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Seymour's Conjecture

R.J. Faudree*
Department of Mathematics
Memphis State University, Memphis, TN 38152, USA

R.J. Gould*
Department of Mathematics and Computer Science
Emory University, Atlanta, Georgia 30322, USA

M.S. Jacobson**
Department of Mathematics
University of Louisville, Louisville, Kentucky 40292, USA

R.H. Schelp***
Department of Mathematics
Memphis State University, Memphis, TN 38152, USA

Abstract

Paul Seymour conjectured that for any positive integer k that any graph G of order n and minimum degree at least $\frac{kn}{k+1}$ contains the k^{th} power of a Hamiltonian cycle. We will give some evidence to support this conjecture by showing for $k = 2$ that for any $\varepsilon > 0$, there is a $C = C(\varepsilon)$ such that if G has order n and minimum degree at least $\frac{(1+\varepsilon)\,3n}{4} + C$, then G has a square of a Hamiltonian cycle. In fact, we will show that between each pair of vertices of the graph G there is a square of a path of length j for each $3 \le j < n$. A corresponding result for general k is also given - for any given $\varepsilon > 0$, there is a $C = C(\varepsilon, k)$ such that if G has order n and minimal degree at least $\frac{(1+\varepsilon)(2k-1)n}{2k} + C$, then G contains the k^{th} power of a path of length j between each pair of vertices for each $k+1 \le j < n$.

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1. Introduction

Let C_n denote a cycle with n vertices. A *chord* of length k in C_n is an edge between two vertices that are at a distance k in C_n. Let C_n^k denote the graph obtained from C_n by adding all chords of length at most k. Thus in C_n^k, which we will call the k^{th} *power* of C_n, a pair of vertices are adjacent if and only if the distance between them on the cycle is at most k. If a graph G of order n contains a copy of a C_n^k, we will say that G contains the k^{th} *power of a Hamiltonian cycle.*

The following was conjectured by Paul Seymour [3].

Conjecture (Seymour). *If G is a graph of order n with minimum degree* $\delta(G) \geq \dfrac{kn}{k+1}$ *for some positive integer k, then G contains the* k^{th} *power of a Hamiltonian cycle.*

Let $K_{n_1, n_2, \ldots, n_r}$ denote the complete r-partite graph with n_i vertices in part i. Thus, $K_{n_1, n_2, \ldots, n_r}$ is a graph with $n = n_1 n_2 \ldots n_r$ $n_1 + n_2 + \cdots + n_r$ vertices, and if $n_1 \leq n_2 \leq \ldots \leq n_r$, then the minimum degree is $n - n_r$. Any cycle C in $K_{n_1, n_2, \ldots, n_r}$ in which all of the chords of length at most $r - 1$ of the cycle C are in the graph $K_{n_1, n_2, \ldots, n_r}$ must have no pair of vertices from the same part within a distance less than r on the cycle. Thus in the cycle C every r^{th} vertex will come from the same part. It follows that the longest $(r-1)^{th}$ power of a cycle in $K_{n_1, n_2, \ldots, n_r}$ is rn_1 $\ldots$

An immediate consequence of the previous observations is that any complete $(k+1)$-partite graph contains the k^{th} power of a Hamiltonian cycle if and only if each of the parts have precisely the same number of vertices. Thus, any complete $(k+1)$-partite graph with minimum degree less than $\dfrac{kn}{k+1}$ will not contain the k^{th} power of a Hamiltonian cycle. Thus, if the Conjecture of Seymour is true, it is the best possible result of this type.

The special case of the Seymour Conjecture when $k = 1$ is that any graph of order n and minimum degree at ~~most~~ $n/2$ contains a least

Hamiltonian cycle. This is, of course, the well known result of Dirac [1].

Closely related to the existence of Hamiltonian cycles in a graph are other Hamiltonian properties, such as Hamiltonian connected (the existence of a Hamiltonian path between each pair of vertices of the graph), and pancyclic (the existence of cycles of each length). With the objective in mind of investigating other Hamiltonian properties related to k^{th} powers, we describe the following specialized notation. Notation not specifically defined will follow that of [2].

If P_n is a path with n vertices, then P_n^k will denote the kth power of P_n, which means that a pair of vertices are adjacent if and only if their distance in P_n is at most k. By a *k-path of length l* between a pair of vertices u and v in a graph G, we will mean a copy of a P_l^k starting at u and ending at v. A graph G of order n is *k-panconnected* if for each pair of vertices of G, there is a k-path of length l for each $k + 1 \leq l < n$, and it is *k-Hamiltonian connected (k-Hamiltonian)* if it contains a k-path of length $n - 1$ between each pair of vertices (a C_n^k). A graph G is *k-pancyclic* if it contains a C_l^k for each $3 \leq l \leq n$.

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We will prove the following results. The first deals with the case of squares of paths and cycles in graphs.

Theorem A. *For any $\varepsilon > 0$, there is a $C = C(\varepsilon)$ such that if G is a graph of order $n \geq 3$ with minimum degree $\delta(G) \geq \frac{(1+\varepsilon)3n}{4} + C$, then G is 2-Hamiltonian connected, 2-panconnected, 2-Hamiltonian, and 2-pancyclic.*

The previous result is a special case of the following more general result.

Theorem B. *For any positive integer $k \geq 3$, and any $\varepsilon > 0$, there is a $C = C(\varepsilon, k)$ such that if G is a graph of order $n \geq k + 1$ with minimum degree $\delta(G) \geq \frac{(1+\varepsilon)(2k-1)n}{2k} + C$, then G is k-Hamiltonian connected, k-panconnected, k-Hamiltonian, and k-pancyclic.*

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2. Results

Before starting the proof of Theorem A, we introduce one additional item of terminology that will be needed. By a *2-path of length l* between a pair of edges uv and xy in a graph G, we will mean a copy of a P_l^2 such that the first two vertices of the path are u and v respectively, and the last two vertices are x and y respectively. Of course, if there is 2-path of the length l between uv and xy, there is certainly such a 2-path of length l between u and y. We will say that a graph G of order n is *edge 2-panconnected* if there is a 2-path of length l between each pair of independent edges of the graph for each $5 \leq l \leq n - 1$. We are now prepared to prove the following. a

Theorem 1. *For any $\varepsilon > 0$, there is a $C = C(\varepsilon)$ such that if G is a graph of order $n \geq 4$ with minimum degree $\delta(G) \geq \frac{(1+\varepsilon)3n}{4} + C$, then G is edge 2-panconnected. Thus, G is 2-Hamiltonian connected, 2-panconnected, 2-Hamiltonian, and 2-pancyclic.*

Proof. The proof will be by induction on n the order of the graph. Note that if $n \leq \frac{4C}{1-3\varepsilon}$, then the graph G is a complete graph, and the result follows trivially. Therefore, we can assume that $n > \frac{4C}{1-3\varepsilon}$.

Assume G does not have the required property, but the addition of any edge will give the property. Let u_1u_2 and v_1v_2 be a pair of independent edges for which for some t there is no 2-path of length t connecting them. We will show that $t > 4C$, but first we will show that there is a 2-path of length 5. Because of the minimum degree, we can select a vertex w_1 that is adjacent to u_1, u_2 and v_1 and distinct from v_2. Again, using the minimum degree, we can find an additional vertex w_2 that is adjacent to u_2, w_1, v_1 and v_2. This gives a 2-path of length 5. Now using the edges u_2w_1 and v_1v_2, one can repeat the procedure avoiding the vertex u_1 to obtain a path of length 6. This procedure can be continued until a 2-path of length at least 4C is obtained, since $\delta(G) > 3n/4 + C$ implies that any four vertices have more than C common neighbors.

Let P be a 2-path of maximal length, say with m vertices, between u_1u_2 and v_1v_2 such that all shorter paths exist. We can assume that $m > 4C$. Let H be the subgraph of order $n - m$ obtained from G by deleting the vertices of P. The maximality of P implies that no vertex of H can be adjacent to 4 consecutive vertices of P. Therefore, each vertex of H has at most $3 + \frac{3(m-3)}{4} = 3(m+1)/4$ adjacencies on the path P. Thus, $\delta(H) \geq 3(1+\varepsilon)n/4 + C - 3(m+1)/4 \geq 3(1+\varepsilon)(n-m)/4 + C$, since $m > 4C > \frac{1}{\varepsilon}$. This implies, by the induction hypothesis, that the graph H is edge 2 - panconnected.

To complete the proof we will need to consider three cases, that depend on the magnitude of m.

Case 1. $m \leq (n+3)/2$.

Select a vertex u_3 in H that is adjacent to both u_1 and u_2. Since u_2 has less than $n/4$ nonadjacencies in H, and u_3 has less than $(n-m)/4$ nonadjacencies, there is a vertex u_4 that is adjacent to both u_2 and u_3. By the same reasoning, there are v_3 and v_4 in H such that v_3 is adjacent to v_1, v_2 and v_4, and v_4 is adjacent to v_1. Also, all of these vertices are distinct. This gives two "links" between u_1u_2 and v_1v_2 and the graph H.

In the graph H there are 2-paths from u_4u_3 to v_3v_4 of each length from 5 to $n - m - 1$. Thus, using the "links" just described, there are 2 - paths from u_1u_2 to v_1v_2 of each length from 9 to $n - m + 3$. Hence, $n - m + 3 \leq m$, and so we can assume that $m \geq (n+3)/2$.

Case 2. $(n+3)/2 < m \leq 2n/3$.

We will count the number of edges of $\overline{G}$ between H and the path P. Each vertex of H has degree at most $(1-\varepsilon)n/4 - C$ in $\overline{G}$, so there are at most $(n-m)((1-\varepsilon)n/4 - C) < (n-m)(n/4 - C)$ edges in $\overline{G}$ between H and P.

Just as in Case 1, we next consider the possibility of "links" using pairs of consecutive vertices of P and adjacent vertices of H. If there are two such "links" using two pairs of vertices that are a distance less than $n - m$, then we can, just as in Case 1, generate 2 - paths between u_1u_2 and v_1v_2 longer than m. Thus, we can assume that the number of pairs of consecutive vertices that support such "links" is small, in fact no more than two disjoint ones can exist.

If a vertex w of P is adjacent to more than $t = ((1-\varepsilon)n/4 - C) + ((1-\varepsilon)(n-m)/4 - C)$ vertices of H, then there is a "link" using w and either of the vertices on the path adjacent to w. Thus, we can assume (with at most two exceptions) that w is nonadjacent to at least $n - m - t > (2n-3m)/4 + 2C$ vertices of H. If a pair of consecutive vertices of P have no common adjacency in H, then as a pair they will have at least $n-m$ nonadjacencies in H. If they have a common adjacency, then there will be a "link" unless one of them has $n - m - ((1-\varepsilon)(n-m)/4 - C) > 3(n-m)/4 + C$ nonadjacencies. Therefore, if there is no "link" associated with the pair, they will have at least $3(n-m)/4 + C + (2n-3m)/4 + 2C = (5n-6m)/4 + 3C$ edges of $\overline{G}$ in to H. Therefore, using the upper and lower bounds on the number of edges in $\overline{G}$ between P and H, we have the following inequality :

$$\left(\frac{m}{2}-2\right)\left(\frac{5n-6m}{4}+3C\right) < (n-m)\left(\frac{n}{4}-C\right).$$

This is equivalent to the inequality :

$$\frac{1}{8}(2n-3m)(2m-n) < (5n-6m)/2 - (n-m+6)C$$

However, for $n/2 \le m \le 2n/3$, the left hand side of the above inequality is nonnegative and the right hand side is negative for C large. This gives a contradiction, so we can assume that $m > 2n/3$.

Case 3. $m > 2n/3$.

Each vertex in H is adjacent to at most $3(m+1)/4$ vertices of P, therefore we have

$$|H| > \frac{3(1+\varepsilon)n}{4} + C - \frac{3(m+1)}{4}.$$

This gives that $|H| > 3|H|/4 + 3(\varepsilon n - 1)/4 + C$, and so $|H| > 4(3(\varepsilon n - 1)/4 + C) \ge 3\varepsilon n + 4C - 1$.

As in the previous cases, the number of edges in $\overline{G}$ between H and P is at most $(n-m)((1-\varepsilon)n/4 - C) < (n-m)(n/4 - C)$. Also, as in the previous case, we consider the "links" between P and H. If there are two "links" using pairs of vertices closer than

$n - m$, then the value of m is not maximal. Therefore, the number of pairs of vertices for which there are "links" is at most $m/(n-m) + 1 < n/(3\varepsilon n) = 1/3\varepsilon$. For any pair of consecutive vertices on P for which there are no more than $(n-m) - ((1-\varepsilon)(n-m)/4 - C)$ nonadjacencies in H, there will be a "link". Hence we can assume that there are at least $(m - \frac{1}{3\varepsilon})/2$ pairs of vertices on P incident to at least $(n-m) - ((1-\varepsilon)(n-m)/4 - C) > 3(n-m)/4$ edges of $\overline{G}$ into H. Therefore, the number of edges of $\overline{G}$ between P and H is at least $(\frac{m - 1/(3\varepsilon)}{2})(3(n-m)/4)$. This gives the following inequality involving the upper and lower bounds on the number of edges of $\overline{G}$ between H and P.

$$\left(\frac{m - 1/(3\varepsilon)}{2}\right)\left(\frac{3(n-m)}{4}\right) < (n-m)(n/4 - C).$$

However, this is equivalent to the following inequality.

$$\frac{1}{8}(3m - 2n)(n-m) < (n-m)\left(\frac{1}{8\varepsilon} - C\right).$$

Each factor in the left hand side of the above inequality is nonnegative for $2n/3 \leq m \leq n$, and the term on the right is negative. This gives a contradiction, which completes the proof of this case and the fact G is edge 2-panconnected.

All that remains is to prove the existence of the appropriate 2-cycles in G. Since the minimum degree condition implies that G contains a complete graph K_5 (see [4]), there are the appropriate cycles of length at most 5. To obtain the longer cycles, note that if the initial vertices u_1, u_2, v_1, v_2 form a complete graph K_4, then the 2-path from $u_1\, u_2$ to $v_1\, v_2$ with m vertices is also a 2-cycle of length m. This completes the proof of Theorem 1. □

If one is only interested in 2-Hamiltonian paths and cycles in G, and not paths and cycles of each possible length, the proof techniques used in Theorem 1 can be used to prove a slightly stronger result. With a considerable increase in tedious arithmetic and an extended case analysis, the minimum degree condition of Theorem 1 can be reduced to just $\delta(G) \geq 3n/4$ to get 2-Hamiltonian paths and cycles. However, this still exceeds the conjectured value of Seymour of $\delta(G) \geq 2n/3$.

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Before discussing the proof of Theorem B, we need an additional item of terminology. By a *k-path of length l* between a pair of disjoint complete subgraphs A = $\{a_1, a_2, ..., a_k\}$ and B = $\{b_1, b_2, ..., b_k\}$ with k vertices in a graph G, we will mean a copy of a P_l^k such that the first k vertices of the path are $a_1, a_2, ..., a_k$ respectively, and the last k vertices are $b_1, b_2, ..., b_k$ respectively. Of course, if there is a k-path of length l between A and B, there is certainly such a k-path of length l between a_1 and b_k. We will say that a graph G of order n is K_k *k-panconnected* if there is a k-path of length l between each pair of disjoint complete subgraphs of the graph for each $2k - 1 \leq l \leq n - 1$. We are now prepared to discuss the following.

Theorem 2. *For any positive integer $k \geq 3$, and for any $\varepsilon > 0$, there is a $C = C(\varepsilon, k)$ such that if G is a graph of order $n \geq 4$ with minimum degree $\delta(G) \geq \frac{(1+\varepsilon)(2k-1)n}{2k} + C$, then G is K_k k-panconnected. Thus, G is k-Hamiltonian connected, k-panconnected, k-Hamiltonian, and k-pancyclic.*

The proof of Theorem 2 has the same outline as that of Theorem 1. Only the arithmetic is different. Let P be a maximal length k-path in a graph G between two complete subgraphs for which there are not k-paths of every possible length, and let H be the subgraph induced by the remaining vertices of G. The graph H can be shown to satisfy the same minimum degree condition as G. Let m be the length of this maximal length path P. In this general situation, there are also three cases to be considered. When $m \leq n/2$, "links" are built between the two complete subgraphs at either end of the path P and the graph H. Then the panconnected properties of the graph H are used. In the last two cases, $n/2 < m < kn/(2k-1)$, and $m \geq kn/(2k-1)$, either there exists appropriate "links" between H and two pairs of k consecutive vertices of P that are sufficiently close on P, or a count on the number of edges of $\overline{G}$ between P and H gives a contradiction. Because of the similar nature of this proof to that of Theorem 1, we do not include the details here, but leave them to the reader.

3. Questions

The question left by the Conjecture of Seymour is still open, since the lower bounds on minimum degree used here exceed the conjectured bounds. Related to the Conjecture of Seymour, is the question of the minimum degree needed to insure that a graph is k-panconnected or k-pancyclic. In particular, is it true that a graph G of order n and minimum degree $\delta(G) \geq \frac{kn}{k+1} + C$ is k-panconnected and k-pancyclic? It is possible that $C = 0$ will suffice for the k-pancyclic property.

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