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Neighborhood Intersections and a Generalization of Ore's Theorem

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ABSTRACT: For sets of vertices, we consider a form of generalized degree based on neighborhood unions. In particular, for a graph G, the degree of the set $S = \{x_1, \ldots, x_k\}$ is defined to be,

$$deg\ S = |\bigcup_{i=1}^{k} N(x_i)|$$

where S is a set of k vertices in G and $N(x)$ denotes the neighborhood of the vertex x. Clearly, for singletons, this is the ordinary degree of a vertex. Many authors have studied what bounds on generalized degrees force a graph to have a given property. In particular, hamiltonian and other path length problems have been extensively studied.

In this paper, we introduce an added restriction involving neighborhood intersections. Let $IC_2(G) \geq t$, mean that for all pairs of nonadjacent vertices, the intersection of their neighborhoods contains at least t vertices. We show for a given generalized degree sum and intersection condition, various path and cycle properties are implied. In particular, we obtain a generalization of the well-known hamiltonian result of Ore. In addition, we consider similar questions for graphs G containing no induced subgraph isomorphic to $K_{1,3}$.

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Section 1. Introduction.

For standard terms and notation not found here see [11].

The study of cycle and path lengths in graphs has been one of the fundamental problems in graph theory since its inception. Degree conditions have been fundamental to this study. Over the last five years a new approach has been introduced, beginning with the work in [7]. Rather than considering degree sums of pairs (or sets) of nonadjacent vertices, it was proposed to study the effects of bounds on the order of the neighborhood union of pairs (or sets) of nonadjacent vertices and relate this parameter to various cycle and path problems. Several other papers followed [1, 3, 6, 8, 9], suggesting a connection between neighborhood unions and other properties of graphs. Most recently, the study of neighborhood unions has evolved into a generalization of the concept of degree. The combination of generalized degree conditions with other graph properties seems capable of producing generalizations of many fundamental theorems involving ordinary degrees, as well as producing new types of results. This paper continues this line of investigation.

Several forms of generalized degree for sets of vertices (where the sets considered satisfy various conditions) have been used to further the study of a variety of graph properties. In [5] and [7], hamiltonian properties were studied using sets of independent vertices of various sizes and a form of generalized (independent) degree. These generalized degrees were also used in [6] to study matchings and extremal path and cycle lengths. In [3], a Turan-type extremal result was obtained. A survey of recent results using several types of generalized degrees (neighborhood unions) can be found in [12].

In the above mentioned papers, as well as in other related works, the following definition is central. We define the *degree* of a set S as,

$$deg\ S = | \bigcup_{v \in S} N(v) |.$$

For convenience, when the set is a singleton, we will abbreviate this notation with $deg\ x$, the standard notation for the degree of a vertex.

Two of the most fundamental results on hamiltonian graphs are due to Dirac [2] and Ore [13].

Theorem A[2]. If G is a graph of order $p \geq 3$ such that $deg\ x \geq \frac{p}{2}$ for each vertex $x \in V(G)$, then G is hamiltonian.

Theorem B [13]. If G is a graph of order $p \geq 3$ such that for each pair of nonadjacent vertices x and y

$$deg\ x + deg\ y \geq p$$

then G is hamiltonian.

Using the idea of generalized degrees, the following generalizations of Dirac's Theorem [2] were shown. Theorem C uses all 2–sets of vertices in the graph, while Theorem D uses all independent 2–sets.

Theorem C [4]. If G is a 2–connected graph of sufficiently large order p satisfying $deg\ S \geq p/2$ for each 2–set S of vertices, then G is hamiltonian.

Theorem D.[7] If G is a 2-connected graph of order n such that $deg\ S \geq \frac{2p-1}{3}$, for each 2–set S of independent vertices in G, then G is hamiltonian.

In [5], it was shown that similar properties of G can be ascertained under weaker degree conditions by including additional structural conditions. A graph G is said to be H-free if H is not an induced subgraph of G. The following was shown in [5].

Theorem E [5]. If G is a $K_{1,3}$-free graph of order p such that

$$deg\ S \geq \frac{n}{3} + 1$$

for all 2-sets S of vertices in G, then G is hamiltonian.

Finally, when discussing cycles and/or paths we will refer to a closed segment, denoted $[a, b]$ (or open segment denoted (a, b)) as the subpath of vertices on the cycle or path between a and b, inclusive of a and b (exclusive of a and b).

Main Results.

Implicit in the hypothesis of Ore's Theorem is the fact that any two independent vertices have at least two common neighbors. In this paper we explore the effects of this neighborhood intersection condition. To say that G satisfies $IC_t(G) \geq k$ means that for any set of t independent vertices, $x_1, x_2, \ldots, x_t$,

$$\left| \bigcap_{i=1}^{t} N(x_i) \right| \geq k.$$

As above, $IC_t \geq k$ will be used when the context is clear. In the next section we give several results that combine independent degree conditions with an intersection condition to attain hamiltonian results.

We begin with a few simple observations concerning the intersection condition. First, observe that if $IC_t(G) \geq k$, then $IC_s(G) \geq k$, for any $s \leq t$. Also note that if $IC_2(G) \geq 1$, then G is connected and has diameter at most two. We next turn our attention to connectivity.

Proposition 1. If $IC_2(G) = k$ $(k \geq 1)$, then G is k-connected.

Proof. Clearly G must be connected. Any cutset of G partitions the remaining vertices into at least two disjoint components. If there is a cutset with fewer than k vertices, then there exists a pair of independent vertices whose neighborhoods intersect in fewer than k vertices, contradicting the fact that $IC_2(G) = k$. Hence, G must be k-connected. ■

This result can be seen to be sharp by considering the class of graphs obtained by identifying two complete graphs at $k - 1$ vertices. The next result shows an Ore-type (Theorem B) relationship between these conditions and hamiltonian graphs. We denote as A_{t_1}, an independent set of t_1 vertices. We say a collection of sets $A_{t_1}, \ldots, A_{t_r}$, where $|A_{t_i}| = t_1$, are *distinct independent sets* if for each i, A_{t_i} is an independent set of vertices and the intersection of any two is empty.

Theorem 2. Let G be a graph of order p satisfying $IC_2(G) \geq k \geq 2$. If for some pair t_1, t_2 of positive integers satisfying $2 \leq \sum_{i=1}^{2} t_i \leq k + 1$, each pair of distinct independent sets A_{t_1}, A_{t_2} satisfies

$$deg\ A_{t_1} + deg\ A_{t_2} \geq p,$$

then G is hamiltonian.

Proof. Since G is k–connected ($k \geq 2$), G must contain a cycle. Let C be a longest cycle in G and suppose C does not span $V(G)$. Choose some $x \in V(G) - V(C)$ which is adjacent to some vertex, say v_1, on C. Let p_1 (also denoted $v_1 - 1$) and s_1 (also denoted $v_1 + 1$) be the predecessor and successor of v_1 on C with respect to some fixed ordering of the vertices of C. Clearly, x is not adjacent to either p_1 or s_1, for then a cycle longer than C would result. Since $IC_2 \geq k$, the pair x and s_1 must have $k - 1$ more (v_1 is already adjacent to both) common neighbors. If any of these common neighbors is off C, a cycle longer than C is immediate. Thus, all common neighbors of x and s_1 are on C.

Let $v_2, v_3, \ldots, v_k \in V(C)$ be $k - 1$ other common neighbors of x and s_1 given in order with respect to the orientation of C. Also, let $p_2, p_3, \ldots, p_k$, and $s_2, \ldots, s_k$ be the corresponding predecessors and successors of $v_2, v_3, \ldots, v_k$.

Note that x is not adjacent to any of the vertices p_i or s_j ($1 \leq i, j \leq k$) or a cycle longer than C results. Also note that s_i and s_j ($i \neq j$) (as well as p_i and p_j) must be nonadjacent, for otherwise, assuming $i < j$

$$x,\ v_1, p_1, \ldots, s_j, s_i, \ldots, v_j, s_1, \ldots, v_i, x$$

would be a cycle longer than C.

We now consider the independent sets

$$A_{t_1} = \{ x, p_2, \ldots, p_{t_1} \} \text{ and}$$
$$A_{t_2} = \{ s_{t_1+1}, \ldots, s_{t_2} \}$$

(Note that if $t_1 + t_2 = k + 1$, then we begin A_{t_2} with vertex s_{t_1} instead.) We proceed by showing that for every vertex in $N(A_{t_2})$, there is a distinct vertex not in $N(A_{t_1})$. To do this we consider each adjacency, say z, of an arbitrary vertex $s_j \in A_{t_2}$. We show that corresponding to each such z, there is a distinct vertex that cannot be in $N(A_{t_1})$. Using this fact, we obtain a contradiction to the degree sum condition. Thus, suppose that s_j is an arbitrary member of A_{t_2}.

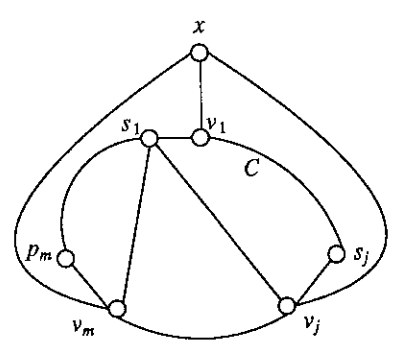

Figure 1. The adjacency situation.

CASE 1. Suppose that $z \in N(s_j) - V(C)$.
We claim that $z \notin N(A_{t_1})$. Clearly, x is not adjacent to z, or a cycle longer than C is immediate. If z is adjacent to some $p_m \in A_{t_1} \cap V(C)$, then

$$x, v_1, p_1, \ldots, s_j, z, p_m, \ldots, s_1, v_m, \ldots, v_j, x$$

is a cycle longer than C, producing a contradiction. Hence, in either situation, $z \notin N(A_{t_1})$. Thus, we may suppose that if z is in $N(A_{t_2})$, then $z \in V(C)$.

CASE 2. Suppose that $z \in N(s_j) \cap V(C)$.
Subcase 1a. Suppose that $z \in (s_j, v_1]$.
Then consider the predecessor $z-1$ of z along C. If $p_m \in A_{t_1} \cap V(C)$ (for some m, $2 \leq m \leq t_1$) is adjacent to $z-1$, then

$$x, v_1, p_1, \ldots, z, s_j, \ldots, z-1, p_m, \ldots, s_1, v_m, \ldots, v_j, x$$

would be a cycle longer than C, a contradiction. If x is adjacent to $z-1$, then again we see that

$$x, z-1, \ldots, s_j, z, \ldots, v_1, \ldots, v_j, x$$

is a cycle longer than C, a contradiction. Hence, $z-1 \notin N(A_{t_1})$.
Subcase 1b. Suppose that $z \in [v_m, v_j]$.
Again consider the predecessor $z-1$ of z along C, and suppose that $z-1$ is adjacent to an arbitrary $p_m \in A_{t_1}$ ($p_m \neq x$). Then we see that

$$x, v_1, \ldots, s_j, z, \ldots, v_j, s_1, \ldots, p_m, z-1, \ldots, v_m, x$$

is a cycle longer than C, a contradiction. Similarly, if x is adjacent to $z-1$, then

$$x, z-1, \ldots, s_1, v_j, \ldots, z, s_j, \ldots, v_1, x$$

is a cycle longer than C. Thus, we again see that $z-1 \notin N(A_{t_1})$.

Subcase 1c. Suppose that $z \in (v_1, v_m)$.
Then if $p_m \in N(A_{t_1})$ ($p_m \neq x$) is adjacent to $z-1$ we see that

$$x, v_1, \ldots, s_j, z, \ldots, p_m, z-1, \ldots, s_1, v_j, \ldots, v_m, x$$

would be a cycle longer than C, a contradiction. If x is adjacent to $z-1$, then

$$x, z-1, \ldots, v_1, \ldots, s_j, z, \ldots, v_j, x$$

is a cycle longer than C, again a contradiction. Thus, we again see that $z-1$ is not in $N(A_{t_1})$.

Thus, in all cases, there is a distinct vertex not in $N(A_{t_1})$ for every vertex in $N(A_{t_2})$. We summarize this relationship with the function $f\colon N(A_{t_2}) \to \overline{N(A_{t_1})}$ (the complement of $N(A_{t_1})$) defined as

$$f(z) = z \quad \text{if } z \in N(A_{t_1}) - V(C) \quad \text{and}$$

$$f(z) = z-1 \quad \text{if } z \in [N(A_{t_1}) \cap V(C).$$

Thus, f maps adjacencies of vertices in A_{t_2} to nonadjacencies of vertices in A_{t_1}. Since the map f is 1–1, we conclude that

$$deg\ A_{t_1} \le p - t_1 - deg\ A_{t_2}.$$

Further, since $t_1 \ge 1$, we see that

$$deg\ A_{t_1} + deg\ A_{t_q} \le p - 1,$$

contradicting our hypothesis. Therefore, G must contain a hamiltonian cycle. ■

Next we show that the bounds on these parameters, both the *deg* and *IC* cannot be reduced. First consider the complete bipartite graph $G = K_{n,n-1}$. This graph has

$$deg\ S_1 + deg\ S_2 = 2n - 2 = |V(G)| - 1.$$

for distinct arbitrary independent sets S_1 and S_2 (subsets of the larger partite set). It also satisfies the condition

$$IC_2 = \frac{|V(G)| - 1}{2}.$$

However, G is not hamiltonian. Hence, even reducing the degree sum condition while increasing the intersection condition will not maintain the hamiltonian property.

With Theorem 2 established, we now point out several natural corollaries. The first is of course Ore's Theorem, and hence Dirac's Theorem as well.

Corollary 3.(Ore[13]) If G is a graph of order p such that for each pair of nonadjacent vertices x and y,

$$deg\ x + deg\ y \ge p,$$

then G is hamiltonian.

Proof. Let $t_1 = 1$ and $t_2 = 1$ and recall that $IC_2 \ge 2$ (see observations before Theorem 2). ■

Corollary 4. If G is a graph of order p which satisfies

$$deg\ A \ge \frac{p}{2} \quad \text{and} \quad IC_2 \ge k,$$

for each independent vertex set A where ($1 \le |A| \le \frac{k}{2}$), then G is hamiltonian.

Corollary 5. (Dirac[2]) If G is a graph of order p satisfying $deg\ x \ge \frac{p}{2}$ for each vertex x, then G is hamiltonian.

Corollary 6. If G is a graph of order p satisfying $IC_2 \ge k - 1$ and if for each pair of distinct independent sets A_1 and A_2 with $|A_1| + |A_2| \le k$,

$$deg\ A_1 + deg\ A_2 \ge p - 1 \quad \text{and}$$

then G is traceable.

Proof. Let G have order p and satisfy the stated hypotheses. Consider the graph $H = G + x$, with order $p + 1$. If A_1 and A_2 are two distinct independent sets of vertices in H, then $\sum_{i=1}^{2} deg\ A_i \ge p + 1$ and

$IC_2(H) \geq k$. Hence, H is hamiltonian by Theorem 2. Now simply remove x from a hamiltonian cycle of H to obtain a hamiltonian path in G. Therefore, G is traceable. ■

These corollaries are also sharp. This can be seen by considering examples similar to those used to show the sharpness of Theorem 2.

In [10], $K_{1,3}$–free graphs and their relation to hamiltonian results were studied. In particular, the following was shown.

Theorem F ([10]). If G is a 2–connected, $K_{1,3}$–free graph with diameter at most 2, then G is hamiltonian.

This result can be restated in our present terminology as follows.

Theorem G ([10]). If G is a 2–connected, $K_{1,3}$–free graph with $IC_2(G) \geq 1$, then G is hamiltonian.

We now extend this result, but first we give the following definition. A graph G of order p is *pancyclic* if G contains cycles of all lengths m, $3 \leq m \leq p$.

Theorem 7. If G has order $p \geq 5$ and is $K_{1,3}$–free with $IC_2 \geq 2$, then G is pancyclic.

Proof. If G is complete, then G is pancyclic. Hence, we assume that w_1 and w_2 are independent vertices in G. Since $IC_2 \geq 2$, there exist vertices w_3 and w_4 that are common neighbors of w_1 and w_2 (hence, a 4–cycle exists). Since G is connected and $p \geq 5$, there exists a vertex w_5 adjacent to one of w_i, $i = 1, 2, 3, 4$. Without loss of generality assume w_5 is adjacent to w_1. But now w_1, w_3, w_4, w_5 induces a $K_{1,3}$ unless one of the edges w_3w_4, w_3w_5 or w_4w_5 is in G. In any case, a triangle is formed.

Let C be any nonhamiltonian cycle of length $l \geq 4$ and suppose that no cycle of length $l+1$ exists in G. Further, suppose that $x \notin V(C)$ but that x is adjacent to at least one vertex on C. Suppose that $y \in V(C)$ and that x is adjacent to y. Further, let $y-1$ be the predecessor of y along C. If x were adjacent to $y-1$, a cycle exactly one vertex longer than C would result. Thus, x and $y-1$ are not adjacent. Since $IC_2 \geq 2$, x and $y-1$ must have an additional neighbor, say z, in common.

CASE 1. Suppose that $z \in V(C)$.
Let $z+1$ and $z-1$ be the successor and predecessor of z along C respectively. Since G is $K_{1,3}$–free, $z+1$ and $z-1$ must be adjacent. But this implies that

$$x,\ z,\ y-1, \ldots, z+1,\ z-1, \ldots, y,\ x$$

is a cycle exactly one vertex longer than C, a contradiction.

CASE 2. Suppose that $z \notin V(C)$.
In this case we need to also consider the symmetric case of the neighborhood intersection between x and $y+1$, where $y+1$ is the successor of y along C. If x and $y+1$ have a common neighbor on C, than Case 1 applies, so we may assume they have a common neighbor, say z', off C. Suppose that $z \neq z'$. Neither z nor z' is adjacent to y, for than a cycle exactly one vertex longer than C would result. But since G is $K_{1,3}$–free, z must be adjacent to z'. This implies that

$$z, z', y+1, y+2, \ldots, y-1, z$$

is a cycle exactly one vertex longer than C, as we have removed y and added z and z'. Now suppose that $z = z'$. Then since G is $K_{1,3}$–free, one of the edges yz, $(y+2)z$ or $y(y+2)$ must be in G. In any case, a cycle one vertex longer than C is easily found.

Hence, cycles of all possible lengths must exist and so G is pancyclic. ∎

To say that a graph has property $P_{m,d}$ means that between every pair of vertices there are at least m vertex disjoint paths of length at most d. Graphs that satisfy property $P_{m,d}$ for some m and d are said to have a *Menger Path System*. In [5], the existence of a Menger Path System in relation to neighborhood unions was studied. We close with a result on Menger Systems using our intersection condition.

Theorem 8. If G has order $p \geq k+1$ and $IC_2 \geq k$, then G has $P_{k,3}$.

Proof. We will show that for every pair of vertices, there are k paths of length at most 3. Choose $x, y \in V(G)$. If x is not adjacent to y, then since $IC_2 \geq k$, there are in fact k paths of length 2 between x and y.

Suppose that x is adjacent to y. Since G contains at least $k+1$ vertices, if x and y are adjacent to all other vertices, there would be at least $k-1$ paths of length 2 and one path of length 1. We may assume that there exists $z \in V(G)$ so that z is not adjacent to y. By the hypothesis, z and y have at least k common neighbors. Since x may be one of these common neighbors, we will only consider the $k-1$ others, say $y_1, y_2, \ldots, y_{k-1}$. We partition them into two sets,

$$A = \{ y_i \mid xy_i \in E(G) \} \quad \text{and} \quad B = \{ y_j \mid xy_j \notin E(G) \}.$$

For each element y_j in B, there is a set of vertices B_j, with at least as many elements as B, that is a common neighbor of x and y_j, and distinct from A. We can find a set of distinct representatives from this set system. Thus, between x and y there is a path of length 1, $|A|$ paths of length 2 and $|B|$ paths of length 3. Consequently, there are k vertex disjoint paths of length at most 3 from x to y, regardless of the situation. Thus, G has property $P_{m,d}$. ∎

Note that although we cannot assure that the intersection condition gives more than k–connectivity (recall Proposition 1), this last result does give a "stronger" form of connectivity condition.

Conclusion. This paper is only a start at investigating the significance of neighborhood intersection conditions. Other properties of graphs can be considered, as has been done with degrees and generalized degrees of several types. Such conditions include matchings, (nonhamiltonian) path and cycle lengths and chromatic number just to mention a few. Additionally, as has been done for generalized degrees, there is no reason to restrict attention to just nonadjacent vertices.

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