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On Rotation Numbers for Digraphs

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Abstract

The rotation number $h(D)$ of a digraph D of order p is the minimum number of arcs in a digraph F of order p such that for every vertex x of D and every vertex y of F, there exists an embedding of D in F with x at y. The rotation number is determined for all asymmetric digraphs whose underlying graph is a star and studied for tournaments as well as for asymmetric digraphs whose underlying graph is a cycle.

1. Introduction

For a graph G of order p rooted at a vertex x, the *rotation number* $h(G, x)$ of this rooted graph is the minimum size of a graph F of order p such that for every vertex y of F, there exists an embedding of G in F with x at y. The notation h in $h(G, x)$ indicates that G can be homogeneously embedded in F.

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This concept was introduced in 1980 by Cockayne and Lorimer [5] and emanates from a problem in broadcasting. Rotation numbers have been investigated for complete bipartite graphs [1, 5], unions of cycles [2, 3], unions of stars [7], a class of rooted graphs [8], generalized stars [6] and unicyclic graphs containing exactly one bridge [4].

For a graph G of order p that is not rooted, the *rotation number* $h(G)$ of G is the minimum size of a graph F of order p such that for every vertex x of G and every vertex y of F, there exists an embedding of G in F with x at y. For example, $h(G) = 9$ for the graph G of Figure 1. The unique graph F of size 9 in which G can be homogeneously embedded is also shown in Figure 1.

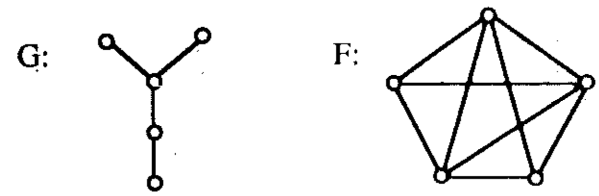

Figure 1

Rotation numbers for digraphs can be defined similarly. Let D be a digraph of order p rooted at a vertex x. A *homogeneous superdigraph* of D is a digraph H of order p such that for every vertex y of H, there exists an embedding of D in H with x at y. A homogeneous superdigraph of minimum size is called an *optimal digraph* for the rooted digraph D. The *rotation number* $h(D, x)$ of this rooted digraph D is then defined as the size of an optimal digraph for D. For example, if D is the digraph of Figure 2, then $h(D, x_1) = 4$ while $h(D, x_2) = 6$. Optimal digraphs F_1 and F_2 for D at x_1 and x_2, respectively, are also shown in Figure 2.

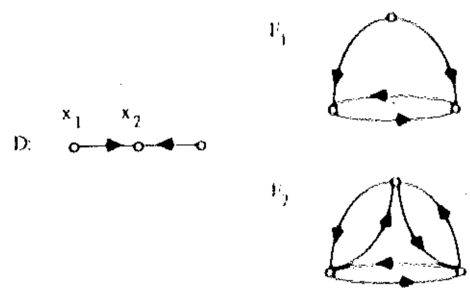

Figure 2

In the case of a digraph D of order p that is not rooted, homogeneous superdigraphs and optimal digraphs for D are defined similarly. Then the *rotation number* $h(D)$ is the size of an optimal digraph for D. For the digraph D of Figure 2, it follows that $h(D) = 6$.

If D is a vertex-transitive digraph, then $h(D)$ equals the size $q(D)$ of D. We define the *rotation ratio* $r(D)$ of a digraph D by $\frac{h(D)}{q(D)}$. Then $r(D) \geq 1$ for every digraph D, and $r(D) = 1$ if and only if D is vertex-transitive. Thus, $r(D)$ provides a measure of the symmetry of D, with the more symmetric digraphs having a rotation ratio close to 1.

2. Rotation Numbers for Stars

When we refer to a *star* in digraph theory, we mean an asymmetric digraph whose underlying graph is a star. We write $S_n(m)$ to indicate a star with n arcs, m of which are directed outwardly from the central vertex (see Figure 3).

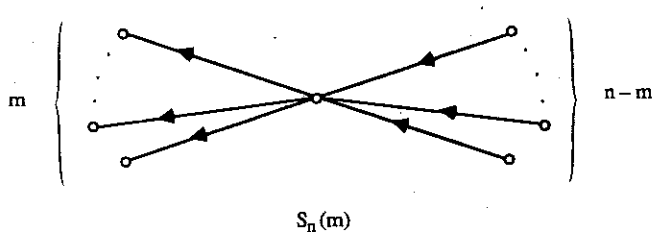

Figure 3

In what follows, the following two well-known theorems from graph theory will be useful (see [9, pp. 216-217]).

Theorem A. *If p (≥ 3) is odd, then the complete graph K_p can be factored into $\frac{p-1}{2}$ hamiltonian cycles.*

Theorem B. *If p (≥ 2) is even, then the complete graph K_p can be factored into $\frac{p-2}{2}$ hamiltonian cycles and a 1-factor.*

Theorem 1. *Let $D \cong S_n(m)$ where $n \geq 1$. If x is the central vertex of D, then*

$$h(D, x) = [\max(m, n-m)](n+1).$$

Proof. Assume, without loss of generality, that $m \geq n - m$. We show that $h(D, x) = m(n+1)$. Let H be an optimal digraph for this rooted digraph. Because D has order $n + 1$, so does H. Since for every vertex y of H, there exists an embedding of D in H with x at y, the outdegree of each vertex of H must be at least m. Thus, H has at least $m(n+1)$ arcs; so $h(D, x) \geq m(n+1)$. To complete the proof, we show that there exists an optimal digraph F for D containing exactly $m(n+1)$ arcs. This is accomplished by showing that there exists an m-regular digraph D (every vertex has outdegree and indegree m) of order $n + 1$.

Since $m \geq n - m$, it follows that $m \geq \frac{n}{2}$. Suppose first that n is even. Since $n + 1$ is odd, the complete graph K_{n+1} can be factored into $\frac{n}{2}$ hamiltonian cycles $C_1, C_2, \ldots, C_{\frac{n}{2}}$. For each $i = 1, 2, \ldots, \frac{n}{2}$, let C_i' be a directed cycle obtained by cyclically directing the edges of C_i, and let C_i'' be the directed cycle whose arcs are directed oppositely to that of C_i'. Thus the complete symmetric digraph K^*_{n+1} can be factored into the cycles $C_1', C_2', \ldots, C_{\frac{n}{2}}', C_1'', C_2'', \ldots, C_{\frac{n}{2}}''$. The digraph F consisting of m of these cycles is m-regular and is therefore optimal for D.

Next, suppose that n is odd. By Theorem B, the complete graph K_{n+1} can be factored into $\frac{n-1}{2}$ hamiltonian cycles $C_1, C_2, \ldots, C_{\frac{n-1}{2}}$ and a 1-factor F_1. Let C_i' and C_i'' $(1 \leq i \leq \frac{n-1}{2})$ be as defined above. Define F to be that digraph consisting of m of the cycles $C_1', C_2', \ldots, C_{\frac{n-1}{2}}', C_1'', C_2'', \ldots, C_{\frac{n-1}{2}}''$ if $m < n$, while define F to be the complete symmetric digraph K^*_{n+1} if $m = n$. □

We now consider $h(S_n(m), x)$, w

Theorem 2. *Let $D \cong S_n(m)$, where outdegree 0, then*

$$h(D, x) =$$

Proof. First we show that $h(D, x$ digraph for D. Then F contains a $S_n(m)$. Let c be the centre of H. Thu where $m > n - m$. Since D can be en exists a subdigraph $H_1 \cong S_n(m)$ centre of H_1 by c_1. Thus (c_1, c) is an

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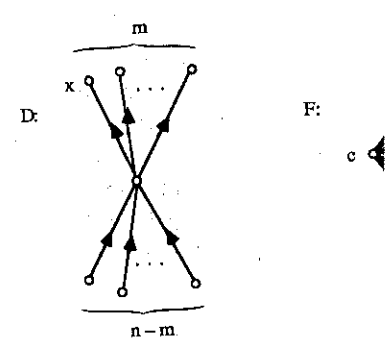

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□

We now consider $h(S_n(m), x)$, where x is not a central vertex.

Theorem 2. *Let $D \cong S_n(m)$, where $m > n-m$. If x is a vertex of outdegree 0, then*

$$h(D, x) = 2n.$$

Proof. First we show that $h(D, x) \geq 2n$. Let F be an optimal digraph for D. Then F contains a subdigraph H isomorphic to $S_n(m)$. Let c be the centre of H. Thus $\text{od}_H c = m$ and $\text{id}_H c = n-m$, where $m > n-m$. Since D can be embedded in F with x at c, there exists a subdigraph $H_1 \cong S_n(m)$ of F with x at c. Denote the centre of H_1 by c_1. Thus (c_1, c) is an arc of F.

Suppose that (c, c_1) is an arc of H. There must be $n-m$ vertices of F (different from c) adjacent to c_1, and there must be m vertices (including c) adjacent from c_1. Thus, F must contain at least n arcs that do not belong to H; so $h(D, x) \geq 2n$.

Suppose now that (c, c_1) is not an arc of H. Then F must contain at least $n-1$ arcs that do not belong to H, namely $m-1$ additional arcs directed away from c_1 and $n-m$ arcs directed toward c_1. If, however, these were the only arcs of F and D was embedded in F with x at c_1, then there is no in-neighbour of c_1 available for the centre in this embedding. Therefore, at least one more arc must be added to produce F, so $h(D, x) \geq 2n$.

That $h(D, x) = 2n$ follows by observing that D can be homogeneously embedded in the digraph F of size $2n$ shown in Figure 4. □

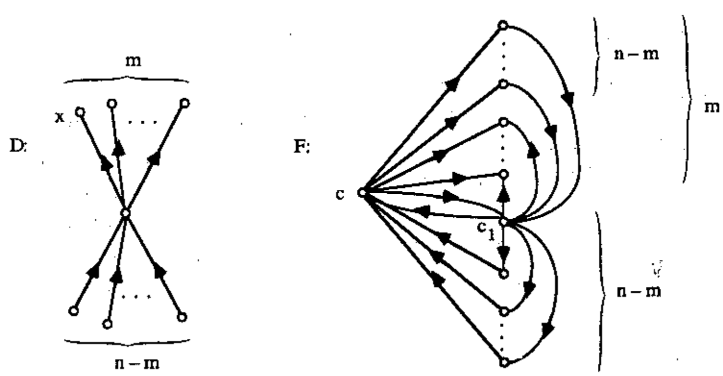


Figure 4

Corollary. *Let $D \cong S_n(m)$, where $n - m > m$. If x is a vertex of indegree 0, then*

$$h(D, x) = 2n.$$

Theorem 3. *Let $D \cong S_n(m)$, where $n - m \geq m$. If x is a vertex of outdegree 0, then*

$$h(D, x) = 3n - 2m + 1.$$

Proof. First we show that $h(D, x) \geq 3n - 2m + 1$. Let F be an optimal digraph for D. Then F contains a subdigraph H isomorphic to $S_n(m)$. Let c be the centre of H and let S be the set of vertices of H adjacent to c. Thus $|S| = n - m$. By hypothesis, D can be embedded in F with x at c. Let $H_1 \cong S_n(m)$ be a subdigraph of F with x at c. Denote the centre of H_1 by c_1.

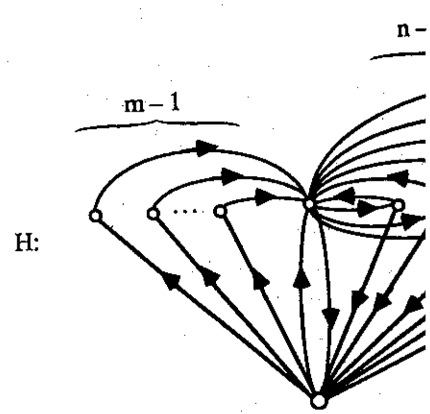
Suppose, first, that (c, c_1) is an arc of H. Thus, $c_1 \notin S$. In F there must be m arcs directed away from c_1, one of which is directed toward c, and $n - m$ arcs directed toward c_1.

Let k be the number of vertices of S that are adjacent to c_1 in H. Thus, $n - 2m + 1 \leq k \leq n - m$. If D is embedded in F with x placed at any of these k vertices of S, then an additional arc directed toward it is required. Thus, the size of F is at least $2n + k$. Since $k \geq n - 2m + 1$, it follows that the size of F is at least $2n + (n - 2m + 1) = 3n - 2m + 1$, that is, $h(D, x) \geq 3n - 2m + 1$.

Next, suppose that (c_1, c) is an arc of H, that is, $c_1 \in S$. In F there must be m arcs directed away from c_1, including (c_1, c), and $n - m$ arcs directed toward c_1.

Again, let k be the number of vertices of S that are adjacent to c_1 in H. Thus, $n - 2m + 1 \leq k \leq n - m$. If D is embedded in F with x placed at any of these k vertices of S, then an additional arc directed toward it is needed. Thus, the size of F is at least $2n - 1 + k$. Suppose D is embedded in F with x at c_1, and let $H_2 \cong D$ with x at c_1. At least one additional arc is required for this embedding, so that the size of F is at least $2n + k$. Here too then, $h(D, x) \geq 3n - 2m + 1$.

That $h(D, x) = 3n - 2m + 1$ follows from the fact that there exists a homogeneous superdigraph H for D of size $3n - 2m + 1$ (see Figure 5). □



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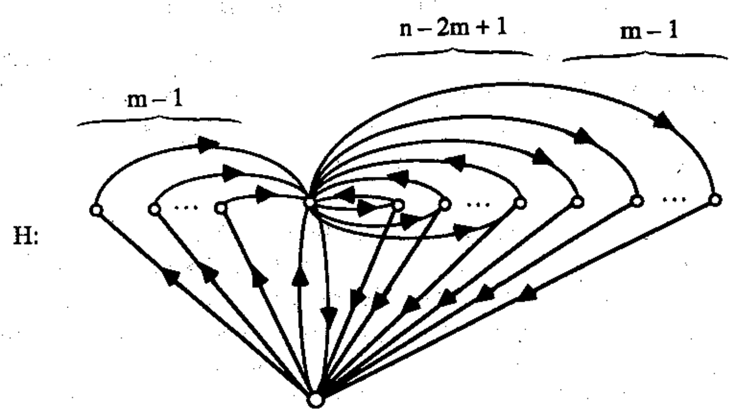


Figure 5

Corollary. *Let* $D \cong S_n(m)$, *where* $n - m \leq m$. *If* x *is a vertex of indegree* 0, *then*

$$h(D, x) = 3n - 2m + 1.$$

Combining all the foregoing results, we have the following.

Theorem 4. *If* $D \cong S_n(m)$, *where* $n \geq 3$, *then*

$$h(D) = [\max(m, n - m)](n + 1).$$

According to Theorem 4 then, if D is a star of size n, then the rotation ratio $r(D)$ satisfies the inequalities

$$\frac{n+1}{2} \leq r(D) \leq n + 1.$$

3. Rotation Numbers for Cycles

We call an asymmetric digraph a *cycle* if its underlying graph is a cycle. A directed cycle of order n will be denoted by $\vec{C}_n$.

Although normally we consider rotation numbers for rooted and unrooted digraphs, the following observation shows that only the rooted version is interesting.

Theorem 5. *Let a digraph D be a cycle of order n. Then* $h(D) = n$ *if* $D \cong \vec{C}_n$, *and* $h(D) = 2n$, *otherwise.*

Proof. If $D \cong \vec{C}_n$, then D itself serves as an optimal digraph.

If D is not isomorphic to $\vec{C}_n$, then D has a vertex of outdegree 2. Therefore, an optimal digraph has at least $2n$ arcs, and $h(D) \geq 2n$. On the other hand, the symmetric cycle C_n^* is an optimal digraph for D of size $2n$, so $h(D) \leq 2n$. □

By Theorem 5, if D is a cycle then either $r(D) = 1$ or $r(D) = 2$. In the same manner, one can show that for the rooted version, $h(D, x) = 2n$ whenever x has indegree or outdegree equal to 2. Consequently, we henceforth consider cycles rooted at x with $\text{od } x = \text{id } x = 1$. For such cycles, the following notation will be useful. Starting from the selected vertex x, while proceeding around the cycle in some direction, we denote each arc by "w" (with) if it is consistent with the direction of the travel, or by "a" (against), otherwise. For example, the pattern $(wwawaw)$ denotes the cycle D in Figure 6.

D: (wwawaw)

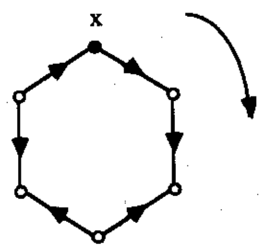


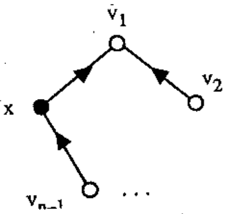
Figure 6

In general, if D is a cycle not isomorphic to $\vec{C}_n$, then $n + 1 \leq h(D, x) \leq 2n$. However, for some special patterns, a better lower bound is possible.

Theorem 6. *If D is a rooted cycle of order n with the pattern $(w\underline{a} \ldots w)$ or $(w \ldots \underline{a}w)$ at the vertex x, then $h(D, x) \geq \frac{3}{2}n$.*

Proof. Let F be an optimal digraph for D. For every vertex v of F, we have $\text{od } v \geq 1$ and $\text{id } v \geq 1$. Suppose that there is a vertex v in F such that $\text{od } v = 1$ and $\text{id } v = 1$. Assume, without loss of generality, that the pattern for D at x is $(w\underline{a} \ldots w)$. Therefore, if $x = v_0, v_1, v_2, \ldots, v_{n-1}$ are the vertices of the cycle D, then

(v_{n-1}, x), (x, v_1) and (v_2, v_1) are arcs
by y and z the unique vertices of F
(see Figure 7).



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Suppose that there exists an en
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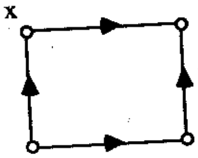

Fig

In general, for a $4k$-cycle D
$(waawwaaw \ldots waaw)$, k re
digraph has size $6k$ and may b
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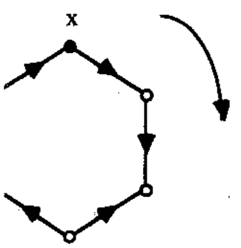

Figure 6

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(v_{n-1}, x), (x, v_1) and (v_2, v_1) are arcs of D (see Figure 7). Denote by y and z the unique vertices of F such that $(y, v), (v, z) \in E(F)$ (see Figure 7).

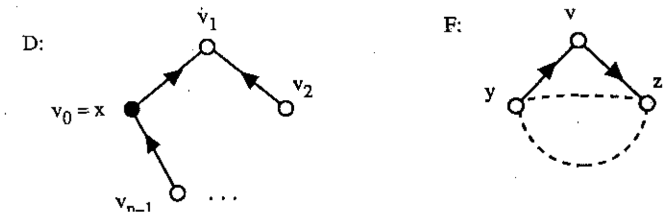


Figure 7

Suppose that there exists an embedding of D in F with x at y. The vertex v_{n-1} cannot correspond to the vertex v, since $(v_{n-1}, x) \in E(D)$ and $(v, y) \notin E(F)$. Therefore, v_1 must correspond to v. Then, however, v_2 must correspond to z, which is impossible since $(v_2, v_1) \in E(D)$ and $(z, v) \notin E(F)$.

From the above observation, it follows that for every vertex v of F, od v + id $v \geq 3$ and, consequently, $q(F) \geq \frac{3n}{2}$. □

The lower bound given by Theorem 6 is sharp. To see that, let us consider a 4-cycle D given by the pattern ($waaw$) (see Figure 8). Its optimal digraph F has size 6.

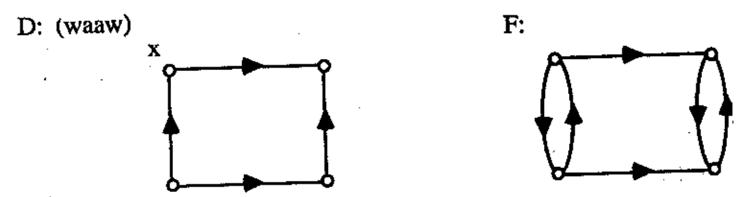


Figure 8

In general, for a $4k$-cycle D ($k = 1, 2, 3, \ldots$) given by a pattern ($waawwaaw \ldots waaw$), k repetitions of $waaw$, an optimal digraph has size $6k$ and may be obtained by adding the reverse arc for every second arc of the digraph D.

Finding the exact value of a rotation number for a cycle seems to be quite difficult, even for cycles with only one arc reversed. For small values of n, a computer search has been performed and the results are summarized in the following table.

size of D	pattern of D	$h(D, x)$	number of optimal digraphs
5	$(wawww)$	10	22
5	$(wwaww)$	9	5 (all isomorphic)
6	$(wawwww)$	12	37
6	$(wwawww)$	12	34
7	$(wawwwww)$	14	?
7	$(wwawwww)$	14	?
7	$(wwwawww)$	13	7 (all isomorphic)

Table. Rotation number for cycles with one arc reversed

We believe that the following conjecture is true.

Conjecture. *Let D be a cycle of length n obtained from a directed cycle by reversing one arc. Let D be rooted at a vertex v with* od v = id v = 1.

(a) *If n is even, then $h(D, x) = 2n$.*

(b) *If n is odd, then $h(D, x) = 2n - 1$ whenever the reversed arc lies opposite to x, and $h(D, x) = 2n$, otherwise.*

Moreover, the only optimal digraphs for the case with $h(D, x) = 2n - 1$ are digraphs obtained by removing one arc from the symmetric cycle C_n^*.

4. Rotation Number for Tournaments

A *tournament* is a digraph obtained by orienting the edges of a complete graph. If T is a tournament of order p, then its rotation number $h(T)$ must lie between $\frac{1}{2}p(p-1)$ and $p(p-1)$. A natural question arises : What are the possible values for rotation numbers of tournaments ?

Theorem 7. *If k and p are positive integers such that $\left\lceil \frac{p-1}{2} \right\rceil \le k \le p-1$, then there exists a tournament T of order p with $h(T) = kp$.*

Proof. Assume first that p is odd. Su

tournament with $V(T_0) = \{v_0, v_1, \ldots,$

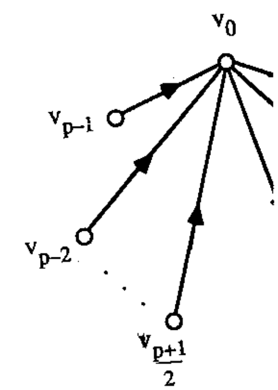
$E(T_0)$ if and only if $0 < (j - i)$ (mod

vertex-transitive and each vertex is

vertices (see Figure 9, where only

vertex v_0 are shown). Since T_0 i

optimal digraph for itself. Thus $h($



Figu

Suppose next that $k = \frac{p-1}{2}$ +

$v_0 v_{p-1}$, that is, T_1 is obtained fr

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$h(T_1) \ge \frac{1}{2}p(p+1)$. On the oth

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$\ldots, v_{p-1}, v_0$ is a homogeneous s

$\frac{1}{2}p(p-1) + p = \frac{1}{2}p(p+1) = pk.$

For higher values of k we u

$k = \frac{p-1}{2} + r$, we reverse the la

Then the outdegree of v_0 fo

a rotation number for a cycle seems r cycles with only one arc reversed. uter search has been performed and n the following table.

h (D, x)	number of optimal digraphs
10	22
9	5 (all isomorphic)
12	37
12	34
14	?
14	?
13	7 (all isomorphic)

for cycles with one arc reversed

ving conjecture is true.

of length n obtained from a directed et D be rooted at a vertex v with

x) = 2n.

x) = 2n − 1 whenever the reversed nd h (D, x) = 2n, otherwise.

imal digraphs for the case with obtained by removing one arc from

Tournaments

obtained by orienting the edges of ı tournament of order p, then its between $\frac{1}{2}p\,(p-1)$ and $p\,(p-1)$. A Vhat are the possible values for nts ?

sitive integers such that $\left\lceil \frac{p-1}{2} \right\rceil \le$ *a tournament T of order p with*

Proof. Assume first that p is odd. Suppose $k = \frac{p-1}{2}$. Let T_0 be a tournament with $V(T_0) = \{v_0, v_1, \ldots, v_{p-1}\}$, and where $(v_i, v_j) \in E(T_0)$ if and only if $0 < (j - i) \pmod p \le \frac{p-1}{2}$ Therefore, T_0 is vertex-transitive and each vertex is adjacent to and from $\frac{p-1}{2}$ vertices (see Figure 9, where only the arcs incident with one vertex v_0 are shown). Since T_0 is vertex transitive, it is the optimal digraph for itself. Thus $h(T_0) = \frac{1}{2}p\,(p-1) = pk$.

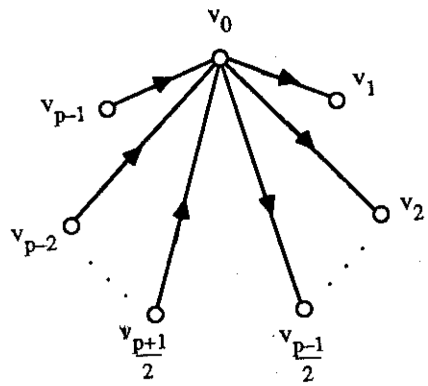

Figure 9

Suppose next that $k = \frac{p-1}{2} + 1$. Define $T_1 \cong T_0 - v_{p-1}v_0 + v_0v_{p-1}$, that is, T_1 is obtained from T_0 by reversing the direction of the arc (v_{p-1}, v_0). Since od $v_0 = \frac{p+1}{2}$, the rotation number $h(T_1) \ge \frac{1}{2}p\,(p+1)$. On the other hand, a digraph F obtained from T_0 by adding the reverse arc for each arc on the cycle $v_0, v_1, \ldots, v_{p-1}, v_0$ is a homogeneous superdigraph for T_1, and has size $\frac{1}{2}p\,(p-1) + p = \frac{1}{2}p\,(p+1) = pk$.

For higher values of k we use a similar method, namely for $k = \frac{p-1}{2} + r$, we reverse the last r arcs incident to the vertex v_0. Then the outdegree of v_0 forces the rotation number of the obtained digraph T_r to be at least $p\left(\frac{p-1}{2}+r\right)$. An optimal digraph for T_r can be obtained from T_0 by adding to each arc of the form (v_i, v_j), when $1 \le (i-j) \pmod{p} \le r$, its reverse arc. Such a digraph has also size $\frac{1}{2}p(p-1)+rp = pk$.

For p even we use a slight modification of the previous construction. If $k = \left\lceil \frac{p-1}{2} \right\rceil = \frac{p}{2}$ then the tournament T_0 has $\frac{p}{2}$ vertices of outdegree $\frac{p}{2}$ (namely $v_0, v_1, v_2, \ldots, v_{\frac{p}{2}-1}$) and $\frac{p}{2}$ vertices of outdegree $\frac{p}{2}-1$ (namely $v_{\frac{p}{2}}, \ldots, v_{p-1}$). On the "main diagonals" we have arcs $(v_0, v_{\frac{p}{2}}), (v_1, v_{\frac{p}{2}+1}), \ldots, (v_{\frac{p}{2}-1}, v_{p-1})$. Other arcs are directed in the same manner as for the case with p odd. An optimal digraph for T_0 is obtained by adding $\frac{p}{2}$ "main diagonal" arcs, that is, the arcs $(v_{\frac{p}{2}}, v_0), (v_{\frac{p}{2}+1}, v_1), \ldots, (v_{p-1}, v_{\frac{p}{2}-1})$. Therefore,

$$h(T_0) = \frac{1}{2}p(p-1) + \frac{p}{2} = p \cdot \frac{p}{2} = pk.$$

For larger values of k we construct the corresponding digraphs from T_0 as in the case with p odd. □

By Theorem 7, the rotation ratio of every tournament lies between 1 and 2. In fact, every rational number in the interval [1, 2] is the rotation ratio of some tournament. In order to see this, let r be a rational number such that $1 \le r \le 2$. Then $r = 1 + \frac{a}{b}$, where b is a positive integer and a is an integer such that $0 \le a \le b$. Define $k = a + b$ and $p = 2b + 1$, and observe that $\left\lceil \frac{p-1}{2} \right\rceil \le k \le p-1$. By Theorem 7, there exists a tournament T of order p such that $h(T) = kp$. Thus,

$$r(T) = \frac{h(T)}{\binom{p}{2}} = \frac{2kp}{p(p-1)} = \frac{2k}{p-1} = \frac{2a+2b}{2b} = 1 + \frac{a}{b} = r.$$

It is natural to ask whether s
mentioned in Theorem 7 are attair
tournaments. The next example
possible.

T:

Figur

Let T be a tournament of order
of orders 4 and 3, where every v
order 4 is adjacent to each vert
order 3. Figure 10 illustrates th
guarantees that the numbers
numbers of some tournaments
tournament T of Figure 10, $h(T)$

Recall that the vertex set
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property of being strong. These
can be ordered in such a way t
vertex in S_i is adjacent to each v

We now examine when a
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mely $v_{\frac{p}{2}}, \ldots, v_{p-1}$). On the "main
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same manner as for the case with
T_0 is obtained by adding $\frac{p}{2}$ "main
rcs $(v_{\frac{p}{2}}, v_0), (v_{\frac{p}{2}+1}, v_1), \ldots, (v_{p-1},$

$1) + \frac{p}{2} = p \cdot \frac{p}{2} = pk.$

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such that $1 \leq r \leq 2$. Then $r = 1 + \frac{a}{b}$,
r and a is an integer such that
d $p = 2b + 1$, and observe that
em 7, there exists a tournament T
Thus,

$$\frac{2k}{p-1} = \frac{2a+2b}{2b} = 1 + \frac{a}{b} = r.$$

It is natural to ask whether some values other than those mentioned in Theorem 7 are attainable by a rotation number for tournaments. The next example shows that this indeed is possible.

T:

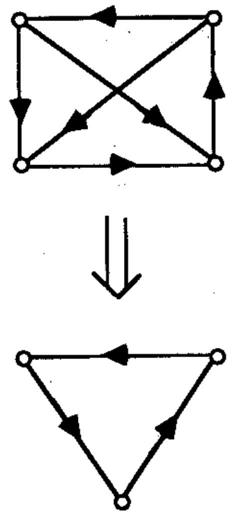

Figure 10

Let T be a tournament of order 7 having two strong components of orders 4 and 3, where every vertex in the strong component of order 4 is adjacent *to* each vertex in the strong component of order 3. Figure 10 illustrates the construction of T. Theorem 7 guarantees that the numbers 21, 28, 35, and 42 are rotation numbers of some tournaments of order 7. However, for the tournament T of Figure 10, $h(T) = 36$.

Recall that the vertex set of every tournament can be partitioned into subsets such that the digraph (tournament) induced by each subset is strong and maximal with respect to the property of being strong. These *strong components* $S_1, S_2, \ldots, S_k$ can be ordered in such a way that whenever $1 \leq i < j \leq k$, every vertex in S_i is adjacent to each vertex from S_j.

We now examine when a tournament T of order p having maximum outdegree (indegree) $p-2$ has the rotation number $h(T) = p(p-2)$. The following theorem gives a characterization of such tournaments.

Theorem 8. *Let T be a tournament of order p, maximum outdegree (or indegree) $p-2$, and strong components $S_1, S_2, \ldots, S_k$ of orders $p_1, p_2, \ldots, p_k$, respectively. The rotation number $h(T) = p(p-2)$ if and only if the following two conditions are satisfied:*

(1) *There exist integers $r_1, r_2, \ldots, r_n$ $(r_j \geq 3)$ and positive integers b_j^i $(j = 1, 2, \ldots, n; i = 1, 2, \ldots, k)$ such that for every $i = 1, 2, \ldots k$,*

$$p_i = \sum_{j=1}^{n} b_j^i r_j.$$

(2) *If $p_i = b_1^i r_1 + b_2^i r_2 + \ldots + b_n^i r_n$, then for every vertex u of S_i and for every number r_j the disjoint union*

$$b_1^i \vec{C}_{r_1} + b_2^i \vec{C}_{r_2} + \ldots + b_n^i \vec{C}_{r_n}$$

of cycles is a subdigraph of S_i, where the vertex u lies on some r_j-cycle.

Proof. Suppose that $h(T) = p(p-2)$ and F is an optimal digraph for T. Then F is $(p-2)$-regular and its complement F is 1-regular, which means that $\overline{F}$ is a disjoint union of directed cycles. The fact that for every $x \in V(T)$ and every $y \in V(F)$ the tournament T is embeddable in F with x at y is equivalent to $\overline{F}$ being embeddable in $\overline{T}$ with y at x. Since $\overline{T}$ is obtained from T by reversing all arcs, $\overline{T}$ has the strong components of the same order as T has, namely the components $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k$. Moreover, two corresponding strong components, S_i from T and $\overline{S}_i$ from $\overline{T}$, have the same cycle structure.

The above facts force the following sequence of statements :

(1) All vertices from any cycle in $\overline{F}$ correspond to vertices from only one strong component in $\overline{T}$.

(2) Vertices of each strong component of $\overline{T}$ (or T) can be decomposed into subsets forming directed cycles. The union of all these cycles for all strong components is isomorphic to $\overline{F}$. Thus $p_i = b_1^i r_1 + b_2^i r_2 + \ldots + b_n^i r_n$, for some integers b_j^i.

(3) Since any vertex y of $\overline{F}$ can
of $\overline{T}$, $b_j^i \geq 1$ and in the afo
can be placed on a direc
$r_1, r_2, \ldots, r_n$.

On the other hand, if the cor
we define

$$\overline{F} \cong \bigcup_{j=1}^{n} (b_j^1 + b_j^2$$

Then F is a homogeneous su
so F is an optimal digraph for T

For tournaments of order
indegree) $p-2$, the following
from Theorem 8.

Corollary 1. *If T has a str*
$h(T) > p(p-2)$.

Corollary 2. *If T is a strong tou*

Corollary 3. *If T has strong c*
then $h(T) = p(p-2)$.

The application of Theore
when the condition (1) is no
immediate answers (namely
with the following orders of st

(a) 3 and 5

(b) 5 and 9

(c) 6 and 10

(d) 8, 11 and 15, and so on.

In the case when the con
it is necessary to investig
components. Consider, for e
presented in Figure 11. Both T
order 6 and 3 and therefore
$3 = 1 \cdot 3$. However, the vertice
T_1 can be partitioned into t
v_5, v_4 which is not possible
$h(T_2) > 63$ (in fact $h(T_2) = 6$

rnament of order p, maximum outdegree
ɔng components $S_1, S_2, \ldots, S_k$ of orders
The rotation number $h(T) = p(p-2)$
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's $r_1, r_2, \ldots, r_n$ ($r_j \geq 3$) and positive
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cycles for all strong components is
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(3) Since any vertex y of $\overline{F}$ can be identified with any vertex x of $\overline{T}$, $b_j^i \geq 1$ and in the afore mentioned decomposition, x can be placed on a directed cycle of any of the lengths $r_1, r_2, \ldots, r_n$.

On the other hand, if the conditions (1) and (2) are satisfied we define

$$\overline{F} \cong \bigcup_{j=1}^{n} (b_j^1 + b_j^2 + \ldots + b_j^k) \ \vec{C}_{r_j}$$

Then F is a homogeneous superdigraph for T of size $p(p-2)$; so F is an optimal digraph for T. □

For tournaments of order p and maximum outdegree (or indegree) $p-2$, the following corollaries can be easily derived from Theorem 8.

Corollary 1. *If T has a strong component of order 1, then $h(T) > p(p-2)$.*

Corollary 2. *If T is a strong tournament, then $h(T) = p(p-2)$.*

Corollary 3. *If T has strong components of the same order (≥ 3), then $h(T) = p(p-2)$.*

The application of Theorem 8 is especially straightforward when the condition (1) is not satisfied. For example, it gives immediate answers (namely $h(T) > p(p-2)$) for tournaments with the following orders of strong components :

(a) 3 and 5
(b) 5 and 9
(c) 6 and 10
(d) 8, 11 and 15, and so on.

In the case when the condition (1) of Theorem 8 is satisfied, it is necessary to investigate the cycle structure of strong components. Consider, for example, two tournaments T_1 and T_2 presented in Figure 11. Both T_1 and T_2 have strong components of order 6 and 3 and therefore we have decomposition $6 = 2\cdot3$ and $3 = 1\cdot3$. However, the vertices of the larger strong component of T_1 can be partitioned into two 3-cycles: v_1, v_3, v_2, v_1 and v_4, v_6, v_5, v_4 which is not possible for T_2. Thus $h(T_1) = 9\cdot7 = 63$ while $h(T_2) > 63$ (in fact $h(T_2) = 65$).

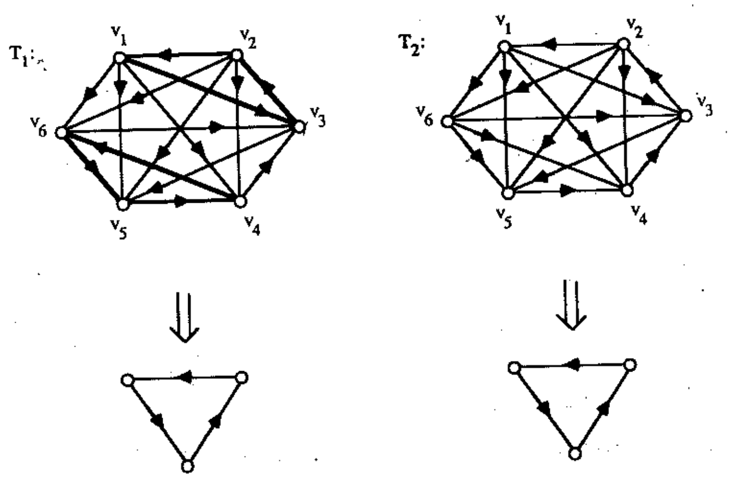

Figure 11

Of course, whenever a tournament T of order p has a maximum outdegree (or indegree) $p-1$, then $h(T) = p(p-1)$ and K_p^* is the only optimal digraph. For a tournament T of order p and maximum outdegree (or indegree) $p-2$, the largest possible value that can be achieved by the rotation number is $h(T) = p(p-2) + \left\lceil \frac{p}{3} \right\rceil$. To see this, let $p(T) = 3l + r$, where $r = 0, 1, 2$. Then a digraph F such that $\overline{F} \cong l \cdot \vec{P}_3 \cup \vec{P}_2$ for $r = 2$ and $\overline{F} \cong l \cdot \vec{P}_3$ for $r = 0, 1$, is a homogeneous superdigraph for T. The value $h(T) = p(p-2) + \left\lceil \frac{p}{3} \right\rceil$ is attained by a tournament T_p, $p \geq 7$, that has strong components of order 3, 1, 1, ..., 1, 3 (see Figure 12).

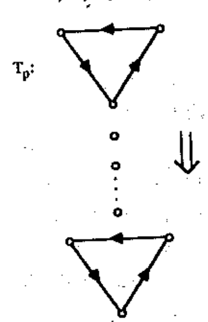

Figure 12

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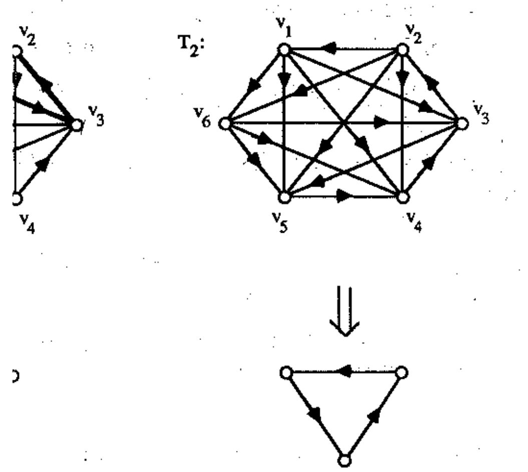

Figure 11

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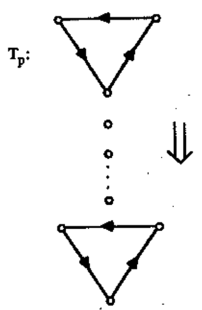

Figure 12

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