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GENERALIZED DEGREES, CONNECTIVITY AND HAMILTONIAN PROPERTIES IN GRAPHS

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ABSTRACT: For sets of vertices, we consider a form of generalized degree based on neighborhood unions. Bounding this generalized degree from below, we obtain results about the connectivity of a graph as well as Dirac-type results about highly hamiltonian properties in such graphs. In particular, we determine lower bounds on the generalized degree sufficient to imply a graph is hamiltonian-connected or pan yclic.

Section 1. Introduction.

For standard terms and notation not found here see [10].

Over the past few years a form of generalized degree condition for sets of vertices (where the sets satisfy various conditions) has been used to further the study of a variety of graph properties. In [5] and [9], hamiltonian properties were studied using sets of independent vertices of various sizes, while in [6], matchings and extremal path and cycle lengths were studied for the same types of sets. In [2], a Turan-type extremal result was obtained. A survey of recent results using generalized degrees can be found in [11].

As these results have emerged, variations on the types of sets used have occurred. A natural combinatorial choice is the collection of all t-subsets of the vertex set V of a graph G. If

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$S = \{ v_1, v_2, \ldots, v_t \}$ is an arbitrary t-set of vertices in V, we define

$$deg\ S = | \bigcup_{i=1}^{t} N(v_i) |.$$

This definition follows those used in the above mentioned papers as well as in other related works.

Using this definition, the following generalization of the classic Hamiltonian Theorem of Dirac [1] was shown.

Theorem A [3]. If G is a 2-connected graph of sufficiently large order n satisfying $deg\ S \geq n/2$ for each 2-subset $S = \{ x, y \}$ of vertices in G, then G is hamiltonian.

For convenience, we will denote by $\delta_2(G)$, the minimum of $deg\ S$, where this minimum is taken over all distinct pairs of vertices $\{ x, y \} = S$ in G. In general then, we define $\delta_k(G)$ to be the minimum $deg\ S$, where the minimum is taken over all k-subsets S of vertices in G.

The purpose of this paper is to explore connectivity and highly hamiltonian properties in graphs satisfying lower bound degree restrictions, usually lower bounds on δ and δ_2.

Section 2. Connectivity

Lower bounds on a generalized degree, unlike ordinary degree bounds, do not necessarily imply any connectivity conditions in a graph. For example, suppose our choice of subset S is the collection of all pairs of nonadjacent vertices. Then in the graph $G = K_{n/2} \cup K_{n/2}$, (n even) any pair of nonadjacent vertices $S = \{ x, y \}$ satisfies $deg\ S = n - 2$, the largest possible value for such a pair. The graph G is clearly disconnected, and thus, no lower bound on the generalized degree for such sets of vertices will ever imply connectivity. However, when the type of subset in use varies, other results can occur. We shall now show that a lower bound on the generalized degree for arbitrary k-sets of vertices along with a minor minimum degree condition can produce a useful connectivity condition.

Theorem 1. If G is a graph of order n satisfying $\delta(G) \geq t$ and $\delta_k(G) \geq \frac{n+t+2-k}{2}$ for $(2 \leq k \leq t+2)$, then G is $(t+2-k)$-connected.

Proof. Suppose that G is not $(t+2-k)$-connected and let C be a vertex cutset with fewer than $t+2-k$ vertices. Let C_1 be a component of $G - C$ of smallest order. Clearly,

$$| V(C_1) | \leq \left\lfloor \frac{n - |C|}{2} \right\rfloor .$$

If $| V(C_1) | < k$ and $x \in V(C_1)$, then

$$deg\ x \leq k-2+|C| \leq k-2+t+1-k = t-1,$$

contradicting the minimum degree condition. Hence, C_1 contains at least k vertices.

Now let $S = \{ x_1, x_2, \ldots, x_k \}$ be an arbitrary k-subset of vertices in C_1. Then

$$deg\ S = | \bigcup_{i=1}^{k} N(x_i) | \le \left\lfloor \frac{n - |C|}{2} \right\rfloor + |C|$$

$$\le \frac{n + |C|}{2} \le \frac{n + t + 1 - k}{2}$$

contradicting the minimum generalized degree condition. Hence, G must be $(t + 2 - k)$-connected. ∎

We note that even stronger connectivity conditions imposed by generalized degrees in conjunction with other properties have been studied in [7] and [8].

As an example of the sharpness of Theorem 1, consider the graphs H_i formed by taking 2 copies of K_p $(p \ge i + 1)$ and identifying i vertices from each copy of K_p. Then $|V(H_i)| = 2p - i = n$ and further, for a set S of k vertices selected in one copy of K_p, by avoiding the i identified vertices, we see that $deg\ S = p = \frac{n+i}{2}$. This is clearly the minimum such degree (when $p - i \ge k$) and H_i is clearly i-connected, but not $(i + 1)$-connected. Thus, the result cannot be strengthened to imply the graph is $(i + 1)$-connected.

The following Corollary is immediate from Theorem 1.

Corollary 2. If G is a graph of order n and if for each set $S = \{ x, y \}$ of distinct vertices of G,

$$deg\ S \ge \frac{n+t}{2} \text{ and } \delta(G) \ge t, \text{ then } G \text{ is } t\text{-connected.}$$

A natural question one might ask is: What connectivity can be guaranteed by just a generalized degree condition for arbitrary k-sets of vertices, that is, without some sort of minimum degree condition also being included? The answer can be seen in the graph formed by joining a vertex x to any one vertex of K_{n-1}. This graph satisfies $\delta_2(G) \ge n - 2$ (the highest δ_2 possible in a noncomplete graph), but has connectivity exactly 1. A minor minimum degree condition, in combination with a generalized degree bound, can provide ample connectivity for highly hamiltonian properties. For example, hamiltonian-connected graphs are known to be 3-connected (at least when $n \ge 4$) and we can obtain this level of connectivity by avoiding vertices of degree at most 2 and bounding the generalized degree from below.

Section 3. Highly Hamiltonian Properties

In this section we examine highly hamiltonian properties in graphs satisfying certain minimum generalized degree conditions. For convenience, we define the following terminology. Suppose that $P: x_1, \ldots, x_k$ is a path and that x_k has a chord (edge off the path) to a vertex x_i on P. Then by a Posa-rotation of the path P and edge $e = x_k x_i$ we mean the creation of a new path

$$P_r: x_1, x_2, \ldots, x_i, x_k, x_{k-1}, \ldots, x_{i+1}.$$

That is, we merely insert the edge $x_k x_i$ and delete the edge $x_i x_{i+1}$. In so doing, we obtain a path on the same vertex set, from x_1 to x_{i+1}, the successor of the end vertex of the chord e. We denote by (r, s), the segment of a path strictly between vertex r and vertex s, while $[r, s]$ denotes the segment of a path including r and s. We now consider the following Lemma.

Lemma 4. Let G be a graph of order $n \geq 6$ satisfying $\delta_2(G) \geq \frac{n+3}{2}$. If $V(G)$ can be partitioned into two disjoint subsets, say A and B, which contain spanning paths

$$P_1: x = x_1, x_2, \ldots, x_k = u \quad \text{which spans } A \text{ and}$$
$$P_2: v = x_{k+1}, \ldots, x_n = y \text{ which spans } B,$$

with the properties that

$$|N(u) \cap V(P_1)| \geq 2 \text{ and } |N(v) \cap V(P_2)| \geq 2,$$

then G contains an $(x-y)$–hamiltonian path.

Proof. Suppose G satisfies the given conditions but has no $(x-y)$–hamiltonian path. Then $V(G) = V(P_1) \cup V(P_2)$ partitions $V(G)$ as described, and also each of u and v have at least one chord to a vertex on their respective paths.

Since u has a chord to a vertex on P_1, say u', we note that if $u_1 = u' + 1$ (the successor of u' on P_1) has a chord in the segment (u', u), then this chord is "shorter" than that of u. Hence, we can perform a Posa-rotation, to obtain an $x - u_1$ path on the same vertex set with endvertices x and u_1, where the endvertex has a shorter chord. We can repeat this procedure a finite number of times to obtain a path on the same vertex set, beginning with x and ending with a vertex, say u, having the shortest possible chord from u. That is, u has a chord to say u' and both $u_1 = u' + 1$ and u have no chords in the segment (u', u). A similar procedure could be applied to P_2 to obtain v' and its predecessor v_1. Thus, we may assume that P_1 and P_2 were chosen with this property.

Note that u and u_1 are not adjacent to v or v_1 or an $(x-y)$–hamiltonian path is immediate. We now consider two cases.

CASE 1: Suppose that u is not adjacent to u_1 on P_1 and v is not adjacent to v_1 on P_2. (That is, the end chords of the paths do not form triangles.)

If u (or u_1) is adjacent to a vertex $w \in (v_1, y)$, the segment of P_2 strictly between the vertices v_1 and y, then both v and v_1 are not adjacent to $w + 1$, the successor of w along P_2, or else

$$x, x+1, \ldots, u, w, w-1, \ldots, v, w+1, w+2, \ldots, y$$

would be an $(x-y)$–hamiltonian path in G. Similar paths can be found if v_1 is adjacent to $w + 1$ or if u_1 is adjacent to w, and either of v or v_1 is adjacent to $w + 1$. We also note that by our choice of v and v', both v and v_1 have no chords in the segment (v, v_1). Hence, their only adjacencies in $[v, v']$ are to $v + 1$, v' and $v_1 - 1$. But recall that u and u_1 are not adjacent to either v or v_1 in this interval.

Hence, the adjacencies of u and u_1 define certain nonadjacencies of v and v_1 (with the possible exceptions of $v + 1$ and $v_1 - 1$) which we can describe with the 1–1 and onto function

$$f_2: V(P_2) - \{v, v_1, y\} \rightarrow V(P_2) - \{v, v_1, v'\}$$

where $f_2(w) = w + 1$ if $w \in [v', y)$ and $f_2(w) = w$ if $w \in (v, v_1)$.

A similar argument shows that for each adjacency w of u or u_1 in $[x, u')$, the vertex $w+1$ is not adjacent to v or v_1. The adjacencies of u and u_1 thus define certain nonadjacencies of v and v_1 described by the 1–1 and onto function

$$f_1: V(P_1) - \{u', u_1, u\} \rightarrow V(P_1) - \{x, u_1, u\}$$

defined by $f_1(w) = w+1$ if $w \in [x, u')$ and $f_1(w) = w$ if $w \in (u_1, u)$.

Since v and v_1 are nonadjacent and both are not adjacent to u and u_1 and since for each adjacency of u or u_1 (with the possible exception of u', u_1+1, $u-1, v+1$, v_1-1 and y), there is a nonadjacency of v and v_1 as described by the functions f_1 and f_2 we see that

$$\begin{aligned} deg\{v, v_1\} &\le (n-4) - [deg\{u, u_1\} - 6] \\ &\le \frac{n+1}{2}, \end{aligned}$$

a contradiction to the generalized degree condition.

CASE 2. Suppose u is adjacent to u_1 on P_1 but v is not adjacent to v_1 on P_2.

Then an argument similar to that of Case 1 shows that

$$f_1: V(P_1) - \{u', u_1, u\} \rightarrow V(P_1) - \{x, u_1, u\}$$

defined as

$$f_1(w) = w+1 \text{ if } w \in [x, u') \text{ and}$$

$$f_2: V(P_2) - \{v, v_1, y\} \rightarrow V(P_2) - \{v, v_1, v'\}$$

defined as

$$f_2(w) = w+1 \text{ if } w \in [v', y) \text{ and}$$
$$f_2(w) = w \text{ if } w \in (v, v_1)$$

are 1–1 and onto. Further, f_1 and f_2 map possible adjacencies of u and u_1 to nonadjacencies of v and v_1 (with the possible exceptions of u', u_1, u, $v+1$, v_1-1 and y). Hence,

$$\begin{aligned} deg\{v, v_1\} &\le (n-4) - [deg\{u, u_1\} - 6] \\ &\le \frac{n+1}{2}, \end{aligned}$$

again contradicting the generalized degree condition.

If u is not adjacent to u_1 on P_1 but v is adjacent to v_1 on P_2, or if v is adjacent to v_1 on P_2 and u is adjacent to u_1 on P_1, then in both situations

$$\begin{aligned} deg\{v, v_1\} &\le (n-2) - [deg\{u, u_1\} - 4] \\ &\le \frac{n+1}{2}, \end{aligned}$$

a contradiction.

In all cases a contradiction is reached and hence, the graph contains an $(x-y)$–hamiltonian path. ∎

Theorem 5. If G is a graph of sufficiently large order n such that $\delta_2(G) \geq \frac{n+3}{2}$ and $\delta(G) \geq 3$, then G is hamiltonian-connected.

Proof. Suppose the result fails to hold and further, suppose that G is a maximal counterexample. That is, suppose the result fails on G, but does hold on $G + e$ where e is any edge not in G.

Since G is not hamiltonian-connected, there exists a pair of vertices, say x and y, not joined by a hamiltonian path in G. Without loss of generality, we may assume $deg\ x \geq \frac{n+3}{4}$. If $deg\ x = n-1$, then consider the graph $H = G - x$. By Corollary 2, G is 3–connected, thus H is 2–connected. But now, by Theorem A, we see that H is hamiltonian. If C is a hamiltonian cycle in H, then in G, the vertex x is adjacent to each vertex of C and an $(x-y)$–hamiltonian path is easily found.

Thus, we may assume that $deg\ x < n-1$. Since G is a maximal counterexample, $G + xu$ contains an $(x-y)$–hamiltonian path, for any edge xu where $u \notin N_G(x)$. Let

$$P : x = v_1,\ u = v_2,\ v_3, \ldots, v_n = y$$

be such an $x-y$ path in $G + xu$. Then in G, we see that x, along with the vertices $v_2, \ldots, v_n$ spans $V(G)$ and we denote by P_1 the subpath of P from v_2 to v_n. We now consider two cases.

CASE 1: Suppose there exists a shortest chord from u (to say u') on P_1, with u' prior to all but at most one neighbor of x on P_1.

Without loss of generality, we may assume that uu' is a shortest chord in this interval (or else we apply Posa-rotations). Let u_1 be the predecessor of u' along P_1 and let x_1 and x_2 be two neighbors of x along P_1 beyond u' and prior to y (this is possible since $deg\ x \geq \frac{n+3}{4}$ and n is sufficiently large). Further, suppose that y_1 and y_2 are the successors of x_1 and x_2 along P_1, respectively. Since $deg\ x \geq \frac{n+3}{4}$ and n is large, x_1 and x_2 may be selected so that $dist(x_1, x_2) \leq 4$.

Note that u and u_1 are not adjacent to y_1 or y_2 or we can easily find an $(x-y)$–hamiltonian path in G, for example, if u is adjacent to y_1 then

$$x, x_1, \ldots, u, y_1, \ldots, y$$

is such a path. Clearly, similar paths exist in the other cases. Also recall that u and u_1 are not adjacent to x.

If y_1 (or y_2) is adjacent to $w \in (y_2 + 1, y]$, then u is not adjacent to $w - 1$, or else

$$x, x_1, \ldots, u, w-1, \ldots, y_1, w, w+1, \ \ldots, y$$

is an $(x-y)$–hamiltonian path in G. We note that similar paths can be found if y_2 is adjacent to w or if u_1 is adjacent to $w-1$.

For every adjacency $w \in (u', x_1)$ of y_1 (or y_2), the vertex $w+1$ is not adjacent to u (or u_1) for otherwise

$$x, x_1, x_1 - 1, \ldots, w+1, u, u+1, \ldots, w, y_1, y_1+1, \ldots, y$$

is an $(x-y)$–hamiltonian path in G. Similar paths can be found if y_2 is adjacent to w or if u_1 is adjacent to $w+1$.

By our choice of u and u', both u and u_1 have no chords to vertices in the segment (u, u_1). Thus, u and u_1 have at most three adjacencies in $[u, u']$, namely $u+1$, $u_1 - 1$, and u'. We note also that not all three of these vertices need be distinct.

If $dist(x_1, x_2) = 4$, then (x_1, x_2) contains exactly the distinct vertices y_1, y_1+1 and $x_2 - 1$. If y_1 or y_2 is adjacent to $x_2 - 1$, then u and u_1 are not adjacent to $y_1 + 1$ for otherwise, by Lemma 4, an $(x-y)$–hamiltonian path would exist. This can be seen using the paths

$$x, x_1, \cdots, u$$
$$y_1, \cdots, y$$

if y_1 is adjacent to $x_2 - 1$; while

$$x, x_1, y_1, y_1+1, u, \cdots, x_1 - 1$$
$$x_2 - 1, x_2, y_2, \cdots, y$$

are the paths if y_2 is adjacent to $x_2 - 1$ and u is adjacent to $y_1 + 1$. Similar paths can be found if u_1 is adjacent to $y_1 + 1$.

Under the assumption that $dist(x_1, x_2) = 4$ and that u, $u+1$, $u_1 - 1$ and u_1 are all distinct vertices, consider the sets

$$R = V - \{ x, u, u+1, u_1 - 1, u_1, u', x_1, y_1, y_1+1, x_2, y_2, y_2+1 \}$$

and

$$D = V - \{ x, u, u+1, u_1 - 1, u_1, u', u'+1, y_1, x_2 - 1, x_2, y_2, y \}.$$

Define the function $f : R \to D$ as:

$$f(w) = w - 1 \text{ for } w \in (y_2 + 1, y]$$
$$f(w) = w + 1 \text{ for } w \in (u', x_1)$$
$$f(w) = w \text{ for } w \in (u, u_1) - \{ u+1, u_1 - 1 \} \text{ and}$$
$$f(x_2 - 1) = y_1 + 1.$$

Now we see that f is 1–1 and onto and maps adjacencies of the pair y_1 and y_2 to nonadjacencies of the pair u and u_1.

Since we know in this case that u and u_1 are not adjacent to five vertices, namely x, y_1, y_2 and each other, the remaining seven vertices are "exceptional", that is, they may have adjacencies to both the sets $\{ y_1, y_2 \}$ and $\{ u, u_1 \}$ simultaneously. Thus, we see from this that

$$deg\ \{ u, u_1 \} \le (n-5) - [deg\ \{ y_1, y_2 \} - 7] \le \frac{n+1}{2},$$

a contradiction to the generalized degree condition. We note that if $dist(x_1, x_2) \le 3$, a count similar to the one above again produces a contradiction (as the number of exceptional vertices can only decrease or remain the same from this change).

If u is adjacent to u_1 on P_1 and $dist(x_1, x_2) = 4$, then (since $u+1 = u_1$ and $u_1 - 1 = u$) there are only three vertices that we can say are not adjacent to u or u_1. However, the number of exceptional vertices also decreases to five, namely u', x_1, y_1+1, x_2 and y_2+1. Thus, in this case we see that

$$deg\ \{u, u_1\} \leq (n-3) - [deg\ \{y_1, y_2\} - 5] \leq \frac{n+1}{2},$$

again a contradiction. Further, if $dist(x_1, x_2) \leq 3$, a similar argument again produces a contradiction. While if $dist\ (u, u_1) = 2$, again a similar argument applies.

CASE 2: There exist at least two neighbors of x preceding u', the first neighbor of u on P_1.

Suppose that x_1 and x_2 are the neighbors of x on P_1 and that y_1 and y_2 are the predecessors of x_1 and x_2, respectively. Also suppose that u_1 and u_2 are two neighbors of u beyond x_2 on P_1 and that, in this case, z_1 and z_2 are the predecessors of u_1 and u_2, respectively. Note that x is not adjacent to z_1 or z_2 or an $(x-y)$–hamiltonian path is immediate. Thus, $x_2 \neq z_1$. If u_1 and u_2 are chosen to be the first two adjacencies of u along P_1 after x_2, then z_1 is not adjacent to u and z_2 is not adjacent to u unless $z_2 = u_1$.

Suppose $w \in (u_2, y]$ is adjacent to y_1 (or y_2), then z_1 (and z_2) is not adjacent to $w-1$, for otherwise

$$x, x_1, \ldots, z_1, w-1, \ldots, u_1, u, \ldots, y_1, w, \ldots, y$$

is an $(x-y)$–hamiltonian path in G. A similar argument applies if $w \in (x_2, z_1)$ is adjacent to y_1 or y_2; while if $w \in (u, y_1)$ is adjacent to y_1 or y_2, we note that $w+1$ is not adjacent to z_1 or z_2 or else an $(x-y)$–hamiltonian path would exist.

Now suppose that $w \in (u_1, z_2)$ is adjacent to y_1 or y_2, then $w-1 \notin N(z_1)$ or $N(z_2)$ or again an $(x-y)$–hamiltonian path would exist.

If, however, there is a neighbor w of y_2 in (x_1, y_2), then clearly $w+1 \notin N(z_1)$ (or $N(z_2)$) or else

$$x, x_2, \ldots, z_1, w+1, \ldots, y_2, w, \ldots, u, u_1, u_1+1, \ldots, y$$

is an $(x-y)$–hamiltonian path in G. If $w \in (x_1, y_2)$ is in $N(y_1)$, then note that the paths

$$Q_1: x, x_2, \ldots, z_1, w+1, \ldots, y_2$$

along with the chord from y_2 to x_2 and the path

$$Q_2: w, \ldots, y_1, \ldots, u, u_1, \ldots, y$$

along with the chord from w to y_1, satisfy the conditions of Lemma 4. Hence, an $(x-y)$–hamiltonian path exists in G, again a contradiction. A similar argument applies when z_2 is adjacent to $w+1$.

Thus, we have defined a function

$$f: V - \{x, u, y_1, x_1, y_2, x_2, z_1, u_1, z_2, u_2\} \rightarrow V - \{x, u, u+1, x_1, x_1+1, z_1-1, z_1, z_2-1, z_2, y\}$$

defined as

$$f(w) = w-1 \text{ for } w \in (u_2, y] \cup (u_1, z_2) \cup (x_2, z_1) \text{ and}$$
$$f(w) = w+1 \text{ for } w \in (u, y_1) \cup (x_1, y_2).$$

that is 1–1 and onto. The function f maps possible adjacencies of y_1 and y_2 to nonadjacencies of v_1 and v_2.

Since z_1 and z_2 are not adjacent to x, y_1, y_2, or u (we assume z_2 was selected that way for now), we again see that the exceptional vertices are potentially x_1, x_2, z_1, u_1, z_2 and u_2. Thus, we have that

$$deg\ \{z_1, z_2\} \leq (n-4) - [deg\ \{y_1, y_2\} - 6] \leq \frac{n+1}{2}.$$

If however, $u_1 = z_2$ (that is, u_1 and u_2 are adjacent on P and so we select $u_1 = z_2$), then we note that y_1 and y_2 cannot be adjacent to u_2 or a hamiltonian $x - y$ path is easily found. Thus, in this case we see that

$$deg\ \{z_1, z_2\} \leq (n-3) - [deg\ \{y_1, y_2\} - 5] \leq \frac{n+1}{2}.$$

Thus, a contradiction is again reached and hence, we conclude that G is hamiltonian-connected. ■

As an example of the sharpness of this result, consider the graph G formed by taking two copies of K_p and identifying two vertices from the first copy with two vertices of the second copy. Then G has order $n = 2p - 2$ and $\delta_2(G) = p = \frac{n+2}{2}$. However, this graph is clearly not hamiltonian-connected as there is no hamiltonian path between the pair of identified vertices. Thus, the value of δ_2 cannot be lowered in Theorem 5 without some other assumptions.

The graph $H = K_3 + 3K_2$ also satisfies the generalized degree condition $\delta_2 \geq 5$ and since H has order $n = 9$, this is a $\delta_2 = \frac{n+1}{2}$ situation. Hence, it is possible that the bound in Theorem 5 can be reduced to $\frac{n+1}{2}$ if n is sufficiently large and if the connectivity of the graph is assumed to be three or more. We conjecture that this is true. However, this cannot be attained from our degree and generalized degree conditions.

In what follows, the graph H of Figure 1 will be helpful.

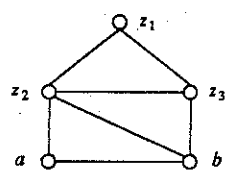

Figure 1. The graph H with vertices as labeled.

Lemma 6. If G is a graph of order $n \geq 6$ with $\delta_2(G) \geq \frac{n+5}{2}$, then H is a subgraph of G.

Proof. If G is complete (or lacks only one edge) the result is immediate, and for $n = 6, 7, 8$, the lower bound on δ_2 implies that G is complete. Thus, we may assume that $n \geq 9$ and that G is not complete. We now show that C_4 is a subgraph of G. Suppose that $xy \notin E(G)$. Then, without loss of generality, we may assume that $deg\ x \geq \frac{n+5}{4}$ (≥ 4 since $n \geq 9$). Note that $N(x)$ can induce at most a matching, for otherwise C_4 is contained in G. Let $x_i, i = 1, 2, 3, 4$ be neighbors of x with x_i not adjacent to x_{i+1}

$(i = 1, 3)$ and suppose that $N'(x_i) = N(x_i) - (N(x) \cup \{x\})$. Further, let $n_i = |N'(x_i)|$, for $1 \le i \le 4$. If $N'(x_i) \cap N'(x_j) \ne \phi$, then C_4 is clearly contained in G. Otherwise, we see that $n_1 + n_2 + 3 \ge \frac{n+5}{2}$ and $n_3 + n_4 + 3 \ge \frac{n+5}{2}$. Hence, $n_1 + n_2 + n_3 + n_4 \ge n - 1$, a contradiction. Thus, C_4 is a subgraph of G.

Now suppose that a, b, c, d, a is a C_4 in G. If either ac or bd is an edge of G, then $K_4 - e$ is a subgraph of G. Suppose not, then

$$|(N(a) \cup N(c)) - \{b, d\}| \ge \frac{n+5}{2} - 2 = \frac{(n-4)+5}{2}$$

and similarly,

$$|(N(b) \cup N(d)) - \{a, c\}| \ge \frac{(n-4)+5}{2}.$$

Thus, in the $n-4$ vertices of $V - \{a, b, c, d\}$, the vertex pairs (a, c) and (b, d) have at least 5 common neighbors (hence, neighbors off the C_4). Then one of the adjacent pairs (a, b), (b, c), (c, d) or (d, a) has at least 2 common neighbors off the C_4. Hence, $K_4 - e$ is a subgraph of G (although it may not be induced).

Suppose that x_1 and x_2 are the (possibly) nonadjacent vertices in the $K_4 - e$ while y_1 and y_2 are the other two vertices of $K_4 - e$. Then

$$|(N(x_1) \cup N(x_2)) - \{x_1, x_2, y_1, y_2\}| \ge \frac{n+5}{2} - 4 = \frac{(n-4)+1}{2}$$

and

$$|(N(y_1) \cup N(y_2)) - \{x_1, x_2, y_1, y_2\}| \ge \frac{(n-4)+1}{2}.$$

Thus, in the $n-4$ vertices of $V - \{x_1, x_2, y_1, y_2\}$, one of the adjacent pairs (y_1, x_1), (x_1, y_2), (y_2, x_2), or (x_2, y_1) has a common neighbor outside of $K_4 - e$. But, then H is clearly a subgraph of G. ■

Theorem 7. If G is a graph of sufficiently large order n with $\delta_2(G) \ge \frac{n+5}{2}$ and $\delta(G) \ge 3$, then G is pancyclic.

Proof. Suppose G satisfies the hypothesis of the Theorem. By Theorem B, G is hamiltonian. If G is not pancyclic, then let t be the largest integer ($4 \le t \le n$) such that for all $s \ge t$, an s-cycle is contained in G, but there is no $(t-1)$-cycle in G. Since G is hamiltonian but not pancyclic, such a t must exist. Also, from Theorem A and Corollary 2, it is easy to see that $G - x$ is hamiltonian for any $x \in V(G)$. By again using Theorem A, or the structure of $H = G - \{x, y\}$ if H is 1-connected, we can also show that H is hamiltonian, hence $t \le n - 2$.

By Lemma 6, H is a subgraph of G. Hence, 3, 4 and 5 cycles are contained in G, thus, $t \ge 7$. Let the vertices of H be a, b, z_1, z_2, z_3 as labeled in Figure 1. Our next goal is to show that G contains a path

$$P: u_1, u_2, \ldots, a, z_2, z_1, z_3, b, \ldots, v_1, v_2$$

containing exactly $t+1$ vertices where at least two distinct vertices of P precede a and two more distinct vertices succeed b.

To see that such a path exists in G, consider the graph $F = (G - \{z_1, z_2, z_3\}) + x$. If $\delta(F) \geq 3$, then since $|V(F)| = n-2$ and $\delta_2(F) \geq \frac{n+5}{2} - 3 + 1 = \frac{(n-2)+3}{2}$, we can apply Theorem 5 to obtain the fact that F is hamiltonian-connected. Thus, there exists an $a-b$ hamiltonian path P^* in F. If we simply choose vertices u_1 and u_2 and v_1 and v_2 along this path so that the segments from $u_1, u_2, \ldots, a$ and $v_1, v_2, \ldots, b$ contain $t-2$ vertices in all, then link these two segments with the path z_2, z_1, z_3, we get the desired path. This is always possible, even if x is adjacent to a or b or at distance two from a or b along the path P^*. In such a case we use an edge from a to the path beyond x and the segment of the path P^* from that point back towards x (see Figure 2). Such an edge exists since $deg_F\, a \geq 3$.

If $\delta(F) < 3$, then there is only one vertex in F with degree less than three, for otherwise the generalized degree condition would be violated. Let y be the vertex with $deg_F\, y < 3$ and consider the following cases.

If $deg_F\, y = 0$, then we form $F' = F - y$. Since y was only adjacent with the vertices z_i, $i = 1, 2, 3$ in G, we see that $\delta(F') \geq 3$, and $\delta_2(F') \geq \frac{(n+5)}{2} - 3 + 1 = \frac{(n-3)+4}{2}$ and so F' is hamiltonian-connected by Theorem 5. Thus, an argument similar to that above produces the desired path P.

Finally, if $deg_F\, y = 1$ or 2, then we form $F'' = G - \{z_1, z_2, z_3, y\}$. Then, $\delta(F'') \geq 3$ and $\delta_2(F'') \geq \frac{(n+5)}{2} - 4 = \frac{(n-4)+1}{2}$. Thus, if F'' is 2-connected, then it is hamiltonian by Theorem A and so $F'' + x$ is hamiltonian-connected and we proceed as before to find the path P. If F'' is only 1-connected (it must be connected from the degree conditions) then since $\delta(F'') \geq 3$, it must be of the form $2K_{(n-5)/2} + w$. Further, the edge ab must lie in one of the two copies of $K_{(n-5)/2}$. It is now easy (since n is large) to find the desired path P. Thus, in all cases the path P can be produced.

Suppose $u_1, u_2, \ldots, a, z_2, z_1, z_3, b, \ldots, v_2, v_1$ is the desired path P. We now consider the distinct pairs u_1 and u_2 and v_1 and v_2.

For each adjacency $w \in (u_1, a]$ of u_1 or u_2, the vertex $w-1$ is not adjacent to v_1 or v_2 or else a $(t-1)$–cycle would be contained in G. This is easily seen since if u_1 is adjacent to w and v_1 is adjacent to $w-1$, a $(t+1)$–cycle is formed using these edges and P. Now replacing z_1 and z_3 by the edge from z_2 to b forms a $(t-1)$–cycle, contradicting our assumption that no $(t-1)$–cycle exists. Similar $(t-1)$–cycles can be formed if u_2 is adjacent to w or if v_2 is adjacent to $w-1$ by omitting 0, 1 or 2 vertices from z_1, z_2 and z_3. Also, similar arguments apply if $w \in (b, v_2]$.

If u_1 or u_2 is adjacent to $z \notin V(P)$, then both v_1 and v_2 are not adjacent to z. If this were not the case, then one of a $(t+2)$–cycle, $(t+1)$–cycle, or t–cycle would be formed from P and the edges from u_i ($i = 1$ or 2) to z and from z to v_j ($j = 1$ or 2). But then replacing the path a, z_2, z_1, z_3, b by the proper subpath from a to b forms a $(t-1)$–cycle, contradicting our assumption.

Finally, if u_1 (or u_2) is adjacent to z_2, then neither v_1 or v_2 can be adjacent to a or a $t-1$ cycle can be found.

Thus, the function

$$f : V - \{z_1, z_3, b, u_1\} \rightarrow V - \{z_1, z_2, z_3, v_2\}$$

defined by

$$f(w) = w - 1 \text{ if } w \in (u_1, a] \text{ or } w \in (b, v_2]$$
$$f(z) = z \text{ if } z \notin V(P_1) \text{ and}$$
$$f(z_2) = a,$$

is 1–1 and onto. Further, this function maps possible adjacencies of the pair u_1 and u_2 onto nonadjacencies of the pair v_1 and v_2 (with the possible exceptions of u_1, z_1, z_3 and b). Thus,

$$\begin{aligned} deg\{v_1, v_2\} &\le n - (deg\{u_1, u_2\} - 4) \\ &\le n - \frac{n+5}{2} + 4 \\ &\le \frac{n+3}{2}, \end{aligned}$$

contradicting the generalized degree condition.

Thus, a contradiction is reached and hence, G is pancyclic. ∎

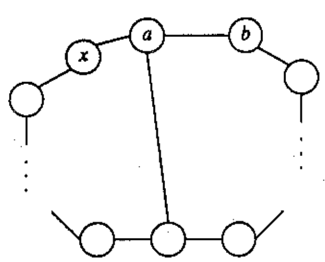

Figure 2 Reshaping to get the path P_1.

Many open questions remain. For example, will higher connectivity assumptions in conjunction with lower generalized degree bounds produce the same conclusions? Another natural problem is to extend these results to δ_k, $k > 2$. Other variations involve changing the type of set S selected. In [4], similar results were found for independent pairs of vertices.

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