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On independent generalized degrees and independence numbers in $K(1, m)$-free graphs

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Abstract

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In this paper we use independent generalized degree conditions imposed on $K(1, m)$-free graphs (for an integer $m \geqslant 3$) to obtain results involving $\beta(G)$, the vertex independence number of G. We determine that in a $K(1, m)$-free graph G of order n if the cardinality of the neighborhood union of pairs of non-adjacent vertices is a positive fraction of n, then $\beta(G)$ is bounded and independent of n. In particular, we show that if G is a $K(1, m)$-free graph of order n such that the cardinality of the neighborhood union of pairs of non-adjacent vertices is at least r, then $\beta(G) \leqslant s$, where s is the larger solution to $rs(s-1) = (n-s)(m-1)(2s-m)$. We also explore the relationship between $\beta(G)$ and $\delta(G)$ (the minimum degree) in $K(1, m)$-free graphs and provide a generalization for degree sums of sets of more than one vertex.

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1. Introduction

In recent years, many results have been obtained involving adjacency conditions in graphs that do not contain a copy of $K(1, 3)$ as an induced subgraph. In [9], for example, Matthews and Sumner utilized a lower bound on the minimum degree of a graph to obtain several types of hamiltonian results. In [6], restrictions on the cardinality of the neighborhood union of pairs of non-adjacent vertices were used to get similar properties. These bounds were improved slightly in [1]. Here we use a form of a generalized degree condition [3] imposed on graphs which do not contain a copy of $K(1, m)$, for an integer $m \geq 3$, as an induced subgraph. We use this condition on these $K(1, m)$-free graphs to obtain results involving $\beta(G)$, the vertex independence number of G. In particular, we prove the following.

Theorem A. *If G is a $K(1, m)$-free graph of order n (with $m \geq 3$) such that the cardinality of the neighborhood union of pairs of non-adjacent vertices is at least r, then $\beta(G) \leq s$ where s is the larger solution to*

$$rs(s-1) = (n-s)(m-1)(2s-m).$$

For example, when $m = 3$ and $r = n/3 + c$ (where c is some small constant), then $\beta(G) \leq 11$. In [4], this fact is used to obtain hamiltonian and hamiltonian related properties in $K(1, 3)$-free graphs with specified connectivity and neighborhood conditions.

Note that when r is a positive fraction of n, $\beta(G)$ is bounded and independent of n. Clearly this is not the case when the $K(1, m)$-free restriction is dropped. For instance, consider the complete bipartite graph $K(t, t)$, $t \geq m$, in which $K(1, m)$'s abound.

We also explore the relationship between $\beta(G)$ and $\delta(G)$ (the minimum degree) in $K(1, m)$-free graphs, and provide a generalization for degree sums of sets of more than one vertex.

Theorem B. *If G is a $K(1, m)$-free graph of order n such that*

$$\min\Big(\sum_{v \in P} \deg(v)\Big) = px$$

where the sum is taken over all independent sets $P \subset V(G)$ such that $|P| = p$, then

$$\beta(G) \leq \frac{(m-1)n}{x+m-1}.$$

For clarification of undefined notation and terms used in this paper see [7].

2. Results

We first prove some preliminary results.

Lemma 1. *Given integers a, b and c with $a \leqslant c \leqslant b$, then*

$$a\left(b-\frac{a}{2}\right) \leqslant c\left(b-\frac{c}{2}\right).$$

Proof. Since $b \geqslant c$ and $a \leqslant c$ certainly $b \geqslant (c+a)/2$. Then

$$(c-a)b \geqslant \frac{(c-a)(c+a)}{2},$$

which implies that

$$a\left(b-\frac{a}{2}\right) \leqslant c\left(b-\frac{c}{2}\right). \qquad \square$$

We are concerned with outcomes involving the generalized degree of sets of independent vertices of a graph G. If $S=\{v_1, v_2, \ldots, v_k\}$ is a k-set of independent vertices of V, we define

$$\deg S = \left|\bigcup_{i=1}^{k} N(v_i)\right|.$$

Then we denote by $I_k(G)$, the minimum $\deg S$, where the minimum is taken over all k-subsets of independent vertices in G. In this paper we will primarily be concerned with lower bound restrictions on the independent generalized degree of pairs of non-adjacent vertices. Thus $I_2(G)$ will be our main consideration.

Theorem 2. *If G is a $K(1, m)$-free graph of order n (with $m \geqslant 3$) such that*

$$\beta(G) \geqslant m \quad \text{and} \quad I_2(G) \geqslant r,$$

then $\beta(G) \leqslant s$, where s is the larger solution to

$$rs(s-1) = (n-s)(m-1)(2s-m).$$

Proof. Suppose $\beta(G)=t$. Let T be a set containing the maximum number of independent vertices. Then $|T|=t$. Denote by S the set containing the remaining $n-t$ vertices.

We obtain our result by counting the number of edges between the sets T and S, and by applying Lemma 1.

We first count edges by considering pairs of vertices in T. Since there are t vertices in T, then there are $\binom{t}{2}$ distinct pairs of vertices in T. With $I_2(G) \geqslant r$, we see that there are at least

$$\binom{t}{2} r + \sum_{v \in S} \binom{\deg_T(v)}{2} - \left[\sum_{v \in S} \deg_T(v)\right](t-2)$$

edges out of T, where $\deg_T(v)$ counts all edges from v into T for $v \in V(S)$. We arrive at this expression for the following reasons: Since $I_2(G) \geq r$, $\binom{t}{2}r$ counts the minimum number of adjacencies for each pair of vertices in T. This number, however, does not include edges that exist from overlapping neighborhoods of vertices in T. Thus

$$\sum_{v\in S} \binom{\deg_T(v)}{2}$$

counts those edges. Moreover, for each vertex u in T, $\deg(u)$ is counted a total of $t-1$ times, thus we compensate by subtracting the excess we have counted for each vertex.

Secondly, the number of edges from S to T is just $\sum_{v\in S} \deg_T(v)$. Hence,

$$\binom{t}{2} r + \sum_{v\in S} \binom{\deg_T(v)}{2} - \left(\sum_{v\in S} \deg_T(v)\right)(t-2) \leq \sum_{v\in S} \deg_T(v)$$

which gives

$$\binom{t}{2} r \leq \sum_{v\in S} \deg_T(v)\left[t - \frac{1}{2} - \frac{\deg_T(v)}{2}\right].$$

Since G is $K(1, m)$-free and T is an independent set of vertices, $\deg_T(v) \leq m-1$. Also from the hypothesis, $\beta(G) \geq m$. Applying Lemma 1, let $a = \deg_T(v)$, $b = t - \frac{1}{2}$, and $c = m - 1$. Then

$$\deg_T(v)\left(t - \frac{1}{2} - \frac{\deg_T(v)}{2}\right) \geq (m-1)\left(t - \frac{1}{2} - \frac{(m-1)}{2}\right),$$

which gives

$$\sum_{v\in S} \deg_T(v)\left(t - \frac{1}{2} - \frac{\deg_T(v)}{2}\right) \leq (n-t)(m-1)\left(t - \frac{1}{2} - \frac{(m-1)}{2}\right).$$

Therefore,

$$\binom{t}{2} r \leq (n-t)(m-1)\left(t - \frac{1}{2} - \frac{(m-1)}{2}\right)$$

and

$$rt(t-1) \leq (n-t)(m-1)(2t-m).$$

Note that any t that satisfies the previous inequality certainly satisfies $t \leq s$, since t must lie between the smaller and larger roots of the quadratic expression. Therefore, we have shown that $\beta(G) = t \leq s$ where

$$rs(s-1) = (n-s)(m-1)(2s-m). \qquad \square$$

The sharpness of the result is illustrated by considering for $m \leq t$ the following graph $G_{m,t}$ of order n (where $n \geq t$ is chosen such that $n - t$ is divisible by $\binom{t}{m-1}$), for n sufficiently large:

$$V(G_{m,t}) = A \cup \{A_{i_1 i_2 i_3 \cdots i_{m-1}} \mid 1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq t\}$$

where $A = \{x_1, x_2, \ldots, x_t\}$ and the sets $A_{i_1 i_2 i_3 \cdots i_{m-1}}$ are disjoint sets with

$$|A_{i_1 i_2 i_3 \cdots i_{m-1}}| = \frac{n-t}{\binom{t}{m-1}}.$$

We define the edge set with the following conditions:

(i) Each of $A_{i_1 i_2 \cdots i_{m-1}}$ is complete.

(ii) For $x_k \in A$, $x_k y \in E$ if and only if $y \in A_{i_1 i_2 \cdots i_{m-1}}$ where $i_j = k$ for some j.

(iii) We have $yz \in E$ if and only if $y \in A_{i_1 i_2 \cdots i_{m-1}}$ and $z \in A_{j_1 j_2 \cdots j_{m-1}}$ where $i_p = j_q$ for some p and q.

To see that $G_{m,t}$ is $K(1, m)$-free, for any k, $1 \leq k \leq t$, the neighborhood of x_k is complete by condition (iii). Thus, x_k cannot be the root of a $K(1, m)$. Also for any $y \in A_{i_1 i_2 \cdots i_{m-1}}$, y is adjacent to only $m-1$ vertices in A. In addition, for any other set $A_{j_1 j_2 \cdots j_{m-1}}$ which contains a vertex $z \in N(y)$, then $i_p = j_q$ for some p and q, which implies $x_{i_p} \in N(z)$ by adjacency condition (ii). Lastly, note that y is adjacent to vertices in only $m-1$ other $A_{i \cdots i}$'s which do not have overlapping subscripts. Thus there are only $m-1$ of these disjoint sets that do not have edges between them. Hence, $G_{m,t}$ is $K(1, m)$-free.

Also note that

$$I_2(G_{m,t}) = \frac{(m-1)(n-t)(2t-m)}{t(t-1)}.$$

In order to see this, we consider two cases:

Case 1: *Consider* $x_i, x_j \in A$.

Observe that the number of $(m-1)$-sets of t labels containing i is $\binom{t-1}{m-2}$. In counting these sets for both i and j, we must then compensate for the duplicate counting that occurs due to the intersection of the neighborhoods of x_i and x_j. But there are $\binom{t-2}{m-3}$, $(m-1)$-sets containing both i and j, thus the number of sets to which x_i and x_j are adjacent is $2\binom{t-1}{m-2} - \binom{t-2}{m-3}$. Since there are $(n-t)/\binom{t}{m-1}$ vertices in each set, we see that

$$|N(x_i) \cup N(x_j)| = \left[2\binom{t-1}{m-2} - \binom{t-2}{m-3}\right] \frac{n-t}{\binom{t}{m-1}}.$$

Since $\binom{t-1}{m-2} = \binom{t}{m-1} \cdot (m-1)/t$ we have

$$\begin{aligned} |N(x_i) \cup N(x_j)| &= \left[2\frac{(m-1)}{t}\binom{t}{m-1} - \frac{(m-1)}{t}\frac{(m-2)}{(t-1)}\binom{t}{m-1}\right] \frac{n-t}{\binom{t}{m-1}} \\ &= \frac{(m-1)(n-t)(2t-m)}{t(t-1)}. \end{aligned}$$

Case 2: *Consider* $y \in A_{i_1 i_2 \cdots i_{m-1}}$.

Then $N(x_{i_1}) \cup N(x_{i_2}) \subset (N(y) \cup \{y\}) - \{x_{i_1}, x_{i_2}\}$. This implies that $|N(x_{i_1}) \cup N(x_{i_2})| < |N(y)|$. From Case 1 we see that $N(y)$ satisfies the independent generalized degree condition, hence any pair of non-adjacent vertices in $V(G_{m,t})$ satisfies the condition for $I_2(G_{m,t})$.

Clearly $\beta(G) = t$ for $G_{m,t}$, thus the result is sharp.

We next offer in Theorem 3 a result that gives an upper bound on $\beta(G)$ in terms of $\delta(G)$ in a $K(1, m)$-free graph. This theorem is a special case of the more general result in Theorem 4 that provides a condition involving degree sums.

Theorem 3. *If F is a $K(1, m)$-free graph of order n with minimum degree $\delta(G)$, then*

$$\beta(G) \leqslant \frac{(m-1)n}{(\delta(G)+m-1)}.$$

The bound in Theorem 3 is sharp for sufficiently large n. Let G be a graph of order n with $\delta(G) = (m-1)(n-t)/t$ defined as follows: Let

$$V(G) = A \cup \{A_i \mid i = 0, 1, \ldots, t-1\},$$

where $|A_i| = (n-t)/t$ for each i, and $A = \{x_1, x_2, \ldots, x_t\}$. Also let

$$\langle A_{i-1} \cup \cdots \cup A_{i+m-3} \rangle$$

be complete for each i, with x_i adjacent to each vertex in each of $A_{i-1}, \ldots, A_{i+m-3}$ (modulo t).

Certainly $\beta(G) = t$ and G is $K(1, m)$-free. Since $\beta(G) = (m-1)(n-t)/t$, then

$$\beta(G) = t = \frac{(m-1)n}{(\beta(G)+m-1)}.$$

Thus the result is sharp.

We now consider a generalization of this result to *p-sets* of independent vertices, $1 \leqslant p \leqslant \beta(G)$. We define

$$\sigma_p = \min\Big(\sum_{v \in P} \deg(v)\Big),$$

where the minimum is taken over all independent sets $P \subset V(G)$ such that $|P| = p$.

Theorem 4. *If G is a $K(1, m)$-free graph of order n such that $\sigma_p = px$, for some p with $1 \leqslant p \leqslant \beta(G)$, then*

$$\beta(G) \leqslant \frac{(m-1)n}{x+m-1}.$$

Proof. Let $\beta(G) = t$. As before, let T be a set containing the maximum number of independent vertices, and let S contain the remaining vertices. Then $|S| = n - t$. We will enumerate edges between the sets T and S by using the p-sets of vertices.

Since T is an independent set, the total number of edges out counting by p-sets is at least $\binom{t}{p}px$. But the degree of each vertex in T is counted an additional $\binom{t-1}{p-1}-1$ times since each vertex appears in $\binom{t-1}{p-1}$ p-sets. Thus since G is $K(1, m)$-free,

$$\binom{t}{p}px-\left[\binom{t-1}{p-1}-1\right]\left(\sum_{v\in T}\deg(v)\right)\leqslant(n-t)(m-1).$$

But $\sum_{v\in T}\deg(v)\leqslant(n-t)(m-1)$, which gives

$$\binom{t}{p}px-\left[\binom{t-1}{p-1}-1\right](n-t)(m-1)\leqslant(n-t)(m-1).$$

Whence,

$$\binom{t}{p}px\leqslant(n-t)(m-1)\binom{t-1}{p-1}.$$

Simplifying,

$$\frac{t}{p}\binom{t-1}{p-1}px\leqslant(n-t)(m-1)\binom{t-1}{p-1}$$

which gives

$$tx\leqslant(m-1)(n-t)\leqslant(m-1)n-(m-1)t$$

so that

$$t\leqslant\frac{(m-1)n}{x+m-1}, \qquad \square$$

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