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Neighborhood unions and highly hamiltonian graphs

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Abstract. In this paper we examine bounds on $|N(x) \cup N(y)|$ (for nonadjacent pairs $x, y \in V(G)$) that imply certain strong hamiltonian properties in graphs. In particular, we show that if G is a 2-connected graph of order n and if for all pairs of distinct nondajacent vertices $x, y \in V(G)$,

a. $|N(x) \cup N(y)| \geq \frac{2n+5}{3}$, then G is pancyclic.
b. $|N(x) \cup N(y)| \geq n - t$ and $\delta(G) \geq t$, then G is hamiltonian.
c. $|N(x) \cup N(y)| \geq n - 2$, then G is vertex pancyclic.

Section 1. Introduction

Adjacency conditions have long been fundamental tools in the study of graph properties, especially those of paths and cycles. Recently, several papers have explored the effects of various neighborhood conditions on a variety of graph properties. In [1], the extremal values for the size of a matching, length of a path and length of a cycle were examined; while in [2] and [3], hamiltonian properties were considered. In this paper we explore the implications of such neighborhood conditions on stronger hamiltonian properties. In particular, we study pancyclic graphs G (graphs containing cycles of all possible lengths for 3 to $|V(G)|$) and vertex pancyclic graphs (graphs for which every vertex lies on a cycle of each length l, $3 \leq l \leq |V(G)|$.)

We denote a path or cycle containing t vertices as P_t and C_t respectively. The cycle C_t will sometimes be called a t-cycle. We also denote the minimum degree of a vertex in a graph G as $\delta(G)$ and the distance between vertices u and v as $d(u, v)$. The *neighborhood* of a vertex v in G is defined to be

$$N(v) = \{u | vu \in E(G)\},$$

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while the *closed neighborhood* $N[v]$ is defined to be

$$N[v] = \{v\} \cup N(v).$$

We say that a graph G satisfies the condition $NC(G) \geq s$, if for all distinct nonadjacent vertices $x, y \in V(G)$, $|N(x) \cup N(y)| \geq s$. When no confusion can arise we will abbreviate this to $NC \geq s$. Terms not found here can be found in [4]. With these definitions in mind, we state the following result from [2].

Theorem A. ([2]) *If G is a 2-connected graph of order $n \geq 3$ with $NC(G) \geq \frac{2n-1}{3}$, then G is hamiltonian.*

We shall also find the following result useful.

Theorem B. ([5]) *If G is graph of order n with the property that $\delta(G) \geq \frac{n}{2} + 1$, then G is panconnected (that is, in G, each pair of vertices x, y is joined by a path of each possible length l, where $d(x, y) \leq l \leq n-1$.)*

Section 2. Main Results

We begin with the following Lemma.

Lemma 1. *Let G be a 2-connected graph of order $n \geq 12$ such that $NC \geq \frac{2n+5}{3}$, and let t be an integer satisfying $t \geq \frac{2n+2}{3}$. Further, let $C_t : x_0, x_1, \cdots, x_{t-1}, x_0$ be a t-cycle in G. Then C_t contains at most two vertices with the property that their degree in $H = < V(C_t) >$ is two. Further, if there are two such vertices, then they are adjacent along C_t. Hence, C_t contains vertices x_i, x_{i+2}, x_{i+4} (subscripts taken $\bmod t$) such that each has at least three adjacencies on C_t.*

Proof: If there exist three or more vertices of degree 2 in H, then there is some pair, say x, y, that are nonadjacent in H. Thus,

$$\begin{aligned} |N_G(x) \cup N_G(y)| &\leq n - t + 4 \\ &\leq n - (\frac{2n+2}{3}) + 4 \\ &= \frac{n+10}{3}, \end{aligned}$$

a contradiction. Clearly, then if there are only two vertices of degree two in H, they must be adjacent and C_t has the desired properties. ∎

In what remains, we denote the graph of Figure 1 as F.

Lemma 2. *Suppose that G is a connected graph of order $n \geq 8$ with $\delta(G) \geq \frac{n+7}{3}$ and $NC(G) \geq \frac{2n+5}{3}$. Then G contains the graph F as a subgraph.*

Proof: Suppose G contains a triangle $T : x, y, z$. Then each of these vertices has at least $\frac{n+1}{3}$ adjacencies off T. Thus, some pair, without loss of generality say x, y

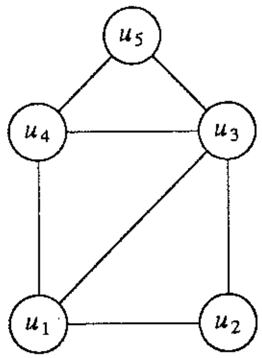

Figure 1. The graph F.

must have a common neighbor w. If $wz \in E(G)$, then some pair of x, y, z, w must have a common neighbor, producing F. If not, then consider the neighbors N of w and z other than $\{x, y, z, w\}$. Since $|N| \geq \frac{2n+5}{3}$, one of x or y has a neighbor in N, forming the subgraph F.

Now to see that G must contain a triangle, let xy be an edge of G. If x and y have a common neighbor we are done. Otherwise, let $N_x = N(x) - \{y\}$ and $N_y = N(y) - \{x\}$. If there is any edge between two vertices in N_x, then a triangle is formed. (A similar statement holds for N_y.) If both N_x and N_y contain no edges, then the minimum degree condition implies that there is an edge from N_x to N_y. Hence, a 4-cycle C is formed. Now, by considering the neighborhoods of the two pairs of nonadjacent vertices on C, we see that two consecutive vertices of C must have a common neighbor, and hence a triangle is formed.

Thus, G contains a triangle and thus, the graph F is a subgraph of G. ∎

Theorem 1. *If G is a 2-connected graph of order $n \geq 19$ and $NC(G) \geq \frac{2n+5}{3}$, then G is pancyclic.*

Proof: Note by Theorem A that G is hamiltonian. Next, suppose that the result fails to hold and suppose that t is the maximum integer ($4 \leq t \leq n$) such that for every $s \geq t$, the graph C_s is a subgraph of G (denoted $C_s \subseteq G$), but that C_{t-1} is not a subgraph of G. Since G is hamiltonian, but by assumption G is not pancyclic, such a t must exist.

Claim 1: Under these conditions, $t \leq \frac{2n+2}{3}$.

Assume to the contrary that $t > \frac{2n+2}{3}$. Then

$$|V(G - C_t)| < n - \frac{2n+2}{3} = \frac{n-2}{3}.$$

Since $n \geq 19$, we see that $t \geq 14$. Now, by Lemma 1, we know that there exists three vertices x_i, x_{i+2}, x_{i+4} (subscripts taken $\mod t$) on $C_t : x_0, x_1, \cdots, x_{t-1}, x_0$ and that each of these vertices has at least three adjacencies on C_t. Without loss of generality say these vertices are x_1, x_{t-1} and x_{t-3}. Suppose further that x_i is the closest adjacency along C_t of x_1 in the set $V(C_t) - \{x_0, x_2\}$. Relabel C_t

if necessary so that $i \leq t/2$. This relabeling is possible by Lemma 1. Thus, we see that

$$N(x_1) \cap \{x_3, x_4, \cdots, x_{i-1}\} = \phi.$$

By our choice of x_1 and x_{t-1}, x_{t-1} must be adjacent to some $x_j \in V(C_t)$, where $j \neq 0, t-2$.

Case 1: Assume that $N(x_{t-1}) \cap \{x_{t-3}, x_{t-4}, \cdots, x_{i+1}\} \neq \phi$.

Consider the cycle

$$C_t : x_0, x_1, \cdots, x_i, \cdots, x_j, \cdots, x_{t-1}, x_0,$$

and recall that the shortest chord from x_1 is the edge $x_1 x_i$. Also, suppose that the shortest chord from x_{t-1} into $\{x_{t-3}, \cdots, x_{i+1}\}$ is $x_{t-1}x_j$. We consider the path $P_{t-1} = C_t - x_0$ along with the additional edges $x_i x_1$ and $x_{t-1}x_j$. Note that if a shorter chord is contained from some potential end vertex, say x_{j+i}, (since it is also the initial vertex of a path of order $t-1$) to x_k where $(j+3 \leq k \leq t-1)$, then we can reorder the vertices to obtain a new path P_{t-1} with shorter end chords. Continuing in this manner, we can construct a path $P = P_{t-1}$ with shortest possible end chords. We also note that neither end chord forms a triangle or the graph C_{t-1} would be contained in G, contradicting our choice of t.

Define the sets

$$N = \{N_P(x_1) \cup N_P(x_{i-1})\} \text{ and}$$
$$N^- = \{x_k \in V(P) \,|\, x_{k+1} \in N\}.$$

Note that $|N| = |N^-|$ and that

$$N^- \cap (N_P(x_{t-1}) \cup N_P(x_{j+1})) \subseteq \{x_{j+2}, x_{i-3}\}.$$

Now, for $N_1 = N_P(x_{t-1}) \cup N_P(x_{j+1})$ we have that

$$|N_1| \geq \frac{2n+5}{3} - (n-t+1),$$
$$|N| \geq \frac{2n+5}{3} - (n-t+1), \text{ and}$$
$$|N| + |N_1| \leq t.$$

Hence we see that

$$t \leq \frac{2n-4}{3},$$

a contradiction. Thus we conclude that $t \leq \frac{2n+2}{3}$.

Case 2: Assume that $N(x_{t-1}) \cap \{x_{t-3}, x_{t-4}, \cdots, x_{i+1}\} \neq \phi$.

Let x_j be the endvertex of the shortest chord from x_{t-1}, and hence, x_j comes from the set

$$\{x_2, \cdots, x_{i-2}\} \cup \{x_i\}.$$

Now assume that

$$N(x_{t-3}) \cap \{x_{t-5}, \cdots, x_{j+1}\} = \phi.$$

Find the minimum subscripted adjacency, say x_k of x_{t-3}, where $1 < k \leq i$. Then neither x_{t-3} nor x_{t-1} are adjacent to x_{k-1}. From the fact that

$$|N(x_{t-1}) \cup N(x_{t-3})| \geq \frac{2n+5}{3},$$

we see that

$$\frac{2n+5}{3} \leq (n-t) + (i-1) + 3 = n - t + i + 2.$$

Then since $i \leq t/2$, we see that

$$\frac{t}{2} \leq \frac{n+1}{3}, \quad \text{and hence,} \quad t \leq \frac{2n+2}{3},$$

a contradiction. If on the other hand,

$$N(x_{t-3}) \cap \{x_{t-5}, \cdots, x_{j+1}\} \neq \phi,$$

then we repeat Case 1 with $x_{t-1} = x_1$ and $x_{t-3} = x_{t-1}$.

Thus, Claim 1 is verified.

Claim 2: $\delta(G) > \frac{n+7}{3}$.

Next, suppose that x is a vertex of minimum degree $\delta(G) = \delta$. Consider the graph $H = G - N[x]$. Then $|V(H)| = n - (\delta + 1)$. Also, if $y \in V(H)$, then x and y are nonadjacent and we see that $\delta(H) \geq \frac{2n+5}{3} - \delta$. Then, by Theorem B, the graph H is panconnected provided

$$\delta(H) = \frac{2n+5}{3} - \delta \geq \frac{n-(\delta+1)}{2} + 1.$$

That is,

$$\frac{n+7}{3} \geq \delta.$$

Thus, if $\delta(G) \leq \frac{n+7}{3}$, the graph H is panconnected. If this is the case, select vertices a and b in H with a adjacent to some u_1 in $N(x)$ and b adjacent to some u_2 in $N(x)$. This can be done, since G is 2-connected. Then using x, u_1, u_2, a, b and the variety of possible paths connecting a and b in H, we obtain cycles of lengths 6 to $n-(\delta+1)+3=n-\delta+2$. Note that H contains the small cycles of lengths $3,4,\cdots,n-\delta-1$.

Now, since $t-1\leq\frac{2n-1}{3}$ and $\delta\leq\frac{n+7}{3}$, then $t-1\leq n-\delta+2$; hence, we see that C_{t-1} is contained in G.

Thus, we see that $\delta>\frac{n+7}{3}$ and the claim in proved.

Next, by Lemma 2, we know that F is a subgraph of G. Further, suppose F is labeled as in Figure 1.

Let P_1 be a maximum path through F with initial vertex y and terminal vertex z. That is, the path P_1 begins at y, eventually enters F, say at u_1, traverses all of F, leaving at u_2, then extends as far as possible.

Claim 3: The path P_1 is a hamiltonian path in G.

Suppose instead that P_1 is not hamiltonian. Let x be a vertex not on P_1. Then since x is not adjacent to y, $|N(x)\cup N(y)|\geq\frac{2n+5}{3}$. Further, both x and y are not adjacent to any successor of a neighbor (other than a vertex of F) of z on P_1, or a path longer than P_1 would exist. But $\deg z\geq\delta(G)>(n+7)/3$ and $NC(G)\geq\frac{2n+5}{3}$, and hence,

$$|N(x)\cup N(y)|\leq n-2-\deg z+5\leq n-2-(\frac{n+7}{3})+5\leq\frac{2n+2}{3},$$

a contradiction. Thus, Claim 3 is established.

Now let $P_1: x_1,x_2,\cdots,u_1,\cdots,u_2,\cdots,x_k$ be a hamiltonian path through F as described. Select vertices x_r and x_s on P_1 such that the subpath

$$P: x_r,x_{r+1},\cdots,x_d=u_1,\cdots,x_{d+4}=u_2,\cdots,x_s$$

contains exactly $t-1$ vertices.

Let

$$S_{x_r}=N(x_r)-N_P(x_r)\text{ and let }S_{x_s}=N(x_s)-N_P(x_s).$$

Suppose there exist nonadjacent vertices $a,b\in S_{x_s}$ and let $x_m\in N(x_r)\cap V(P)-V(F)$. Then note that $x_{m-1}\notin N(a)\cup N(b)$ or else the cycle (when $m<d$)

$$x_r,x_{r+1},\cdots,x_{m-1},a(\text{ or }b),x_s,x_{s-1},\cdots,u_2,u_3,u_4,u_1,\cdots,x_m,x_r$$

would be a C_{t-1}. A similar cycle exists when $m>d+4$. Now, if $z\in N(x_r)-V(P)-V(F)$, then $z\notin N(a)\cup N(b)$ or else the cycle

$$x_r,z,a(\text{ or }b),x_s,x_{s-1},\cdots,u_2,u_3,u_1,\cdots,x_r$$

is a C_{t-1}. As before, since $\delta>\frac{n+7}{3}$, we see that this implies that

$$|N(a)\cup N(b)|<(n-2)-\frac{n+7}{3}+5=\frac{2n+2}{3},$$

a contradiction.

Since a and b were arbitrary nonadjacent vertices in S_{x_s}, we see that $< S_{x_s} >$ must be complete. Analogously, $< S_{x_r} >$ is complete.

Now suppose that there exists some vertex $u \notin V(P) \cup S_{x_r} \cup S_{x_s}$. Since G is connected and P_1 is a hamiltonian path, we may assume without loss of generality that u is adjacent to some vertex $a \in S_{x_s}$. Further, by our choice, it is clear that $ux_s \notin E(G)$. Now, with an argument analogous to the one just completed, we can show that

$$|N(x_s) \cup N(u)| < \frac{2n+2}{3}.$$

But this implies that the edge ux_s must be present in G; in other words

$$V(G) - V(P) = S_{x_r} \cup S_{x_s}.$$

Now consider a new subpath P_2 of P_1 beginning with the vertices of S_{x_r} and following P_1 until a total of $t-1$ vertices are included. Say that x_1 is the initial vertex of P_2 and let x_{t-1} be its final vertex. As before, we consider the neighbors of x_{t-1} off P_2, say $S_{x_{t-1}}$ and since $n \geq 19$ we can show that $< S_{x_{t-1}} >$ is complete. Further, we can again show that $S_{x_{t-1}} = V(G) - V(P_2)$. Analogously, we can show that the same set properties holds for the subpath consisting of the final $t-1$ vertices of P_1. That is, the path P_1 has the property that its initial $n-t+2$ vertices induce a complete subgraph of G and its final $n-t+2$ vertices also induce a complete subgraph of G. Call these subgraphs K^1 and K^2 respectively.

Since $t-1 \leq \frac{2n-1}{3}$ then $|V(K^1)| \geq \frac{n+4}{3}$ and $|V(K^2)| \geq \frac{n+4}{3}$; and since G is 2-connected, we see that the subgraphs K^1 and K^2 are joined by at least two disjoint paths. Thus, it is a simple matter to construct the desired C_{t-1}. Hence, G is pancyclic. ∎

We next turn our attention to a combination of neighborhood conditions and minimum degree conditions to obtain hamiltonian-type results. We begin with the following.

Theorem 2. *If G is a 2-connected graph of order n and t is an integer such that $\delta(G) \geq t$ and $NC(G) \geq n-t$, then G is hamiltonian.*

Proof: Suppose that G is not hamiltonian. Let $C : x_0, x_1, \cdots, x_k, x_0$ be a cycle of maximum length in G. Then, there exists a vertex x not in C that is joined to C by at least two paths disjoint except at x. Say these paths first intersect C at x_i and x_j where $i < j$. It is also clear that $i < j-1$ or a cycle longer than C would exist.

Suppose that $u = x_{i+1}$ and $v = x_{j+1}$. Then u and v are easily seen to be nonadjacent or a cycle longer than C would exist. Now, if $z \in N(x) - V(C)$, then u and v are not adjacent to z or again a longer cycle would exist. Further, if $xx_k \in E(G)$ and $x_k \in V(C)$, then both u and v are not adjacent to x_{k+1} or a longer cycle would exist. Hence, for every adjacency of x, there is a distinct vertex that cannot be adjacent to either u or v. Also, neither u nor v is adjacent to x. This implies that

$$|N(u) \cup N(v)| \leq n - (\delta(G) + 1) = n - t - 1,$$

a contradiction. Therefore, G is hamiltonian. ▮

Corollary 3. *If G is a connected graph of order n and t is an integer such that $\delta(G) \geq t - 1$ and $NC(G) \geq n - t$, then G is traceable.*

Proof: Consider the graph $H = G + x$, that is, the graph obtained by inserting a new vertex x and an edge from x to each vertex of G. Then it is clear that H is 2-connected, has order $n + 1$, $\delta(H) \geq t$ and $NC(H) \geq n + 1 - t$. Thus, from Theorem 2 we see that G is hamiltonian. Let C be a hamiltonian cycle of H. Then, $P = C - x$ is a hamiltonian path of G and hence, G is traceable. ▮

The final property we consider is the following: A graph G of order n is said to be *vertex pancyclic* if each of its vertices lies on a cycle of each possible length l, $3 \leq l \leq n$. Using neighborhood and minimum degree conditions, we obtain the following result about vertex pancyclic graphs.

Theorem 4. *If G is a 2-connected graph of order n such that $\delta(G) \geq 3$ and $NC(G) \geq n - 2$, then G is vertex pancyclic.*

Proof: We recognize two cases.

Case 1: Suppose that G has connectivity 2. Then let $C = \{x, y\}$ be a cut set of size 2. Since $\delta(G) \geq 3$, we see that each of the (at least) two components of $G - C$ must contain at least two vertices. Further, suppose that A and B are two components of $G - C$. Note that since $NC(G) \geq n - 2$, each of A and B will induce a clique. Without loss of generality we assume that

$$|A| \geq |B|.$$

With these observations in mind, we now handle the low order cases. Suppose that $n = 4$. Then, since $\delta(G) \geq 3$, we see that $G = K_4$, so clearly G is vertex pancyclic. If $n = 5$, then since $\delta(G) \geq 3$, each vertex has at most one nonadjacency. But, then $G \subseteq K_5 - e_1 - e_2$ where e_1 and e_2 are independent edges of G. Now it is easy to see that G is vertex pancyclic.

If $n = 6$, then $< A >=< B >= K_2$ and the vertices x and y are each adjacent to all vertices of A and B. Now it is again straightforward to see that G is vertex pancyclic.

If $n = 7$, then withour loss of generality, let $|A| = 3$ and $|B| = 2$. Then the adjacencies of x and y with B are as before. An analysis of the possibilities with

A shows in each case that G is vertex pancyclic. A similar approach applies when $n = 8$.

Now suppose that $n \geq 9$. We observe that $\deg_B(x) \geq 2$ and $\deg_B(y) \geq 2$. If this were not the case, we would merely alter the cut set, by replacing either (or both) x and y by the vertex in B adjacent to them, respectively.

Next, suppose that u and v are vertices of A. If $|A| \geq 5$, then consider the graphs $G - u$ and $G - v$, which by induction are seen to be vertex pancyclic. But, clearly G is hamiltonian, thus we see that u and v lie on cycles of all possible lengths. Thus, G is vertex pancyclic. .

Now suppose that $|A| = 4$. Then, for an appropriate choice of vertices u and v in A, and by again considering $G - u$ and $G - v$ and applying induction, the result will follow.

Case 2: Suppse that $\kappa(G) \geq 3$.

If G has two vertices, say u and v which are not adjacent to any vertex of degree 3 in G, then we apply induction to $G - u$ and $G - v$ as before. Thus, the vertices of degree three must cover all but at most one vertex of $V(G)$. However, in this situation, we are merely left with an analysis of the low order cases $n \leq 8$ and so G is vertex pancyclic. ∎

Examples: We now consider several examples showing the bounds or conditions in the Theorems are sharp.

i. Construct a graph G by taking three copies of K_{t-1} and identifying them at a vertex. The graph G has order $n = 3(t-2)+1$ and satisfies $NC \geq n-t$, $\delta = t-2$, and is 1-connected, but not traceable. Thus the minimum degree condition is sharp in Corollary 3.
ii. Let G be the graph this time obtained by selecting a pair of vertices in each of three copies of K_t and identifying these three pairs. The graph G has order $n = 3(t-2)+2$ and is clearly 2-connected. It also satisfies the properties $NC \geq n-t$ and $\delta = t-1$. However, G is clearly not hamiltonian. Thus, the minimum degree condition in Theorem 2 is sharp.
iii. Finally, consider the graph G obtained by taking a vertex x of degree $t (t \leq \frac{n}{2})$ and joining each of the t independent neighbors of x to each vertex of a K_{n-t-1}. This graph has order n, is t-connected and satisfies $NC \geq n-t$ and $\delta = t$. However, G is not vertex pancyclic.

We conjecture that the conditions of example 3 are best possible, that is, we conjecture that if G has order n, connectivity t and satisfies $NC \geq n-t$ and $\delta \geq t+1$, then G is vertex pancyclic.

References

1. R. J. Faudree, R. J. Gould, M. S. Jacobson and R. H. Schelp, *Extremal problems involving neighborhood unions*, J. Graph Theory **11** (1987), 555–564.
2. R. J. Faudree, R. J. Gould, M. S. Jacobson and R. H. Schelp, *Neighborhood unions and hamiltonian properties in graphs*, J. Combin. Theory B **47** (1989), 1–9.
3. P. Fraisse, *A new sufficient condition for hamiltonian graphs*, J. Graph Theory **10** (1986), 405–409.
4. R. J. Gould, "Graph Theory", Benjamin/Cummings Pub. Co., Menlo Park, CA, 1988.
5. J. Williamson, *Panconnected graphs II*, Period. Math. Hungar. **8** (1977), 105–116.