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Neighborhood unions and a generalization of Dirac's theorem

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Abstract

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Dirac proved that if each vertex of a graph G of order $n \geq 3$ has degree at least $n/2$, then the graph is Hamiltonian. This result will be generalized by showing that if the union of the neighborhoods of each pair of vertices of a 2-connected graph G of sufficiently large order n has at least $n/2$ vertices, then G is Hamiltonian. Other results that are based on neighborhood unions of pairs of vertices will be proved that give the existence of cycles, paths and matchings. Also, Hamiltonian results will be considered that use the union of neighborhoods of more than 2 vertices.

1. Introduction

Dirac [2] proved that if the neighborhood of each vertex of a graph of order n has at least $n/2$ vertices then the graph is Hamiltonian. Numerous generalizations have followed this fundamental result of Dirac. Conditions on the sum of degrees of non-adjacent vertices such as Ore's [12], conditions using connectivity and minimal degree [9], and degree conditions on all pairs of vertices a distance

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two apart [3] are examples of such extensions. We will generalize the Dirac result by showing that if the union of the neighborhoods of each pair of vertices in a graph of order n has at least $n/2$ vertices, then the graph is Hamiltonian. Other properties of graphs that follow from similar neighborhood union conditions will be investigated. Recently, there have been several papers (see [4–8, 11]) that use neighborhood conditions on independent sets of vertices to insure the existence of special subgraphs such as paths and cycles. A survey of neighborhood unions and graphical properties can be found in [10].

We start by defining the neighborhood conditions we will use. Given an integer $t > 0$, a graph G satisfies property $\mathrm{NC}_t(G) \geq s$, if for each set X of t vertices of G,

$$\left| \bigcup_{x \in X} N(x) \right| \geq s,$$

where $N(x)$ is the set of vertices of G adjacent to x, (neighborhood of x). Therefore, $\mathrm{NC}_1(G) \geq s$ just means that each vertex has degree at least s. We will deal mainly with the case $t = 2$, and so $\mathrm{NC}_2(G)$ will usually be written as just $\mathrm{NC}(G)$.

In Section 2, we will investigate the cycles, paths, and matchings in a graph G that are implied by the neighborhood condition $\mathrm{NC}(G)$. In particular, we will prove the following result, where P_m and C_m denote paths and cycles, respectively, with m vertices, and mK_2 denotes a matching with m edges.

Theorem A. *Let G be a 2-connected graph of order n with $\mathrm{NC}(G) \geq s$ for some $3 \leq s \leq n/2$. Then, G contains a P_{2s}, sK_2, and a C_t for $t \geq 2s - 2$.*

Similar results for connected graphs or graphs with no connectivity conditions will be proved.

In Section 3 the neighborhood condition $\mathrm{NC}_t(G)$ needed to insure that the graph G has a Hamiltonian cycle will be studied. The following two theorems will be proved.

Theorem B. *Let $t \geq 2$ be an integer, and let G be a graph of order n with $\delta(G) \geq t$ that satisfies $\mathrm{NC}_t(G) \geq n/2 + c$ for some $c = c(t)$. Then, for n sufficiently large, G is Hamiltonian.*

When $t = 2$, a sharp result can be proved.

Theorem C. *Let G be a 2-connected graph of order n that satisfies $\mathrm{NC}(G) \geq n/2$. Then, for n sufficiently large, G is Hamiltonian.*

2. Cycles, paths, and matchings

Before proving the results of this section, we will introduce some frequently used notation. Notation not specifically mentioned will follow that in [1]. We denote the edge between vertices u and v as uv, and for simplicity denote its existence as $uv \in G$. A path P with m vertices $\{x_1, x_2, \ldots, x_m\}$ will be expressed as $(x_1, x_2, \ldots, x_m)$, and the cycle C with the same vertices (and same order) will be written as $(x_1, x_2, \ldots, x_m, x_1)$. In the path P an edge x_1x_i (x_mx_j) is called an *endchord* of the path, and the endchords x_1x_i and x_mx_j are said to overlap if $j < i$. For a vertex x_i on the path, the predecessor x_{i-1} and the successor x_{i+1} along the path will be denoted by x_i^- and x_i^+ respectively. If S is a collection of vertices of $P - x_1$, then S^- will be the set of predecessors of the vertices of S. Also, the successor set S^+ of S is defined in an analogous way.

Theorem 1. *Let G be a graph of order n with* $\mathrm{NC}(G) \geq s$ *for some* $3 \leq s \leq n$. *Then, G contains a P_s, $\lfloor s/2 \rfloor K_2$, and a C_t for some $t \geq s$. The result is sharp in the sense that longer paths, or cycles are not implied by the conditions.*

Proof. A graph that is the disjoint union of complete graphs K_s verifies that the conclusion of Theorem 1 cannot be strengthened for paths or cycles.

The existence of a cycle C_t for $t \geq s$ implies that $\lfloor s/2 \rfloor K_2 \subset G$, and $P_s \subset G$. Thus, it is sufficient to prove that G contains a cycle of length at least s. Let $P = (x_1, x_2, \ldots, x_m)$ be a maximal length path in G. Then, all of the adjacencies of x_1 and x_r are on P. Both of the endvertices of P cannot have degree 1, so we assume, with no loss of generality, that $x_1x_k \in G$ for $k > 2$. Note that x_{k-1} is also an endvertex of a maximum length path, and all of its adjacencies are also on P. Since $|N(x_1) \cup N(x_{k-1})| \geq s$, we can assume (without loss of generality) that $x_1x_j \in G$ for some $j \geq s$. This gives the desired cycle, and completes the proof of Theorem 1. □

If more is known about the connectivity of G, then more can be said about the paths, cycles, and matchings implied by the neighborhood condition $\mathrm{NC}(G) \geq s$. The following theorem gives this.

Theorem 2. *Let G be a 2-connected graph of order n with* $\mathrm{NC}(G) \geq s$ *for some* $3 \leq s \leq n/2$. *Then, G contains a C_d for some $d \geq 2s - 2$. For $s \leq (n+4)/3$, the result is sharp in the sense that longer cycles are not implied by the conditions.*

Proof. For $3 \leq s \leq (n+4)/3$, consider the graph $H = K_2 + t_1K_{s-2}$, with $n = t_1(s-2) + 2$ and $t_1 \geq 3$. The graph H is 2-connected, satisfies $\mathrm{NC}(H) = s$, and contains a C_{2s-2}, but no longer cycle. This verifies that Theorem 1 is sharp.

Let $P = (x_1, x_2, \ldots, x_m)$ be a path of maximum length in G. The maximality of m implies that no endvertex of a path of length m is adjacent to a vertex not on the path. Let x_1x_r be the shortest endchord from x_1 and let x_mx_t be the shortest endchord from x_m.

The proof of the existence of a large cycle will be broken into several cases that depend on the nature of endchords of maximal length paths.

Case 1: *A path in which all endchords overlap.*

By assumption $t<r$, and we can assume that $r-t$ is minimal over all possible choices of maximum length paths and shortest endchords. Note that there is a maximal length path from x_{r+1} to x_{t-1} that uses these shortest chords. The minimality of $r-t$ implies that neither x_{r+1} nor x_{t-1} is adjacent to a vertex x_k with $t<k<r$. Consider the cycle

$$C=(x_1, x_r, x_{r+1}, \ldots, x_m, x_t, x_{t-1}, \ldots, x_1),$$

which contains all of the vertices of P except for those between x_t and x_r. We will show that C contains at least $2s$ vertices.

Let x_1x_l be the longest endchord from x_1 (note that $l\neq m$), and let $x=x_1$, $x'=x_{l+1}$, $y=x_m$, and $y'=x_{t-1}$. Let $N=N(y)\cup N(y')$. It is easily seen that each vertex of N is contained in C, since y' can have no adjacencies between x_t and x_r. (For otherwise, the minimality of $r-t$ would be contradicted.) Let

$$M=\{x_{k-1}: k\neq r, \text{ and } xx_k \text{ or } x'x_k\in G\}\cup\{y\}.$$

The set M has the same number of elements as $N(x)\cup N(x')$, and M is contained in C. It is straightforward to verify that if $M\cap N\neq\emptyset$, then there is a cycle containing all of the vertices of P. For example, if $x_j\in M\cap N$, with $xx_{j+1}\in G$, then (by assumption $y'x_j\in G$)

$$(x, x_{j+1}, x_{j+2}, \ldots, y, x_t, x_{t+1}, \ldots, x_j, y', \ldots, x_2, x)$$

is such a cycle. Similar cycles exist for the other cases of vertices in $M\cap N$. The maximality of P implies that any cycle of length m must be a Hamiltonian cycle in G, which would complete the proof. Hence, we assume $M\cap N=\emptyset$. Thus, C contains two disjoint sets each with at least s vertices, which completes the proof of this case.

Case 2: *A path with both overlapping and nonoverlapping chords.*

In this case we can assume that $r\leq t$, and that some endchord from x_1 overlaps some endchord from x_m. Let $x=x_1$, $x'=x_{r-1}$, $y=x_m$, and $y'=x_{t+1}$. Note that x' and y' are endvertices of a maximum length path of G. We can assume, with no loss of generality, that there exists an l with $l<r$ such that $x_mx_l\in G$, and no vertex between x_l and x_r on P is adjacent to x, y or y'. Also, x' has only one adjacency between x_l and x_r, namely x_{r-2}. Both of these observations follow from the minimality of r and the maximality of t.

Consider the cycle

$$C=(x_1, x_r, x_{r+1}, \ldots, x_m, x_l, x_{l-1}, \ldots, x_1),$$

which contains all of the vertices of P except for those between x_l and x_r. We will show that C contains at least $2s-2$ vertices in the same way as it was done in Case 1. Let $N=N(y)\cup N(y')$. It is easily seen that each vertex of N is contained in C. Let

$$M=\{x_{k-1}: k\neq r-1, k\neq 1 \text{ and } xx_k \text{ or } x'x_k\in G\}.$$

The set M has as many as $|N(x) \cup N(x')| - 2$ elements, and M is contained in C. It is straightforward to verify that if $M \cap N \neq \emptyset$, then there is a cycle containing all of the vertices of P. The maximality of P implies this would be a Hamiltonian cycle, which would complete the proof. Thus, $M \cap N = \emptyset$, and the cycle C contains two disjoint sets M and N with at least $2s - 2$ vertices. This completes the proof of this case.

Case 3: *None of the endchords from* $\{x_1, x_{r-1}\}$ *overlap endchords from* $\{x_m, x_{t+1}\}$.

Both x_{r-1} and x_{t+1} are also endvertices of a path of length m, and we are assuming that none of the adjacencies of $\{x_m, x_{t+1}\}$ preceed, along the path P, any of the adjacencies of $\{x_1, x_{r-1}\}$. With no loss of generality, we can assume that $x_1x_l \in G$ and $x_mx_u \in G$ with $l \leqslant u$ and that for $l < j < u$, x_j is not adjacent to x_1, x_{r-1}, x_m, or x_{t+1}. Let $A = \{x_1, x_2, \ldots, x_l\}$ and $B = \{x_u, x_{u+1}, \ldots, x_m\}$.

Since G is 2-connected, there are at least two vertex disjoint paths from A to B, and there is the path $(x_l, \ldots, x_u)$. From these three paths we can generate two vertex disjoint paths Q and Q' from A to B (only endvertices are in A or B) such that the endvertices in A of the two paths are x_i, x_l and the endvertices in B are x_u, x_j for $i < l$ and $u < j$. It can easily be shown that in A there is a path R from x_i to x_l that contains all of the vertices of $N(x_1) \cup N(x_{r-1})$. For example, if $i < k$, $x_1x_k \in G$, and there is no vertex of $N(x_1) \cup N(x_{r-1})$ between x_i and x_k, then

$$(x_i, x_{i-1}, \ldots, x_1, x_k, x_{k+1}, \ldots, x_l)$$

is such a path. Also, there is a corresponding path R' in B from x_u to x_j that contains $N(x_{t+1}) \cup N(x_m)$. Each of the paths R and R' have at least s vertices, and a cycle C can be constucted using the paths R, R', Q, and Q'. If $l < u$, then clearly G has a cycle with at least $2s$ vertices. If $l = u$, then C has at least $2s - 1$ vertices. This completes the proof of this case and Theorem 2. □

Note that in the proof of Case 3 of Theorem 2 when $l = u$ and $s = n/2$, the path P has n vertices, and the path Q is $x_u = x_l$ and Q' contains just the edge x_ix_j. Under these conditions, either x_i or x_j is the endvertex of a Hamiltonian path with an overlapping chord, which implies that either Case 1 or Case 2 applies. Therefore, for $s = n/2$, either G contains a Hamiltonian cycle or Case 2 applies. This fact will be used in the proof of Theorem 6 of the next section.

There are several immediate consequences to Theorem 2. It is straightforward to verify that if G is 2-connected graph of order n with $\mathrm{NC}(G) \geqslant s$ for some $3 \leqslant s \leqslant n/2$, then, G contains a P_{2s} and an sK_2. Note that for $s \leqslant n/2$ the graph $K_s + \bar{K}_{n-s}$ is a s-connected graph of order n with $\mathrm{NC}(G) \geqslant s$ that does not contain either a P_{2s+2} or a $(s+1)K_2$. Thus, only an improvement of 1 on the length of the path is possible. It is also easy to verify that if G is a connected graph of order n with $\mathrm{NC}(G) \geqslant s$ for some $3 \leqslant s < n/2$, then, G contains a P_{2s-1}, an $(s-1)K_2$, and a C_t for some $t \geqslant s$. This follows from the observation that $H = K_1 + G$ is a 2-connected graph, and Theorem 2 applied to H will give the result for G.

3. Hamiltonian cycles

We start with a general result about Hamiltonian cycles that uses the union of the neighborhoods of subsets of t vertices.

Theorem 3. *Let $t \geqslant 2$ be an integer, and let G be a graph of order n with $\delta(G) \geqslant t$ that satisfies $\mathrm{NC}_t(G) \geqslant n/2 + c$ for some $c = c(t)$. Then, G is Hamiltonian.*

Proof. We will suppose that G is not Hamiltonian and $c = 8t^3$, and show that this leads to a contradiction. Clearly $n \geqslant 16t^3$, for otherwise the neighborhood condition would not hold. We will assume that G is edge maximal with respect to the property of not being Hamiltonian, so $G + e$ is Hamiltonian for any edge $e \notin G$. Thus, there is a Hamiltonian path between each pair of non-adjacent vertices of G.

If x and y are non-adjacent vertices in G, then $d(x) + d(y) < n$. This follows from the fact that if P is a Hamiltonian path from x to y, then x is not adjacent to itself or the successor along P of any adjacency of y. Thus, $d(x) \leqslant n - 1 - d(y)$, which verifies the stated inequality.

Claim. *There exists a non-adjacent pair of vertices x, y of G with $|N(x) \cap N(y)| \geqslant 2t$.*

Assume this is not true. If G has $2t$ independent vertices, say $x_1, x_2, \ldots, x_{2t}$, then $\mathrm{NC}_t(G) \geqslant n/2 + c$ implies

$$\left(\bigcup_{i=1}^{t} N(x_i)\right) \cap \left(\bigcup_{i=t+1}^{2t} N(x_i)\right) \geqslant 2c + 2t.$$

Also, by assumption, $N(x_i) \cap N(x_j) \leqslant 2t - 1$ for $i \neq j$, which implies

$$\left(\bigcup_{i=1}^{t} N(x_i)\right) \cap \left(\bigcup_{i=t+1}^{2t} N(x_i)\right) \leqslant t^2(2t-1).$$

This implies $c < t^3$, a contradiction, so G does not have $2t$ independent vertices. Select a maximum independent set $X = \{x_1, x_2, \ldots, x_m\}$ of independent vertices, so $2 \leqslant m < 2t$. Partition the vertices of $G - X$ into $m + 1$ sets $\{A_1, A_2, \ldots, A_m, B\}$, where A_i is the set of vertices adjacent to just x_i, and B is the set of vertices adjacent to at least two vertices of X. Since $N(x_i) \cap N(x_j) \leqslant 2t - 1$ for $i \neq j$, B has at most $\binom{m}{2}(2t-1) < 4t^3$ vertices. Any vertex of an A_i can have at most $2t - 1$ adjacencies in an A_j for $i \neq j$, so any set of t vertices of an A_i has at most $(m-1)t(2t-1) + 4t^3 < 8t^3$ adjacencies outside of A_i. Therefore, any A_i with at least t vertices, has more than $n/2 + c - 8t^3 \geqslant n/2$ vertices. Hence, there is at most one A_i, say A_1, that has as many as t vertices, which implies that $|A_1| > n - (m-1)t - 4t^3$. If there are as many as t vertices not in $N(x_1) \cup \{x_1\}$, then clearly one of these t vertices would be adjacent to at least $2t$ vertices of $N(x_1)$. This implies $d(x_1) \geqslant n - t$, which gives a contradiction, since $d(x_1) + d(x_2) \geqslant n$. This completes the proof of the claim. □

Select non-adjacent vertices x and y such that $|N(x) \cup N(y)| \geq 2t$. Let $P = (x = x_1, x_2, \ldots, x_n = y)$ be a Hamiltonian path from x to y. Since there are $2t$ vertices on P that are simultaneously adjacent to x and y, there are $t-1$ endchords from each of x and y such that the endvertices of the endchords from x preceed on P, the endvertices of the endchords from y.

Let $\{xx_{r_1}, xx_{r_2}, \ldots, xx_{r_{t-1}}\}$ be the $t-1$ shortest endchords from x with $2 < r_1 < r_2 < \cdots < r_{t-1}$. Note that for any $j, x' = x_{r_j}^-$ is the endvertex of a Hamiltonian path. If x' has t adjacencies that precede $x_{r_{t-1}}$, then the path P can be replaced by a path P' from x' to y. This can be repeated using P', but eventually this process must stop. Hence, there is no loss of generality in assuming that for any j, $(1 \leq j \leq t-1)$, $x_{r_j}^-$ has at most $t-1$ adjacencies less than $x_{r_{t-1}}$. Let $A = \{x, x_{r_1}^-, x_{r_2}^-, \ldots, x_{r_{t-1}}^-\}$, and let A' be the adjacencies in $\{x, x_2, \ldots, x_{r_{t-1}}\}$ of vertices in A. Thus, $|A'| \leq t^2$. In the same way, we can select the shortest endchords from y to obtain corresponding sets $B = \{y, x_{s_1}^+, x_{s_2}^+, \ldots, x_{s_{t-1}}^+\}$ and B' with $|B'| \leq t^2$.

Let $N = \bigcup_{a \in A} N(a) - A'$, N^- be the predecessors of N along the path P, and $N^* = N^- - B'$. Hence $|N^*| \geq n/2 + c - 2t^2 \geq n/2$. Let $M = \bigcup_{b \in B} N(b)$, so $|M| \geq n/2 + c$. Observe that $M \cap N^* \neq \emptyset$, for otherwise G would contain $|N^*| + |M| > n$ vertices. Thus, assume $x_k \in N^* \cap M$ with $x_{r_i}^- x_{k+1}, x_{s_j}^+ x_k \in G$ for some i and j. The following is a Hamiltonian cycle of G.

$$(x, x_2, \ldots, x_{r_i}^-, x_{k+1}, \ldots, x_{s_j}, y, x_{n-1}, \ldots, x_{s_j}^+, x_k, \ldots, x_{r_i}, x).$$

This gives a contradiction that completes the proof of Theorem 3. □

Note that $\delta(G) \geq t$ is necessary in Theorem 3, because the graph $G = \bar{K}_{t-1} + (\bar{K}_{t-1} \cup K_{n-2t+2})$ has $\delta(G) = t-1$, satisfies $\mathrm{NC}_t(G) \geq n/2 + c$, but is not Hamiltonian. No attempt was made in Theorem 3 to find the smallest choice of c for which the result is true, and clearly $c = 8t^3$ used in the proof of Theorem 3 is not the smallest choice. This was done since the proof technique used will not give the sharpest result. However, in the case when $t = 2$, a sharp result can be obtained with $c = 0$, which is the following theorem.

Theorem 4. *Let G be a 2-connected graph of order n that satisfies* $\mathrm{NC}(G) \geq n/2$. *Then, for n sufficiently large, G is Hamiltonian.*

Note that the Petersen graph P of order 10 is not Hamiltonian, but that $\mathrm{NC}(P) \geq 5$. Therefore, $n > 10$ is necessary in the hypothesis of Theorem 4. However, the proof requires that n be larger than 10.

Proof. We will suppose that G is not Hamiltonian, and show that this leads to a contradiction. We can assume that the addition of any edge to G will result in a Hamiltonian cycle, so there is a Hamiltonian path between each pair of non-adjacent vertices.

Let $P=(x_1, x_2, \ldots, x_n)$ be a Hamiltonian path between $x=x_1$ and $y=x_n$. Let xx_r be the shortest endchord from x, and let yx_s be the shortest endchord from y. Thus, $x'=x_{r-1}$ and $y'=x_{s+1}$ are also endvertices of a Hamiltonian path. With no loss of generality, we can assume that P is chosen, from all of the Hamiltonian paths of G, such that the endchords from the endvertices are as small as possible. Hence, x' is not adjacent to any of the vertices on P that preceded it except for x_{r-2}, and the corresponding statement is true for y'.

The remainder of the proof is patterned after the proof of Theorem 2. If Case 2 of Theorem 2 does not hold, then for each Hamiltonian path either all of the endchords overlap, or none of the endchords overlap, and so G does contain a Hamiltonian cycle. This follows directly from the proofs of Case 1 and Case 3 of Theorem 2 and the remark that followed the proof of Theorem 2. We therefore assume that Case 2 of Theorem 2 is satisfied.

Note that if there is some k such that both the edges xx_{k+1} and yx_k are in G, then G contains the Hamiltonian cycle

$$(x, x_{k+1}, x_{k+2}, \ldots, y, x_k, x_{k-1}, \ldots, x).$$

Similar statements can be made for x' and y'. With this in mind we define the following subsets of vertices of the path P. Let

$$\begin{aligned}
&M = N(x) \cup (N(x') - x_{r-2}),\\
&M^- = \text{predecessors of } M \text{ along } P,\\
&N = N(y) \cup (N(y') - x_{s+2}), \quad \text{and}\\
&N^+ = \text{successors of } N \text{ along } P.
\end{aligned}$$

Each of the sets M, M^-, N, N^+ has at least $n/2-1$ vertices. The above note implies that $M^- \cap N = \emptyset$ (and likewise, $M \cap N^+ = \emptyset$). We will use these sets to partition the vertices of P.

Since, $y \notin M^- \cup N$, $|M^- \cup N| = n-1$ or $n-2$. Therefore, every vertex of $P-y$, with one possible exception, is in either M^- or N. For some positive integer t, the vertices of N can be partitioned into t 'intervals' (consecutive vertices of P) on P, say $\{B_1, B_2, \ldots, B_t\}$, such that the vertices in B_i precede those in B_j if $i<j$, and $B_i \cup B_{i+1}$ is not an interval. Likewise, there is a partition of M into sets $\{A_1, A_2, \ldots, A_t\}$. In this case possibly $A_j = \emptyset$ for some one j. This will occur when the number of neighborhood intervals associated with the two vertices y and y' is one larger than the number of intervals associated with the vertices x and x'. (With no loss of generality we can assume that the number of intervals associated with y and y' is at least as large as the number of intervals associated with x and x'.) Associated with each set A_i are the sets A_i^- and A_i^+, which are the predecessors and successors of A_i along P respectively. There are the corresponding sets B_i^- and B_i^+ related to B_i. Note, by the appropriate selection of the A_i, that the vertices of A_i^- are between B_{i-1} and B_i on the path.

First, consider the case when $|M^- \cup N| = n-1$, which we will call the regular case. Then, for each i $(1 \leq i \leq t)$, $|A_i \cap B_i| = 1$, and we will denote this element by

w_i. Also, there are $t-1$ vertices $\{z_1, z_2, \ldots, z_{t-1}\}$ such that z_i is the vertex of P between B_i and A_{i+1}. When $|M^- \cup N| = n-2$, which we will call the exceptional case, the pattern is the same, except there is some k such that either there are two vertices, say z'_k and z''_k, between B_k and A_{k+1}, or $A_k \cap B_k = \emptyset$. From this decomposition of the vertices of P it is clear that in the regular case that $t \leqslant (n-1)/2$, and in the exceptional case $t < (n-2)/2$. For the remainder of the proof, this notation will be used.

There are two patterns for generating Hamiltonian cycles in the graph G using the path P that will be useful in determining properties of the sets A_i and B_i. If there exist integers $j \leqslant k$ such that the edges $\{xx_j, yx_k, x_{j-1}x_{k+1}\} \subset G$, then there is a Hamiltonian cycle that uses these edges and the edges of P except for $x_{j-1}x_j$ and x_kx_{k-1}. Also, the existence of integers $j > k$ with $\{xx_j, yx_k, x_{j-1}x_{k-1}\} \subset G$, implies that there is a Hamiltonian cycle using these edges and the edges of P except for $x_{j-1}x_j$ and $x_{k-1}x_k$. As a consequence of this, no vertex in A_j^- can be adjacent to any vertex in B_k^+ for $k \geqslant j$ or any vertex in B_k^- for $k < j$. Thus, all of the vertices in A_j^- are non-adjacent to the same set of at least $n/2 - 1$ vertices. Therefore, no pair of vertices in A_j^- can be non-adjacent, for otherwise this pair would be non-adjacent to at least $n/2 + 1$ vertices, contradicting $\mathrm{NC}(G) \geqslant n/2$. Hence, the vertices of each A_i^-, and also by symmetry each B_i^+, form complete graphs. Another direct consequence of these observations is that each z_i is non-adjacent to all of the vertices in any A_j^+ for $j > i$, A_j^- for $j \leqslant i$, B_j^+ for $j > i$, and B_j^- for $j \leqslant i$, and thus z_i has degree at most t if $|M^- \cup N| = n-1$, and $t+1$ in the exceptional case.

If $t \geqslant 4$, then there exist distinct z_i and z_j, and there will be at most $t+3$ vertices in $N(z_i) \cup N(z_j)$ by the observations of the previous paragraph. Hence $t \geqslant n/2 - 3$. This implies that all but at most 4 of the A_i and B_i have at most 1 vertex. Since n is large, we can choose z_i and z_{i+1} such that $|A_i| = |A_{i+1}| = |A_{i+2}| = 1$, and thus it is easy to verify that $N(z_i) \cup N(z_{i+1})$ has at most t vertices when $|M^- \cup N| = n-1$ and $t+1$ vertices in the exceptional case. Thus in the regular case, $t \geqslant n/2$, which contradicts the previous fact that $t \leqslant (n-1)/2$. In the exceptional case, $t \geqslant n/2 - 1$, which implies that $t = n/2 - 1$. However, in this case, $|N(x) \cup N(x')| \leqslant t$ or $|N(x) \cup N(x')| \leqslant n/2 - 1$, which gives a contradiction. Thus, we can conclude that $t \leqslant 3$.

We have already considered the subcase $t = 1$, so the only remaining cases are $t = 2$ or 3. We start with the case $t = 2$, so the number of vertices between B_1 and A_2 will be either 1 or 2.

We first consider the case when z_1 is the single vertex in this interval. Recall that A_2^- is a complete graph, so if A_2^- has at least 3 vertices, then a Hamiltonian cycle can be constructed using chords xz_1^+, yz_1^-, and $z_1z_1^{++}$ and edges from P. Thus, $|A_2| \leqslant 2$, and likewise $|B_1| \leqslant 2$. If $|A_2| = 2$, then note that z_1^+ has no adjacencies in B_2^+, and any adjacency of z_1^+ in $A_1^- - \{x\}$ will give a Hamiltonian cycle. This implies that there are at most 6 vertices adjacent to either z_1 or z_1^+, a contradiction. Therefore, A_2 (and also B_1) has just 1 vertex. However, we can assume with no loss of generality that $|A_1| \leq |B_2|$, and so $|A_1| \leq (n-5)/2$. On the other hand, $|N(x) \cup N(z_1)| \leq |A_1| + 2 \leq (n-1)/2$, a contradiction.

Thus, we can assume that there are two vertices $\{z, z^+\}$ between B_1 and A_2. Since $N(x) \cup N(x') \subset A_1 \cup A_2 \cup \{x\}$, $|A_1 \cup A_2| \geq n/2 - 1$, and by the same argument, $|B_1 \cup B_2| \geq n/2 - 1$. Thus, clearly we must have equality in both cases, for P would have more than n vertices otherwise. Also, $N(z) \cup N(z^+) \subset B_1 \cup A_2 \cup \{z, z^+\}$, so $|B_1| + |A_2| \geq n/2 - 2$. If $|A_1| \geq 3$ and $|A_2| \geq 2$, then there must be an edge between $A_1 \cap A_1^-$ and $A_2 \cap A_2^-$ for, if not, the neighborhood condition would be contradicted for some vertex in $A_1 \cap A_1^-$ paired with some vertex in $A_2 \cup A_2^-$. With no loss of generality, we can assume this edge is $w_1^- w_2^-$. Using this edge and edges $\{xw_2, yw_1\}$, a Hamiltonian cycle can be constructed using all but two of the edges of the path P. Therefore, we can assume that either $|A_1| = 2$ or $|A_2| = 1$, and likewise $|B_1| = 1$ or $|B_2| = 2$. By symmetry and the fact that both B_1 and A_2 cannot be small, we need only consider the two subcases when $|A_1| = 2$ and either $|B_2| = 2$ or $|B_1| = 1$. In the first subcase, we can assume with no loss of generality that there is an edge between x' and $A_2 \cap A_2^-$, because of the neighborhood condition applied to $\{x', y'\}$. This results in a Hamiltonian cycle like the one just generated. In the second subcase consider the pair of vertices $\{x', z\}$. By the previous argument, x' cannot be adjacent to a vertex in $A_2 \cap A_2^-$ and, in general, z is not adjacent to a vertex in A_2^+. This implies that z must be adjacent to numerous vertices in B_2^-, so we can assume $zw_2^+ \in G$. This gives the Hamiltonian cycle

$$(x_1, \ldots, w_1, y, \ldots, w_2^+, z, \ldots, w_2, x_1).$$

This is a contradiction, which completes the proof of the $t = 2$ case.

We now assume that $t = 3$. We have already shown that if there exists z_1 and z_2 along P, then $t \geq n/2 - 3$. Thus, with no loss of generality, we can assume that there exists $\{w_1, w_2, w_3\}$, z_1, and the pair $\{z, z^+\}$ between B_2 and A_3. Recall that z_1 is not adjacent to any vertex in $A_1^- \cup A_2^+ \cup A_3^+ \cup B_1^- \cup B_2^+ \cup B_3^+$. Therefore, z_1 has degree at most 3. Note also that the vertex w_1^- is not adjacent to any vertex in $A_2^+ \cup A_3^+ \cup B_1^+ \cup B_2^+ \cup B_3^+$. Since $|N(w_1^-) \cup N(z_1)| \geq n/2$, we must have $|A_1| \geq n/2 - 4$. The same argument implies w_3^+ is not adjacent to any vertex in $A_1^- \cup A_2^- \cup A_3^- \cup B_1^+ \cup B_2^+ \cup B_3^+$, and so we must have $|B_3| \geq n/2 - 4$. However, there are at least 6 vertices not in the disjoint sets A_1 and B_3. This is a contradiction that completes the proof of this case and of Theorem 4. □

If the 2-connected condition is deleted from the hypothesis of Theorem 4, then the graph is no longer insured of being Hamiltonian. However, there are only a few exceptional graphs, as the following Corollary 5 indicates. Let H be the graph obtained from $K_1 + (K_{(n-1)/2} \cup K_{(n-1)/2})$ by deleting one edge between the K_1 and each of the $K_{(n-1)/2}$.

Corollary 5. *Let G be a graph of order n that satisfies* $\mathrm{NC}(G) \geq n/2$. *If n is sufficiently large, then either*

(1) *G is Hamiltonian,*

(2) $H \leqslant G \leqslant K_1 + (K_{(n-1)/2} \cup K_{(n-1)/2})$ *with n odd, or*
(3) $K_{n/2} \cup K_{n/2} \leqslant G \leqslant K_1 + (K_{(n-2)/2} \cup K_{n/2})$ *with n even.*

Proof. If G is a 2-connected graph, then G is a Hamiltonian by Theorem 6. If G is not connected, then each component with at least 2 vertices, must have at least $n/2$ vertices. Hence, in the disconnected case, n is even and $G = K_{n/2} \cup K_{n/2}$. Therefore, we assume that G is 1-connected.

Let x be a cutvertex of G. Then, $G - x$ has precisely two components, and each component has at least $\lfloor (n-1)/2 \rfloor$ vertices. Note that any component of $G - x$ with a pair of non-adjacent vertices must have order at least $n/2 + 1$, which cannot occur. Thus, each component is a complete graph. If n is odd, then each component of $G - x$ is a $K_{(n-1)/2}$ with x adjacent to all of the vertices in the component except for possibly one vertex. This gives (2). If n is even, then one component is a $K_{(n-2)/2}$, the other is a $K_{n/2}$, and x is adjacent to each vertex of the smaller component. Hence (3) is satisfied, and the proof of Corollary 5 is complete. □

4. Question

It would be worthwhile to sharpen some of the results in Section 2. For example, in a 2-connected graph G or order $n > 2s$, does $\mathrm{NC}(G) \geqslant s$ imply that $P_{2s+1} \subset G$? One interesting question left from Section 3 is to determine the smallest value of c for which Theorem 5 is still valid.

References

[1] G. Chartrand and L.Lesniak, Graphs and Digraphs (Wadsworth and Brooks/Cole, Monterey, CA, 1986).
[2] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.
[3] G.H. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory Ser B 39 (1984) 221–227.
[4] R.J. Faudree, R.J. Gould, M.S. Jacobson and L.M. Lesniak, Neighborhood unions and Pancyclic graphs, to appear.
[5] R.J. Faudree, R.J. Gould, M.S. Jacobson and R.H. Schelp, Extremal problems involving neighborhood unions, J. Graph Theory 11 (1987) 555–564.
[6] R.J. Faudree, R.J. Gould, M.S. Jacobson and R.H. Schelp, Neighborhood unions and hamiltonian properties, J. Combin. Theory Ser B 47 (1989) 1–9.
[7] R.J. Faudree, R.J. Gould and R.H. Schelp, Neighborhood conditions and edge disjoint hamiltonian cycles, Congr. Numer. 59 (1987) 55–68.
[8] P. Raisse, A new sufficient condition for hamiltonian graphs, J. Graph Theory 10 (1986) 405–410.
[9] R. Häggkvist and G.G Nicoghossian, A remark on hamiltonian cycles, J. Combin. Theory Ser B 30 (1981) 118–120.
[10] L.M. Lesniak, Neighborhood unions and graphical parameters, in: Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk, eds., Graph Theory Combinatorics, and Applications (Wiley, New York, 1991) 783–800.
[11] T.E. Lindquester, The effects of distance and neighborhood union conditions on hamiltonian properties in graphs, J. Graph Theory 13 (1989) 339–352.
[12] O. Ore, A note on hamiltonian circuits, Amer. Math. Monthly 67 (1960) 55.