. Hence the generalized wheel $W(m,s;(1,p,p,1))$ has m vertices in the center and along the rim there is an alternating pattern of two single vertices followed by two complete graphs with p vertices.

With this notation we now describe an example that illustrates that the bound on δ_t in Theorem 1 has the correct order of magnitude.

Let n, m, d, and t be positive integers such that $5n>5m+2d-6$ and such that $(d+2)(t+1)$ divides $5n-5m-2d+6$. Let $r=(5n-5m-2d+6)/((d+2)(t+1))$. The generalized wheel graph G defined as

$$G=W(m-2,(d+2)/5;(1,r,(t-1)r,r,1))$$

(see Fig. 1) has n vertices, is m-connected and has

$$\delta_t(G)=\left(\frac{t}{t+1}\right)\left(\frac{5n-5m+6-2d}{d+2}\right)+m-1.$$

However, $P_{d,m}(G)$ is not satisfied.

Our next result involves values of $t=2,3,4$.

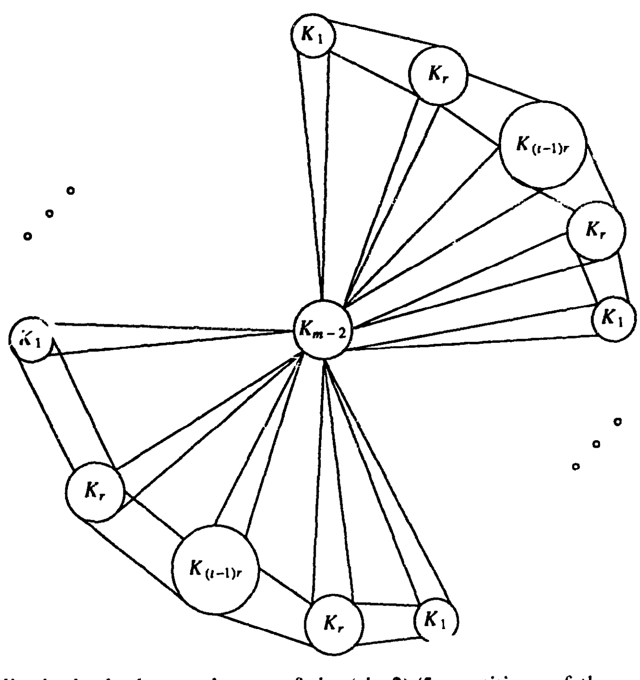

Fig. 1. Generalized wheel where only two of the $(d+2)/5$ repetitions of the pattern are shown.

Theorem 2. *Let $t, d, m \geq 2$ be fixed integers with $2 \leq t \leq 4$, and $d > t$. If G is an m-connected graph of order n with $\delta_t(G) \geq 4n/(d+4-t)+(m-1)d+3t^2$, then for n sufficiently large, $P_{d,m}(G)$ is satisfied.*

Proof. Assume, to the contrary, that G is an m-connected graph of order n with $\delta_t(G) \geq 4n/(d+4-t)+(m-1)d+3t^2$ that does not satisfy $P_{d,m}(G)$ but that $G+uv$ does satisfy $P_{d,m}(G+uv)$ for each pair u, v of nonadjacent vertices of G. Since G does not satisfy $P_{d,m}(G)$, there are vertices x and y of G for which G does not satisfy $P_{d,m}(x,y)$. By the lemma, G contains a collection of m internally disjoint $x-y$ paths, each of length at most $\max\{d+1, \lfloor (d+4-t)/2 \rfloor + t\} = d+1$. Among all such collections, let $P_1, P_2, \ldots, P_m$ be one, the sum of whose lengths is minimum.

Assume, without loss of generality, that P_1 has length $d+1$, say P_1: $x = x_1, x_2, \ldots, x_{d+2} = y$. As in the proof of the previous theorem, define the blocks of G. We observe that every block of length $s \not\equiv 1 \bmod 4$ contains at least $sn/(d+4-t)$ vertices and every block of length $s \equiv 1 \bmod 4$ contains at least $(s-1)n/(d+4-t)$ vertices. Thus, if we show for all but at most $t-2$ blocks of length $s \equiv 1 \bmod 4$ that there are at least $n/(d+4-t)$ additional vertices per block not yet counted in $N_1 \cup N_2 \cup \cdots \cup N_{d+2}$, we would arrive at the contradiction

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| \geq ((d+2)-(t-2))\left(\frac{n}{d+4-t}\right) = n.$$

Clearly, the number of blocks of length 1 is at most $t-1$; otherwise, G contains a set S of t vertices with

$$|N_G(S)| \leq 3t^2 + (m-1)d.$$

Furthermore, if the number of blocks of length 1 in G is denoted by k, then at most $\lfloor (t-k-1)/2 \rfloor$ blocks of length $s \equiv 1 \bmod 4$, $s \geq 5$, contain fewer than $sn/(d+4-t)$ vertices. For otherwise, suppose

$$\begin{array}{l} N_{i_1}, N_{i_1+1}, \ldots, N_{j_1}, \\ N_{i_2}, N_{i_2+1}, \ldots, N_{j_2}, \\ \vdots \\ N_{i_p}, N_{i_p+1}, \ldots, N_{j_p} \end{array}$$

are such blocks of lengths $s_1, s_2, \ldots, s_p$ with $p = \lfloor (t-k-1)/2 \rfloor + 1$. Let $N_{i_{q+1}}, N_{i_{q+2}}, \ldots, N_{i_{q+k}}$ be the blocks of G of length 1. Then $x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_p}, y_{j_p}$ together with $t-2p$ vertices from $x_{i_{q+1}}, x_{i_{q+2}}, \ldots, x_{i_{q+k}}$ form a set S of t vertices. Thus $|\bar{N}(S)| \geq 4n/(d+4-t)+3t^2$ and, consequently we can assume, without loss of generality, that $|N_{j_1-1}| \geq (1/t)(4n/(d+4-t)) \geq n/(d+4-t)$. Since $|N_{i_1} \cup N_{i_2} \cup \cdots \cup N_{j_1-2}| \geq ((s_1-1)n)/(d+4-t)$, we have that

$$|N_{i_1} \cup N_{i_2} \cup \cdots \cup N_{j_1}| \geq \frac{s_1 n}{d+4-t}.$$

Finally, if $t-k\leq 2$, then in fact every block of length $s\geq 2$ contains at least $((s+1)n)/(d+4-t)$ vertices since the t-set S consisting of at most one vertex from each of the blocks of length 1 plus the endpoints of the block satisfies

$$|\bar{N}(S)|\geq \frac{4n}{d+4-t}+3t^2.$$

A case by case analysis of $2\leq t\leq 4$ and $0\leq k\leq t-1$ yields the desired contradiction

$$|N_1\cup N_2\cup\cdots\cup N_{d+2}|\geq n.$$

This completes the proof of the theorem. □

Let n, m, d and t be positive integers such that $2n>2m-4+t+d$ and $d+4-t$ divides $2n-2m+4-t-d$. Let G be the generalized wheel defined by

$$G=W(m-2,t-2;1,(d+4-t)/4;(1,r,r,1)),$$

where $r=(2n-2m-t-d+4)/(d+4-t)$. Then G is an m-connected graph of order n. Furthermore, $\delta_t(G)=(4n-4m-2t-2d+8)/(d+4-t)+m-2+t$; however, $P_{d,m}(G)$ is not satisfied. Thus, the result of Theorem 2 is of the correct order of magnitude.

Our final result is concerned with values of t for which $t\geq 5$, but $d<5t^2$. Here, however, no corresponding examples are presently known.

Theorem 3. *Let $t\geq 5$, $d>t$, and $m\geq 2$ be fixed integers. If G is an m-connected graph of order n with $\delta_t(G)\geq (t/(t+1))(5n/(d+4-t))+(m-1)d+3t^2$, then for n sufficiently large, $P_{d,m}(G)$ is satisfied.*

Proof. Assume, to the contrary, that G is an m-connected graph with $\delta_t(G)\geq (t/(t+1))(5n/(d+4-t))+(m-1)d+3t^2$, that does not satisfy $P_{d,m}(G)$ but that $G+uv$ does satisfy $P_{d,m}(G+uv)$ for each pair u, v of nonadjacent vertices of G. Since G does not satisfy $P_{d,m}(G)$, there are vertices x and y of G for which G does not satisfy $P_{d,m}(x,y)$. By the lemma, G contains a collection of m internally disjoint $x-y$ paths, each of length at most $\max\{d+1,\lfloor 2((t+1)/t)((d+4-t)/5)\rfloor+t\}\leq d+1$. Among all such collections, let $P_1,P_2,\ldots,P_m$ be one, the sum of whose lengths is minimum. Assume, without loss of generality, that P_1 has length $d+1$, say P_1: $x=x_1,x_2,\ldots,x_{d+2}=y$. As in the proofs of the previous two theorems, define the blocks of G. As before, every block of length $s\not\equiv 1 \bmod 4$ contains at least $sn/(d+4-t)$ vertices, and every block of length $s\equiv 1 \bmod 4$ contains at least $(s-1)n/(d+4-t)$ vertices. In fact, each block of length $s\equiv 1 \bmod 4$, $s\geq 5$ contains at least $sn/(d+4-t)-(1/(t+1))(5n/(d+4-t))$ vertices. Furthermore, G has at most $t-1$ blocks of length 1.

We first observe that if the number of blocks of length 1 in G is denoted by k,

$0 \le k \le t-2$, then at most $\lfloor (t-k-1)/2 \rfloor$ blocks of length $s \equiv 1 \bmod 4$, $s \ge 5$, contain fewer than $sn/(d+4-t)$ vertices. For otherwise, suppose

$$\begin{aligned} &N_{i_1}, N_{i_1+1}, \ldots, N_{j_1},\\ &N_{i_2}, N_{i_2+1}, \ldots, N_{j_2},\\ &\vdots\\ &N_{i_p}, N_{i_p+1}, \ldots, N_{j_p} \end{aligned}$$

are such blocks of lengths $s_1, s_2, \ldots, s_p$ with $p = \lfloor (t-k-1)/2 \rfloor + 1$. Let $N_{i_{q+1}}$, $N_{i_{q+2}}, \ldots, N_{i_{q+k}}$ be the blocks of G of length 1. Then $x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_p}, y_{j_p}$ together with $t-2p$ vertices of $x_{i_{q+1}}, x_{i_{q+2}}, \ldots, x_{i_{q+k}}$ form a set S of t vertices. Thus $|\bar{N}(S)| \ge (t/(t+1))(5n/(d+4-t)) + 3t^2$ and, consequently we can assume that $|N_{j_1-1}| \ge (1/(t+1))(5n/(d+4-t))$. Since

$$|N_{i_1} \cup N_{i_2} \cup \cdots \cup N_{j_1-2}| \ge \frac{s_1 n}{d+4-t} - \left(\frac{1}{t+1}\right)\left(\frac{5n}{d+4-t}\right),$$

we have that

$$|N_{i_1} \cup N_{i_2} \cup \cdots \cup N_{j_1}| \ge \frac{s_1 n}{d+4-t}.$$

Thus, the total number of blocks of G of length s with fewer than $sn/(d+4-t)$ vertices is at most $t-2$ and so

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > ((d+2)-(t-2))\left(\frac{n}{d+4-t}\right) = n,$$

and we obtain a contradiction.

Thus we may assume that G has exactly $t-1$ blocks of length 1. Then, however, every other block of G of length $s > 1$ contains at least $(s+1)n/(d+4-t)$ vertices and again we have

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| \ge n.$$

But this is a contradiction and hence the proof of the theorem is complete. □

Corollary 4. *Let $t, d, m \ge 2$ be fixed integers and $5 \le t < d$. If G is an m-connected graph of order n with $\delta_t(G) \ge (t/(t+1))(5n/(d+2-\lfloor (t-1)/2 \rfloor)) + (m-1)d + 3t^2$ and $\delta(G) \ge (m-1)d + 3t$, then for n sufficiently large, G satisfies $P_{d,m}(G)$.*

Proof. If G were a counterexample, then as in the proof of Theorem 3, we would have that if G has k blocks of length 1, then at most $\lfloor (t-k-1)/2 \rfloor$ blocks of length $s \equiv 1 \bmod 4$, $s \ge 5$, contain fewer than $sn/(d+4-t)$ vertices. The minimum degree condition, however, implies that $k=0$ and the result follows. □

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