Discrete Mathematics 91 (1991) 33–43
North-Holland

Neighborhood conditions and edge-disjoint perfect matchings

R.J. Faudree*
Department of Mathematical Sciences, Memphis State University, Memphis, TN 38152, USA

R.J. Gould*
Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

L.M. Lesniak
Department of Mathematics and Computer Science, Drew University, Madison, NJ 07940, USA

Received 2 February 1988
Revised 1 December 1989

Abstract

Faudree, R.J., R.J. Gould and L.M. Lesniak, Neighborhood conditions and edge-disjoint perfect matchings, Discrete Mathematics 91 (1991) 33–43.

A graph G satisfies the neighborhood condition $\mathrm{ANC}(G) \geq m$ if, for all pairs of vertices of G, the union of their neighborhoods has at least m vertices. For a fixed positive integer k, let G be a graph of even order n which satisfies the following conditions: $\delta(G) \geq k+1$; $\mathcal{K}_1(G) \geq k$; and $\mathrm{ANC}(G) \geq n/2$. It is shown that if n is sufficiently large then G contains k edge-disjoint perfect matchings.

A *matching* in a graph is a set of edges of which no two have a common incident vertex. An *s-matching* is a matching with s edges and a *perfect matching* in a graph of order n is a matching with $n/2$ edges. The classic theorem of Tutte [8] characterizing those graphs with perfect matchings states that a nontrivial graph G has a perfect matching if and only if, for every proper subset S of $V(G)$, the number of components of $G-S$ with an odd number of vertices is at most $|S|$. Anderson's proof of Tutte's Theorem [1] employs Hall's Theorem [5], one form of which can be stated as: Let G be a bipartite graph with partite sets V_1 and V_2, where $|V_1| = |V_2|$. Then G contains a perfect matching if and only if for every subset S of V_1,

$$|N_G(S)| \geq |S|,$$

where $N_G(S)$ denotes the set of all vertices adjacent to at least one vertex of S.

* Research supported under ONR Contract #N00014-88-K-0070.

Recently, a number of 'neighborhood conditions' guaranteeing s-matchings in graphs have been obtained. For a vertex x of a graph G, let $N_G(x)$ denote $N_G(\{x\})$. In [2] it was shown that if $|N_G(x) \cup N_G(y)|$ is sufficiently large for every pair x, y of non-adjacent vertices of G, then G contains an s-matching. ('Sufficiently large' is a function of s and the number of vertices of G.) Later in [3], a related result gave a condition on neighborhood unions of pairs of nonadjacent vertices that guarantees many edge-disjoint perfect matchings in a graph. In [4], it was shown that if G is a connected graph of order n and $|N_G(x) \cup N_G(y)| \geqslant s$ for *all* pairs x, y of vertices of G, $1 \leqslant s \leqslant n/2$, then G contains an s-matching. In particular, if G is connected and $|N_G(x) \cup N_G(y)| \geqslant n/2$ for *all* pairs x, y of vertices of G, then G has a perfect matching. Here we extend this result.

A graph G satisfies the *all pairs neighborhood condition* $\mathrm{ANC}(G) \geqslant m$ if, for each pair x, y of vertices of G, we have

$$|N_G(x) \cup N_G(y)| \geqslant m.$$

Theorem 1. *Let k be a positive integer and G a graph of even order n which satisfies the following conditions:*

the minimum degree $\delta(G)$ of G is at least $k+1$; (1)
the edge-connectivity $\mathcal{K}_1(G)$ is at least k; and (2)
$\mathrm{ANC}(G) \geqslant n/2$. (3)

Then if n is sufficiently large, G contains k edge-disjoint perfect matchings.

The following examples illustrate that each of conditions (1), (2), and (3) is necessary for G to contain k edge-disjoint perfect matchings. If G is the complete bipartite graph $K(n/2-1,\ n/2+1)$, then for n sufficiently large G satisfies conditions (1) and (2), but not (3). In this case, G contains no perfect matchings. Next, let G be the graph obtained by adding $k-1$ edges between two disjoint copies of the complete graph $K_{n/2}$. Then for $n \equiv 2 \pmod 4$ and n sufficiently large, G satisfies (1) and (3) but not (2), and the maximum number of edge-disjoint perfect matchings in G is $k-1$. Finally, let G be any graph obtained by identifying one vertex of a copy of $K_{n/2}$ with one vertex of another copy of $K_{n/2}$ and then adding a vertex x of degree k so that in the resulting graph, x is adjacent to the only vertex of degree $n-1$. Then for $n \equiv 0 \pmod 4$ and n sufficiently large, G satisfies (2) and (3) but not (1), and the maximum number of edge-disjoint perfect matchings in G is $k-1$.

The following results will be useful in the proof of Theorem 1.

Theorem A [4]. *If G is a 2-connected graph of order n sufficiently large which satisfies* $\mathrm{ANC}(G) \geqslant n/2$, *then G is hamiltonian.*

Theorem B [7]. *If* ex$(n, K(s, s))$ *denotes the maximum number of edges in a graph of order n which does not contain the complete bipartite graph* $K(s, s)$, *then*

$$\mathrm{ex}(n, K(s, s)) \leqslant \tfrac{1}{2}(s-1)^{1/s} n^{(2-1/s)} + \mathrm{O}(n).$$

Theorem C [6]. *If G is a spanning subgraph of the complete bipartite graph* $K(n/2, n/2)$ *and* $\delta(G) \geqslant n/4$, *then G has a perfect matching.*

Lemma 1. *Let k be a fixed positive integer and G a graph of odd order n which satisfies* $\mathrm{ANC}(G) \geqslant (n-1)/2 + 2k$. *Then for any sequence* $u_1, u_2, \ldots, u_k$ *of vertices of G (where the* u_i*'s are not necessarily distinct), there are k edge-disjoint matchings* $M_1, M_2, \ldots, M_k$ *such that for* $i = 1, 2, \ldots, k$, M_i *is a perfect matching of* $G - u_i$.

Proof. We first observe that $G - u_1$ has order $n-1$ and satisfies $\mathrm{ANC}(G - u_1) \geqslant (n-1)/2 + 2k - 1 \geqslant (n-1)/2 + 1$. This implies that $G - u_1$ is connected and, as indicated earlier, that $G - u_1$ has a perfect matching. Assume now that for some t, $1 \leqslant t < k$, we have constructed the desired matchings $M_1, M_2, \ldots, M_t$. Consider

$$G' = G - \left(\bigcup_{i=1}^{t} M_i\right) - u_{t+1}.$$

Then G' has order $n-1$ and satisfies

$$\begin{aligned}\mathrm{ANC}(G') &\geqslant (n-1)/2 + 2k - (2t+1) \geqslant (n-1)/2 + 2k - (2(k-1)+1)\\ &\geqslant (n-1)/2 + 1.\end{aligned}$$

Again, this implies that G' has a perfect matching M_{t+1} and the proof is complete. □

Lemma 2. *Let t and k be positive integers and let G be a graph of order n satisfying* $\mathrm{ANC}(G) \geqslant t$. *Then for n sufficiently large, G contains k edge-disjoint t-matchings.*

Proof. Since $\mathrm{ANC}(G) \geqslant t$ it follows, of course that $|N_G(x) \cup N_G(y)| \geqslant t$ for every pair x, y of non-adjacent vertices of G. It follows from Theorem 1(a) and (b) of [2] that for n sufficiently large, G contains at least one t-matching M_1. Suppose, then, that edge-disjoint t-matchings $M_1, M_2, \ldots, M_p$ have been constructed, $p < k$, and let $G' = G - \bigcup_{i=1}^{p} M_i$. Let F be a maximum matching in G'. We wish to show that $|F| \geqslant t$. Suppose, to the contrary, that $|F| < t$. Let W be the set of vertices of G' incident with no edge of F. Then, by the maximality of F, no two vertices of W are adjacent in G'. Now, since at most $2tp$ vertices of W are incident (in G) to edges in $\bigcup_{i=1}^{p} M_i$, it follows that for n sufficiently large there are at least four vertices in W incident with none of the edges in $\bigcup_{i=1}^{p} M_i$. Let W'

be the set of these vertices. Thus $|N_{G'}(u) \cup N_{G'}(v)| \geq t$ for every u, v in W'. This implies, however, that for some edge $e = xy$ in F and some u, v in W', both ux and vy are edges of G'. But then $F - \{xy\} \cup \{ux, vy\}$ is a matching in G', which contradicts the maximality of F. □

Proof of Theorem 1. Assume first that G has a cutvertex v. Then, since G satisfies $\mathrm{ANC}(G) \geq n/2$, we must have that $G - v$ consists of exactly two complete components A and B, one of order $n/2 - 1$ and the other of order $n/2$. Since $\mathcal{K}_1(G) \geq k$, if follows that in G the vertex v is adjacent to at least k vertices of A and k vertices of B. Certainly, for n sufficiently large, G has k edge-disjoint perfect matchings. Thus we may assume that G is 2-connected. By Theorem A, the graph G is hamiltonian and so contains at least two edge-disjoint perfect matchings. Thus if the result fails to hold, $k \geq 3$ and we may assume that G is an edge-maximal counterexample.

Let x, y be non-adjacent vertices of G. The maximality of G implies that the graph $G + xy$ contains k edge-disjoint perfect matchings $M_1, M_2, \ldots, M_k$ with $xy \in M_k$. Furthermore, if H is the graph obtained from G by removing $M_1, M_2, \ldots M_{k-1}$, then H contains an $(n/2 - 1)$-matching but no perfect matching. It follows, from Tutte's Theorem and the maximality of G, that there is a proper subset S of $V(H)$ such that $H - S$ has exactly $s + 2$ odd components, where $s = |S| \geq 0$ and x and y are in different components of $H - S$. Furthermore, $\deg_H z = \deg_G z - (k - 1)$ for every vertex z of H. In particular, if u and v are any two vertices of H, then $|N_H(u) \cup N_H(v)| \geq n/2 - 2(k - 1)$, i.e., $\mathrm{ANC}(H) \geq n/2 - 2(k - 1)$.

Assume first that $s = 0$. Thus H has exactly two odd components and, perhaps, some even components. Since $\delta(G) \geq k + 1$, it follows that $\delta(H) \geq 2$ and so each component of H has at least three vertices. Let C be any component of H and let u, v be two vertices in C. Since $\mathrm{ANC}(H) \geq n/2 - 2(k - 1)$, it follows that C has at least $n/2 - 2(k - 1)$ vertices. This implies, for n sufficiently large, that H has exactly two components C_1 and C_2, each of odd order at most $n/2 + 2(k - 1)$, and $\mathrm{ANC}(C_i) \geq n/2 - 2(k - 1)$ for $i = 1, 2$. Thus, for n sufficiently large,

$$\mathrm{ANC}(C_i) \geq n/2 - 2(k - 1) \geq (n/2 + 2(k - 1) - 1)/2 + 2k,$$

and so Lemma 1 applies to each of C_1 and C_2. Finally, since $\mathcal{K}_1(G) \geq k$, there are at least k edges in G between the vertices of C_1 and C_2. This, however, together with Lemma 1, implies that G has k edge-disjoint perfect matchings, producing a contradiction. Thus, $s \geq 1$.

Let $C_1, C_2, \ldots, C_{s+2}$ be the odd components of $H - S$, where $n_i = |C_i|$ for $1 = 1, 2, \ldots, s + 2$, and $n_1 \leq n_2 \leq \cdots \leq n_{s+2}$.

We first show that $n_1 = 1$. Assume, to the contrary, that $n_1 \geq 3$. Then

$$n \geq \sum_{i=1}^{s+2} n_i + s \geq 3(s + 2) + s,$$

implying that $s < n/4$. Now, let $u, v \in V(C_1)$. Since

$$|N_H(u) \cup N_H(v)| \geq n/2 - 2(k-1)$$

and

$$N_H(u) \cup N_H(v) \subseteq V(C_1) \cup S,$$

it follows that $n_1 + s \geq n/2 - 2(k-1)$. Furthermore, since C_1 is the smallest odd component of $H - S$, necessarily $n_1 \leq (n-s)/(s+2)$. Consequently,

$$\frac{(n-s)}{(s+2)} + s \geq \frac{n}{2} - 2(k-1).$$

Simplifying, we find that

$$ns \leq 2s^2 + 2s + 4(k-1)(s+2).$$

Since $s \geq 1$, have

$$n \leq 2s + 2 + \frac{4(k-1)(s+2)}{s}.$$

Since $s < n/4$, for n sufficiently large,

$$2s + 2 + \frac{4(k-1)(s+2)}{s} < n,$$

and we reach a contradiction. Thus, $n_1 = 1$.

Now, consider the case $n_2 > 1$. We first show that $s = 1$. Assume, to the contrary, that $n_2 \geq 3$ and $s \geq 2$. Since $n_1 = 1$ and $n_2 \geq 3$, we have

$$n \geq \sum_{i=1}^{s+2} n_i + s \geq 3(s+1) + s + 1,$$

implying that $s < n/4$. Choose $u, v \in V(C_2)$. Since

$$N_H(u) \cup N_H(v) \subseteq V(C_2) \cup S$$

and

$$|N_H(u) \cup N_H(v)| \geq n/2 - 2(k-1),$$

we have $n_2 + s \geq n/2 - 2(k-1)$. Also, since $n_2 \leq n_3 \leq \cdots \leq n_{s+2}$, it follows that $n_2 \leq (n-1-s)/(s+1)$. Thus,

$$\frac{(n-1-s)}{(s+1)} + s \geq \frac{n}{2} - 2(k-1),$$

and so

$$n(s-1) \leq 2s^2 - 2 + 4(k-1)(s+1).$$

Since $s \geqslant 2$, we have

$$n \leqslant \frac{(2s^2-2)}{(s-1)} + \frac{4(k-1)(s+1)}{(s-1)}.$$

As before, $s < n/4$ implies that for n sufficiently large,

$$\frac{(2s^2-2)}{(s-1)} + \frac{4(k-1)(s+1)}{(s-1)} < n.$$

Thus, if $n_2 > 1$, then $s = 1$. But then in H, the single vertex z of C_1 has degree at most one, so that $\deg_G z \leqslant k$. This contradicts $\delta(G) \geqslant k+1$. We conclude that $n_2 = 1$.

Thus $n_1 = n_2 = 1$. Let u, v be the vertices in C_1 and C_2. Then $|N_H(u) \cup N_H(v)| \geqslant n/2 - 2(k-1)$ and $N_H(u) \cup N_H(v) \subseteq S$, so that $s \geqslant n/2 - 2(k-1)$. Furthermore, since $H - S$ has at least $s+2$ vertices, we have $s < n/2$.

We may assume then that for every pair x, y of non-adjacent vertices of G there are $k-1$ perfect matchings $M_1, M_2, \ldots, M_{k-1}$ whose removal from G results in a graph H with the following properties:

(i) there is a set S of s vertices of H whose removal results in a graph with exactly $s+2$ odd components;

(ii) $n/2 - 2(k-1) \leqslant s < n/2$;

(iii) x and y belong to different components of $H - S$.

For each such pair x, y choose one such graph and denote it by $H_{x,y}$ and let $S_{x,y}$ denote the corresponding set S.

We next observe that

$$xy \notin E(G) \Rightarrow |N_G(x) \cup N_G(y)| < n/2 + 4(k-1). \tag{4}$$

This follows by considering $H = H_{x,y}$ and $S = S_{x,y}$. Then

$$|N_G(x) \cup N_G(y)| \leqslant 2(k-1) + |N_H(x) \cup N_H(y)|.$$

Also, if C_x is the component of $H - S$ containing x and C_y is the component of $H - S$ containing y, then

$$|N_H(x) \cup N_H(y)| \leqslant s + |C_x| + |C_y| - 2.$$

However, $|C_x| + |C_y| \leqslant n - 2s$ and, since $s \geqslant n/2 - 2(k-1)$, it follows that

$$|N_H(x) \cup N_H(y)| \leqslant n/2 + 2(k-1) - 2.$$

Thus,

$$|N_G(x) \cup N_G(y)| < n/2 + 4(k-1).$$

Select non-adjacent vertices u, v of G and consider $H = H_{u,v}$. Let $A = S_{u,v}$. Then $n/2 - 2(k-1) \leqslant |A| < n/2$. Furthermore, $H - A$ has $|A| + 2$ odd components. Suppose that m of these odd components contain three or more vertices.

Then

$$n \geq 3m + (|A| + 2 - m) + |A|,$$

so that

$$m \leq (n - 2|A| - 2)/2 \leq (4(k-1) - 2)/2 = 2(k-1) - 1.$$

Thus $H - A$ has at least $|A| + 2 - m$ isolated vertices, where

$$\begin{aligned} |A| + 2 - m &\geq n/2 - 2(k-1) + 2 - 2(k-1) + 1 \\ &= n/2 - 4(k-1) + 3. \end{aligned}$$

Let B be the set of isolated vertices in $H - A$ that have degree at least $n/4$ in G. Since $\mathrm{ANC}(G) \geq n/2$, at most one vertex of G has degree less than $n/4$. Thus

$$|B| \geq n/2 - 4(k-1) + 2.$$

We conclude that G has $k - 1$ edge-disjoint perfect matchings whose removal results in a graph H with disjoint sets A and B of vertices such that:

(i) $n/2 - 2(k-1) \leq |A| < n/2$;

(ii) $|B| \geq n/2 - 4(k-1) + 2$;

(iii) in H, each vertex of B is adjacent only to vertices of A; and

(iv) in H, each vertex of B is adjacent to at least $n/4 - (k-1)$ vertices of A.

Now, let $\bar{G}(A, B)$ denote the bipartite graph with vertex set $A \cup B$ and edge set $\{ab \mid a \in A, b \in B, \text{ and } ab \notin E(G)\}$. Then $K(2(k-1), 2(k-1))$ is not a subgraph of $\bar{G}(A, B)$; otherwise, select two vertices x and y of B that are vertices in the copy of $K(2(k-1), 2(k-1))$ in $\bar{G}(A, B)$. Then, by (i) and (iii) above,

$$|N_H(x) \cup N_H(y)| < n/2 - 2(k-1),$$

which contradicts $\mathrm{ANC}(G) \geq n/2$.

Thus, an application of Theorem B yields that $\bar{G}(A, B)$ has fewer than $c_k n^{2-1/(2(k-1))}$ edges, where c_k is a constant depending on k.

Let x be a vertex of A which is adjacent, in G, to at least $n/4 + 5(k-1)$ vertices of B, and let y be any vertex of B. Then

$$\begin{aligned} |N_G(x) \cup N_G(y)| &\geq |N_G(x) \cup N_H(y)| \\ &\geq \frac{n}{4} + 5(k-1) + \frac{n}{4} - (k-1) \\ &= \frac{n}{2} + 4(k-1). \end{aligned}$$

By (4), it follows that $xy \in E(G)$. Thus, in G, if a vertex of A is adjacent to at least $n/4 + 5(k-1)$ vertices of B, then it is adjacent to every vertex of B.

Let m denote the number of vertices of A which are adjacent in G to fewer than $n/4 + 5(k-1)$ vertices of B. Thus, in $\bar{G}(A, B)$, each of these vertices of A is adjacent to more than $n/4-9(k-1)+2$ vertices of B, so that $\bar{G}(A, B)$ contains more than $m(n/4-9(k-1)+2)$ edges. However, $\bar{G}(A, B)$ has fewer than $c_k n^{2-1/(2(k-1))}$ edges. Thus, $m \leqslant c_k' n^{1-1/(2(k-1))}$, where c_k' is a constant depending only on k. Let A' denote the vertices in A which are adjacent in G to all vertices of B. Then $|A'| \geqslant n/2 - d_k n^{1-1/(2(k-1))}$, where d_k is a constant depending only on k, $|B| \geqslant n/2-4(k-1)+2$, and, in G, every vertex of A' is adjacent to every vertex of B. Consider any vertex $x \in V(G)-(A' \cup B)$ with $\deg_G x \geqslant n/4$. Then x is adjacent in G to at least $n/48$ vertices of A' or $n/48$ vertices of B for n sufficiently large. This follows from the fact that $|A' \cup B| \geqslant n/3+n/2-4(k-1)+2$ for n sufficiently large. Thus $|V(G)-(A' \cup B)| \leqslant n/6+4(k-1)-2$. Since $\deg_G x \geqslant n/4$, it must be that x is adjacent to at least

$$n/4-(n/6+4(k-1)-2)=n/12-4(k-1)+2>n/24$$

vertices of $A' \cup B$. If x is adjacent to at least $n/48$ vertices of A', then x is adjacent to every vertex of A'; otherwise, there is a vertex z in A' which is not adjacent to x in G but for which

$$|N_G(x) \cup N_G(z)| \geqslant \frac{n}{48}+\frac{n}{2}-4(k-1)+2 \geqslant \frac{n}{2}+4(k-1),$$

for n sufficiently large, contradicting (4). Similarly, if x is adjacent to $n/48$ vertices of B, then x is adjacent to every vertex of B. Inductively, we conclude that there are disjoint sets A'' and B'' of vertices of G such that:

(i) $n-1 \leqslant |A'' \cup B''| \leqslant n$;

(ii) in G, every vertex of A'' is adjacent to every vertex of B''; and

(iii) $n/2-d_k n^{1-1/(2(k-1))} \leqslant |A''| \leqslant |B''|$.

Note that there may be adjacent vertices of B''.

We next show that A'' and B'' can be chosen satisfying (i), (ii) and (iii) and satisfying $|B''|-|A''|<12k$. Suppose, to the contrary, that for all disjoint sets A'' and B'' of vertices satisfying (i), (ii) and (iii), we also have $|B''|-|A''| \geqslant 12k$. Choose one such pair A'', B'' for which $|B''|-|A''|$ is minimum. Since $|B''|-|A''| \geqslant 12k$ and n is even, it follows that $|B''| \geqslant n/2+6k$. Now, the subgraph of G induced by A'', denoted $\langle A'' \rangle_G$, is complete; otherwise there are non-adjacent vertices x, y in A'' for which

$$|N_G(x) \cup N_G(y)| \geqslant |B''| \geqslant n/2+6k,$$

which contradicts (4). Furthermore, by the minimality of $|B''|-|A''|$, no vertex y in B'' is adjacent to every other vertex of B''; otherwise, y could be added to A''. This implies that each vertex of B'' is adjacent to fewer than $d_k n^{1-1/(2(k-1))}+4(k-1)$ vertices of B''; otherwise, there are nonadjacent vertices x and y of B'' for which

$$|N_G(x) \cup N_G(y)| \geqslant n/2-d_k n^{1-1/(2(k-1))}+d_k n^{1-1/(2(k-1))}+4(k-1),$$

which contradicts (4).

Now, let x and y be nonadjacent vertices of B'', and consider the graph $H = H_{x,y}$ with corresponding set $S = S_{x,y}$. Since $n/2 - 2(k-1) \leq |S| < n/2$ and $H - S$ has at least $|S| + 2$ components, it follows that each component of $H - S$ has at most $4(k-1) - 1$ vertices. Consequently, for each vertex z in $V(H) - S$, we have $\deg_H z < n/2 + 4(k-1)$. However, in G, each vertex of A'' has degree at least $n - 2$ since $\langle A'' \rangle_G$ is complete and $|A'' \cup B''| \geq n - 1$, so that every vertex of A'' has degree at least $(n-2) - (k-1)$ in H. Since, for n sufficiently large, $(n-2) - (k-1) > n/2 + 4(k-1)$, we conclude that $A'' \subseteq S$. Furthermore, since $|S| \geq n/2 - 2(k-1)$, it follows that $|V(H) - S| \leq n/2 + 2(k-1)$. But $|B''| \geq n/2 + 6k$. Thus, $|B'' \cap S| \geq 4k$. Choose $4k$ vertices of $B'' \cap S$; call this set D.

We now count the number of edges in G between D and $V(G) - S$. As observed, each vertex of B'' is adjacent in G to fewer than $d_k n^{1-1/(2(k-1))} + 4(k-1)$ other vertices of B''. Since $A'' \subseteq S$, it follows that the number of edges in G between D and $V(G) - S$ is at most

$$(4k)(d_k n^{1-1/(2(k-1))} + 4(k-1)).$$

Furthermore, for n sufficiently large,

$$(4k)(d_k n^{1-1/(2(k-1))} + 4(k-1)) \leq \frac{n}{2} - 1.$$

However, $|V(G) - S| \geq n/2 + 1$. Thus there are two vertices z, w of $V(G) - S$ which are adjacent, in G, to none of the vertices of D. But then

$$N_H(w) \cup N_H(z) \subseteq (S - D) \cup V(C_w) \cup V(C_z),$$

where C_w and C_z are the components of H containing w and z, respectively. (Note that we may have $C_w = C_z$.) Then

$$\begin{aligned} |N_H(w) \cup N_H(z)| &\leq |S| - 4k + |C_w \cup C_z| \leq |S| - 4k + (n - 2|S|) \\ &= n - |S| - 4k \leq n - \left(\frac{n}{2} - 2(k-1)\right) - 4k \\ &< \frac{n}{2} - 2(k-1). \end{aligned}$$

This, however, contradicts $\mathrm{ANC}(G) \geq n/2$. Thus, G has disjoint sets A'' and B'' of vertices satisfying (i), (ii), (iii) and $|B''| - |A''| < 12k$.

Assume first that $|A'' \cup B''| = n$. Then $|A''| = n/2 - t$ and $|B''| = n/2 + t$, where $t < 6k$. Since $\mathrm{ANC}(G) \geq n/2$, it follows that $\mathrm{ANC}(\langle B'' \rangle_G) \geq t$. Since t is bounded by $6k$, we may apply an argument like that given in Lemma 2 to conclude that $\langle B'' \rangle_G$ contains k edge-disjoint t-matchings $N_1, N_2, \ldots, N_k$. Let $V_1, V_2, \ldots, V_k$ be the sets of vertices of B'' incident with edges in $N_1, N_2, \ldots, N_k$, respectively. Note that these sets of vertices are not necessarily disjoint. Consider the complete bipartite subgraph G_1 of G with partite sets $B'' - V_1$ and A''. Certainly, G_1 has a perfect matching M_1, which together with N_1 produces a perfect matching M_1' of

G. Consider now $G - M_1'$, and the bipartite subgraph G_2 of $G - M_1'$ with partite sets $B'' - V_2$ and A''. Although G_2 is not a complete bipartite graph, it is true that for every $w \in V(G_2)$ we have $\deg_{G_2} w \geq |A''| - 1 \geq n/2 - t - 1$. However, G_2 has order $n - 2t$, and $n/2 - t - 1 \geq (n - 2t)/4$ for n sufficiently large. Thus, G_2 has a perfect matching M_2 by Theorem C, and then $M_2' = M_2 \cup N_2$ is a perfect matching of G, disjoint from M_1'. We continue in this fashion to produce edge-disjoint perfect matchings of G. Suppose $M_1', M_2', \ldots, M_p'$ have been constructed, where $p < k$. Consider $G - \bigcup_{i=1}^{p} M_i'$, and the bipartite subgraph G_{p+1} of $G - \bigcup_{i=1}^{p} M_i'$ with partite sets $B'' - V_{p+1}$ and A''. Then $\deg_{G_{p+1}} w \geq n/2 - t - p$ for every $w \in V(G_{p+1})$. Again, G_{p+1} has order $n - 2t$ and $n/2 - t - p \geq (n - 2t)/4$ for n sufficiently large, so that G_{p+1} has a perfect matching M_{p+1}. Then $M_{p+1}' = M_{p+1} \cup N_{p+1}$ is a perfect matching of G, disjoint from $M_1', M_2', \ldots, M_p'$. Thus G contains k edge-disjoint perfect matchings, contradicting our assumption that no such matchings exist. It follows that, necessarily, $|A'' \cup B''| = n - 1$.

Since $|A'' \cup B''| = n - 1$ and $|B''| - |A''| < 12k$ we have that $|B''| = n/2 + t$ and $|A''| = n/2 - t - 1$ where $t < 6k$. Let x be the vertex of G not in $A'' \cup B''$. Then $k \leq \deg_G x < n/4$. Let $a_1, \ldots, a_m, b_{m+1}, \ldots, b_k$ be k vertices of G adjacent to x where each $a_i \in A''$ and each $b_i \in B''$. Since $\mathrm{ANC}(G) \geq n/2$, it follows that $\mathrm{ANC}(\langle B'' \cup \{x\} \rangle_G) \geq t + 1$. Since t is bounded by $6k$ and $\deg_G x < n/4$ we may apply an argument like that given in Lemma 2 to conclude that $\langle B'' \cup \{x\} \rangle_G$ contains k edge-disjoint $(t + 1)$-matchings $N_1, N_2, \ldots, N_k$, none of which contains an edge incident with x. Let $V_1, V_2, \ldots, V_k$ be the sets of vertices in B'' incident with the edges in $N_1, N_2, \ldots, N_k$, respectively. Consider the complete bipartite graph G_1 of G with partite sets $B'' - V_1$ and $A'' - \{a_1\}$. Then G_1 has a perfect matching M_1 which together with $N_1 \cup \{xa_1\}$ is a perfect matching M_1' of G. Suppose edge-disjoint perfect matchings $M_1', M_2', \ldots, M_p'$ have been constructed, where $p < m$. Consider the bipartite subgraph G_{p+1} of $G - \bigcup_{i=1}^{p} M_i'$ with partite sets $B'' - V_{p+1}$ and $A'' - \{a_{p+1}\}$. Then $\deg_{G_{p+1}} w \geq n/2 - t - 2 - p$ for every vertex w of G_{p+1}. Also, G_{p+1} has order $n - 2t - 4$ and

$$n/2 - t - 2 - p \geq (n - 2t - 4)/4$$

for n sufficiently large, so that G_{p+1} has a perfect matching M_{p+1}. Then $M_{p+1}' = M_{p+1} \cup N_{p+1} \cup \{xa_{p+1}\}$ is a perfect matching of G disjoint from $M_1', M_2', \ldots, M_p'$. Thus G contains edge-disjoint perfect matchings $M_1', M_2', \ldots, M_m'$ where $xa_i \in M_i'$ for $i = 1, \ldots, m$.

For $i = m + 1, \ldots, k$, let N_i' be a t-matching contained in N_i such that no edge of N_i' is incident with b_i and let V_i' be the vertices incident with the edges in N_i'. Consider the bipartite subgraph G_{m+1} of $G - \bigcup_{i=1}^{m} M_1'$ with partite sets $B'' - \{b_{m+1}\} - V_{m+1}'$ and A''. Then $\deg_{G_{m+1}} w \geq n/2 - t - 1 - m$ for every w in G_{m+1}. Also, G_{m+1} has order $n - 2t - 2$ and

$$n/2 - t - 1 - m \geq (n - 2t - 2)/4$$

for n sufficiently large. Thus G_{m+1} has a perfect matching M_{m+1} and $M'_{m+1} = M_{m+1} \cup N'_{m+1} \cup \{xb_{m+1}\}$ is a perfect matching of G disjoint from $M'_1, M'_2, \ldots, M'_m$. Clearly we can continue to construct k edge-disjoint perfect matchings of G. Thus our assumption that a maximal counterexample exists leads us to a contradiction and the proof is complete. □

References

[1] I. Anderson, Perfect matchings of a graph, J. Combin. Theory Ser. B 10 (1971) 183–186.

[2] R.J. Faudree, R.J. Gould, M.S. Jacobson and R.H. Schelp, Extremal problems involving neighborhood unions, J. Graph Theory 4 (1987) 555–564.

[3] R.J. Faudree, R.J. Gould and R.H. Schelp, Neighborhood conditions and edge disjoint hamiltonian cycles, Congr. Numer. 59 (1987) 55–68.

[4] R.J. Faudree, R.J. Gould, M.S. Jacobson and L.M. Lesniak, A generalization of Dirac's theorem, Discrete Math., to appear.

[5] P. Hall, On representations of subsets, J. London Math. Soc. 10 (1935) 26–30.

[6] W. Jackson, Hamilton cycles in regular 2-connected graphs, J. Combin. Theory Ser. B 29 (1980) 27–46.

[7] T. Kovari, V.T. Sós and P. Turán, On a problem of Zarankiewicz, Colloq. Math. 3 (1954) 50–57.

[8] W.T. Tutte, A short proof of the factor theorem for finite graphs, Canad. J. Math. 6 (1954) 347–352.