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Hamiltonian Properties and Adjacency Conditions in K(1, 3)–Free Graphs

Ralph J. Faudree*
Memphis State University

Ronald J. Gould*
Emory University

Terri E. Lindquester
Rhodes College

ABSTRACT

We investigate several hamiltonian related properties in K(1, 3)–free graphs by imposing a bound on the neighborhood union of pairs of nonadjacent vertices. We show that the basic results concerning neighborhood unions and hamiltonian properties in graphs can be improved for graphs containing no induced K(1, 3) as a subgraph.

A graph is said to be *K(1, 3)–free* if it does not contain a copy of the complete bipartite graph, K(1, 3), as an induced subgraph. There have been many results in recent years dealing with K(1, 3)–free graphs. For example, see [6] and [11]. The assumption that a graph is K(1, 3)–free is quite strong, and it provides the structure needed to obtain some interesting results involving longest paths and cycles in graphs. Terms not defined here can be found in [5].

In [10], M. Matthews and D. Sumner proved that if G is a connected K(1, 3)–free graph with $\delta \geq (p-2)/3$, then G is traceable. They also showed that any 2–connected K(1, 3)–free graph with the same minimum degree condition is hamiltonian. This imposition of a lower bound on δ accounts for a more even edge distribution in the graph.

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Recently, a different approach was taken by Zhang [14]. With a lower bound on the degree sums of sets of vertices, he was able to avoid restricting the graph with a minimum degree condition. More specifically, he proved:

Theorem (Zhang [14]) *If G is a k–connected $(k \geq 2)$ $K(1, 3)$–free graph of order p such that*

$$\sum_{v \in I} deg(v) \geq p - k$$

for any independent $(k + 1)$–set I, then G is hamiltonian.

To further reduce the edge density of the graph, we consider the neighborhood unions of pairs of nonadjacent vertices in $K(1, 3)$–free graphs. We find the basic results concerning neighborhood unions and hamiltonian properties in graphs [2] can be improved for graphs containing no induced $K(1, 3)$. As in [1], we define

$$NC = \min |N(u) \cup N(v)|,$$

where the minimum is taken over all pairs of nonadjacent vertices u, v. The first result is one involving traceability.

Theorem 1 *If G is a 2–connected $K(1, 3)$–free graph of order p such that*

$$NC \geq (p - 2)/2,$$

then G is traceable.

Proof Assume that G is not traceable. Let $P : x_1, x_2, ..., x_t$ be a longest path in G with $t \leq p - 1$, and let $x \in V(G - P)$. Since G is 2–connected, there exist at least two paths, disjoint except at x, from x to P. Let $x_i \in V(P)$ be the end vertex of a path, P_1, from x to P such that i is minimized. Then choose the path, P_2, from x to P so that if x_j is the end vertex of P_2, $i < j$, there are no paths from x to P with end vertices in $\{x_{i+1}, ..., x_{j-1}\}$. Observe that this is possbile, since x cannot be the end vertex of two paths in $V(G - P)$ that have consecutive end vertices on P.

Since P is a longest path in G, certainly x_1 and x_t have no adjacencies off P. Also since G is $K(1, 3)$–free, the edges $x_{i-1}x_{i+1}$ and $x_{j-1}x_{j+1}$ are in $E(G)$. (Note that there exists no path—disjoint from $V(P - \{x_2, x_{i+2}, x_{t-1}\})$—from x to x_2, x_{i+2}, or x_{t-1}, since these paths and the forbidden subgraph property would force the existence of the edges x_1x_3, $x_{i+1}x_{i+3}$, or $x_{t-2}x_t$, respectively. Any of these situations would create a path longer than P.)

We consider the following adjacencies of x_1:

$$S_1 = \{x_{k-1} : x_k \in N(x_1) \cap V(P)\}.$$

We will show that $S_1 \cap N(x_{j-1}) = \varnothing$ and $S_1 \cap N(x_t) = \varnothing$.

Now for $x_k \in N(x_1)$, $j+3 \le k \le t-1$, (observe $k \ne j+1$, $j+2$, and t) we see that $x_{k-1} \notin N(x_{j-1})$, or the path

$$x_t, \ldots, x_k, x_1, \ldots, x_{j-1}, x_{k-1}, x_{k-2}, \ldots, x_j, \ldots, x,$$

is longer than P. Similarly, $x_{k-1} \notin N(x_t)$ or the path

$$x, \ldots, x_j, x_{j-1}, \ldots, x_1, x_k, x_{k+1}, \ldots, x_t, x_{k-1}, \ldots, x_{j+1},$$

is longer than P. Also for $x_k \in N(x_1)$, $i+3 \le k \le j-3$ and $2 \le k \le i-1$, $x_{k-1} \notin N(x_t)$ or the path

$$x, \ldots, x_j, \ldots, x_t, x_{k-1}, \ldots, x_1, x_k, \ldots, x_{j-1},$$

is a path longer than P. Lastly, $N(x_{j-1}) \cap S_1 = \varnothing$ for $i+3 \le k \le j-3$ and $2 \le k \le i-1$ since the paths

$$x_t, \ldots, x_j, \ldots, x, \ldots, x_i, \ldots, x_{k-1}, x_{j-1}, \ldots, x_k, x_1, \ldots, x_{i-1}, \text{ and}$$

$$x_t, \ldots, x_j, \ldots, x, \ldots, x_i, \ldots, x_{j-1}, x_{k-1}, \ldots, x_1, x_k, \ldots, x_{i-1},$$

are longer than P.

Similarly, we now consider the set S_2:

$$S_2 = \{y : y \in N(x) - V(P)\} \cup \{x_{k-1} : x_k \in N(x) \cap V(P), j+3 \le k \le t-2\}.$$

Then clearly $S_2 \cap N(x_{j-1}) = \varnothing$ and $S_2 \cap N(x_t) = \varnothing$. (Note that x is not adjacent to x_{j+2}.) By the choice of P_1 and P_2 there are no other adjacencies of x.

We define the function f: $N(x) \cup N(x_1) \rightarrow N(x_{j-1}) \cup N(x_t)$ by:

$$f(y) = y, \text{ for } y \notin V(P);$$

$$f(x_{k-1}) = x_k, \text{ for } 2 \le k \le t-1.$$

From the previous arguments, clearly f is injective; therefore, we have

$$f(N(x) \cup N(x_1)) \cap (N(x_{j-1}) \cup N(x_t)) = \varnothing.$$

Also

$$f(N(x) \cup N(x_1)) \cup (N(x_{j-1}) \cup N(x_t)) \subseteq V(G) - \{x_i, x_t, x\}.$$

Since $NC \ge (p-2)/2$, this implies that

$$(p-2)/2 + (p-2)/2 \le p-3, \text{ a contradiction.}$$

Therefore, G is traceable. ❑

Figure 1 illustrates that the connectivity condition in Theorem 1 cannot be lowered.

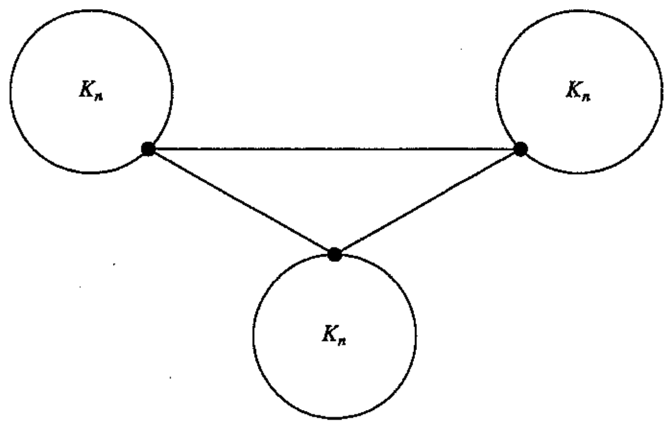

Figure 1 A 1–connected K(1, 3)–free graph that is not traceable.

The bipartite graph K(n, n–2) has order $p = 2n - 2$, while $NC \geq (p-2)/2$. However, K(n, n–2) is not traceable since induced K(1, 3)'s abound.

We next present a series of lemmas which will allow us to prove analogous results for the hamiltonian property and a "highly hamiltonian" property–pancyclicity. A graph of order p is *pancyclic* if it contains a k–cycle for every k such that $3 \leq k \leq p$. It was shown in [3] that if G is a 2–connected graph of order p such that $NC \geq \frac{2p-1}{3} + 2$, then G is pancyclic. We were able to improve this bound by assuming that G contains no induced K(1, 3).

First, it will be useful to define a related concept. A connected graph G of order p is said to be *panconnected* if for each pair u, v of distinct vertices of G there exists a uv–path of length t, for each t satisfying $dist(u, v) \leq t \leq p - 1$. In [13], Williamson proved that by restricting δ in G, panconnectedness can be achieved.

Theorem (Williamson [13]) *If G is a graph of order $p \geq 4$ such that $deg(v) \geq p/2 + 1$ for every vertex v of G, then G is panconnected.*

First we grasp some basic structural ideas:

Lemma 1 *Let H be a graph of order $p \geq 2$ with independence number $\alpha(H) \leq 2$, and let $G = K_1 + H$. Then either G is panconnected or $H = K_r \cup K_{p-r}$ for some $1 \leq r < p$.*

Proof If $p \leq 5$, the result holds trivially, so assume $p \geq 6$ and proceed by induction on $p + 1$, the order of G. Select a vertex v of maximum degree in H and consider the graph $G - v$ which is either panconnected or $H - v = K_r \cup K_{p-r}$ for some $1 \leq r < p - 1$.

If $G - v$ is panconnected, then for any pair of vertices x and y, there are xy–paths of each length from 2 to $p - 1$. Let $P = (x = x_1, x_2, \ldots, x_p = y)$ be an xy–path of length $p - 1$. Since v has maximum degree in H, $\deg(v) \geq 3$. Thus, $vx_i, vx_j \in E(G)$ with $i < j < p$. Since $\alpha(H) \leq 2$, one of the edges vx_{i+1}, vx_{j+1}, or $x_{i+1}x_{j+1} \in E(G)$, which implies that there exists an xy–path in G of length p.

If $H - v = K_r \cup K_{p-r-1}$, then v must be adjacent, without loss of generality, to each vertex of K_{p-r-1}, for otherwise there would be three independent vertices. If v has an adjacency in K_r, it is easily verified that G is panconnected. If not, then $H = K_r \cup K_{p-r}$, and the proof is complete. ❑

Lemma 2 *If G is a 2–connected $K(1,3)$–free graph of order $p \geq 11$ that satisfies*

$$NC \geq \frac{(2p-1)}{3} \text{ and } \delta(G) \leq \frac{(p-5)}{3},$$

then G is pancyclic.

Proof Let v be a vertex of degree $\delta = \delta(G)$, and let $H = G - v - N(v)$. Then $\delta(H) \geq (2p - 1)/3 - \delta$. Since $\delta \leq (p - 5)/3$, we have

$$\delta(H) \geq (p - \delta - 1)/2 + 1 = |H|/2 + 1,$$

so that H is panconnected by [13]. Therefore, H has cycles of each length from 3 to $p - \delta - 1$. Let L be the graph spanned by $v + N(v)$. Since G is $K(1,3)$–free, $\alpha(L) \leq 2$. Because G is 2–connected, there are distinct vertices $u_1, u_2 \in V(L)$ and $v_1, v_2 \in V(H)$ such that $u_iv_i \in V(G)$ for $i = 1, 2$. From Lemma 1, we consider two possibilities.

If L is panconnected, then there are paths in L between u_1 and u_2 of each length from 2 to δ. Also, there are paths in H of each length from 2 to $p - \delta - 2$ between v_1 and v_2. Using those paths, and the edges u_1v_1 and u_2v_2, cycles of each length from 6 to p can be obtained which implies that G is pancyclic.

Suppose that L is not panconnected. Then $L = v + (K_r \cup K_s)$ with $r + s = \delta$ and $r, s \geq 1$. Without loss of generality, assume $u_1 \in K_r$ and $u_2 \in K_s$. Again paths in L between u_1 and u_2 exist of each length from 2 to δ. Thus, as in the previous case, G is pancyclic. ❑

As a corollary to the proof of Lemma 2, using the result of Ore [12] on hamiltonian–connected graphs, we have the following

Corollary 1 *If G is a 2–connected $K(1,3)$–free graph of order $p \geq 11$ that satisfies*

$$NC > (2p-3)/3 \text{ and}$$
$$\delta(G) \leq (p-5)/3,$$

then G is hamiltonian.

Lemma 3 *Let G be a 2–connected $K(1,3)$–free graph of order $p \geq 14$ that satisfies*

$$NC \geq (2p-1)/3 \text{ and}$$
$$\delta(G) \geq (p-4)/3.$$

If G contains a C_m for some $m \geq (2p+5)/3$, then G contains cycles of each length from m to p.

Proof Select t such that G contains a C_t but no C_{t+1} for some $t \geq m$. Let $C_t = (x_1, x_2, \ldots, x_t, x_1)$ and let $H = G - C_t$. Note that if $y \in V(H)$ and $yx_i \in E(G)$, then $yx_{i-1}, yx_{i+1} \notin E(G)$, otherwise a C_{t+1} would exist. The $K(1,3)$–free property implies $x_{i-1}x_{i+1} \in E(G)$. Using this, it is easy to verify that $yx_i \in E(G)$ implies that yx_{i+2} and $yx_{i+3} \notin E(G)$, for otherwise a C_{t+1} would exist in G.

Consider the case when H contains nonadjacent vertices y_1 and y_2. By the previous remarks, each of y_1 and y_2 can be adjacent to at most $t/4$ vertices of C_t, so that

$$\frac{(2p-1)}{3} \leq |N(y_1) \cup N(y_2)| \leq t/2 + (p-t-2).$$

This implies $t \leq (2p-10)/3$, a contradiction. Therefore H is the complete graph K_{p-t}.

If $t = p-1$, then the vertex in H has degree at most $(p-1)/4$. Thus we get $(p-4)/3 \leq (p-1)/4$, which is a contradiction for $p \geq 14$. Hence, H has at least two vetices. Each vertex of H has at least $s \geq (p-4)/3 - (p-t-1) \geq t - (2p+1)/3$ adjacencies on C_t, so clearly $s \geq 2$.

If $yx_i \in E(G)$ and $y'x_j \in E(G)$ for $y, y' \in V(H)$, then $j > i + p - t + 2$, for otherwise G would contain a C_{t+1}. Using the fact that y has at least s adjacencies on C_t and that each pair of these adjacencies are at a distance at least four on C_t, there are at least $4(s-1) + 2(p-t+2)$ vertices of C_t that are not adjacent to y'. Since y' has s adjacencies on C_t, we have

$$s + 4(s-1) + 2(p-t+2) \leq t.$$

This implies $t \leq (4p+5)/6$, a contradiction. ❑

As a corollary to the proof of Lemma 3, using Corollary 1, we get the following

Lemma 4 *Let G be a 2–connected $K(1,3)$–free graph of order $p \geq 14$ that satisfies*

$$NC > (2p-3)/3 \text{ and}$$
$$\delta(G) > (p-5)/3.$$

If G contains a C_m for some $m \geq (2p+4)/3$, then G is hamiltonian.

Lemma 5 *If G is a 2–connected graph of order $p \geq 9$ with $\alpha(G) < 3$, then G is pancyclic.*

Proof Since the Ramsey number $r(K_3, C_t) = 2t-1$ (c.f. [5] pp. 272), G contains a cycle of each length $t \leq (p+1)/2$. Select m such that G contains a cycle of each length from 3 to m, but no cycle of length $m+1$. Let $C = (x_1, x_2, \ldots, x_m, x_1)$ be a cycle of length m and let $H = G - C$.

If $y \in V(H)$ and $yx_i, yx_j \in E(G)$, then yx_{i+1}, yx_{j+1}, and $x_{i+1}x_{j+1} \notin E(G)$ or we create a C_{m+1}. Since $\alpha(G) < 3$, each vertex of H must be adjacent to at most one vertex of C, and H is complete.

Similarly, $x_ix_j \in E(G)$ unless each vertex of H is adjacent to either x_i or x_j. Therefore, either C forms a complete graph K_m or a $K_m - e$, or there exists a vertex x of C which is adjacent to each vertex of H, while $C - x$ forms a complete graph. However, in all cases, there are two disjoint subgraphs that are complete (or one has a missing edge) that span G. Since G is 2–connected there are two independent edges between these subgraphs. Under those conditions it is easily verified that G is pancyclic.

With these lemmas at hand, we can now prove the main theorem.

Theorem 2 *If G is a 2–connected $K(1,3)$–free graph of order $p \geq 14$ with*

$$NC > (2p-2)/3,$$

then G is pancyclic.

Proof Assume G is not pancyclic. By Lemma 2 we can assume that $\delta(G) \geq (p-4)/3$ and by Lemma 5 that $\alpha(G) \geq 3$.

Consider pairwise nonadjacent vertices x, y, and z and assume that $\deg(z) \geq \deg(x) \geq \deg(y)$. Then from the hypothesis, $\deg(z) \geq \deg(x) \geq (2p-1)/6$. We claim that z is adjacent to at least two vertices in $N(x)$ or $N(y)$. If not then,

$$(2p-1)/3 + (2p-1)/6 - 2 \leq p-3,$$

a contradiction. This proves the claim. Therefore we can assume there exist subgraphs A and B with $A \cap B = \{u, v\}$ and with $|A \cup B| \geq (2p-1)/3$. Also x is adjacent to each vertex of A and z is adjacent to each vertex of B.

We will subsequently show that G contains cycles of each length from 3 to $\lceil (2p+5)/3 \rceil$. We can then conclude that G is pancyclic from Lemma 4. To prove the existence of these cycles we must consider the graph H spanned by $A \cup B \cup \{x, y\}$.

Note that from Lemma 1, $A + x$ (respectively $B + z$) is panconnected or $A = A_1 \cup A_2$, where A_1 and A_2 are disjoint complete graphs (respectively $B = B_1 \cup B_2$, where B_1 and B_2 are disjoint complete graphs). We now consider various cases depending on the structure of A and B and the location of u and v, and show the existence of the required cycles.

Case 1 Suppose $A + x$ is panconnected or $u \in V(A_1)$ and $v \in V(A_2)$, and $B + z$ is panconnected or $u \in V(B_1)$ and $v \in V(B_2)$.

In $A + x$ (and $B + z$) there is a path from u to v of each length from 2 to $|A|$ (and $|B|$), thus H has cycles of each length from 4 to $|H| \geq (2p + 5)/3$. Clearly, $A + x$ (and $B + z$) contains a triangle.

Case 2 Suppose $A + x$ is panconnected and $u, v \in B_2$.

Since G is 2–connected, there is a path in $G - y$ between B_1 and $B_2 \cup A$. Let $P = (w = w_0, w_1, \ldots, w_t = w')$ be such a path of minimal length. We claim that $t \leq 2$. If this is not true, then consider the independent vertices w_0, w_2, and x, and observe that $N(x) \cap (N(w_0) \cup N(w_2)) \neq \emptyset$. This verifies the claim. It is easily seen that $H \cup P$ contains cycles of each length from 3 to $|H| + t - 1$, independent of whether w' is in A or B_2. Note that all cycles of length greater than $|A \cup B_2| + 2$ will use the path P.

If $u \in A_1 \cap B_1$ and $v \in A_2 \cap B_2$, then we get the same result. That is, $H \cup P$ contains cycles of each length from 3 to $|H| + t - 1$, independent of whether w' is in $A_1 \cup A_2$ or B_2.

Case 3 The vertices $u, v \in A_2 \cap B_2$.

Since G is 2–connected, there is a path in $G - y$ between B_1 and $A \cup B_2$. Let $P = (w = w_0, w_1, \ldots, w_t = w')$ be such a path of minimal length. We claim that $t \leq 2$. If this is not the case, then consider the independent vertices w_0, w_2, and x, and observe that $N(x) \cap (N(w_0) \cup N(w_2)) \neq \emptyset$. This verifies the claim.

Subcase i Let $w' \in A_1$.

Just as in the previous cases, $H \cup P$ contains cycles of each length from 3 to $|H| + t - 1$, since $p \geq 14$. The remaining cycles of length ≥ 6 can be constructed using the path P.

Subcase ii Let $w' \in A_2 \cup B_2$.

A repeat of the argument exhibiting P implies the existence of a corresponding path P' from A_1 to $A_2 \cup B_2$ of length $t \leq 2$. There are several possibilities that must be considered, but it is straightforward to verify that $H \cup P \cup P'$ contains cycles of each length from 3 to $|H| + t + t' - 2$.

This completes the proof of Theorem 2. ❑

Theorem 3 *If G is a 2–connected K(1,3)–free graph of order $p \geq 14$ with $NC > (2p-3)/3$, then G is hamiltonian.*

Proof Consider nonadjacent vertices x and y. Then $|N(x) \cup N(y)| > (2p-3)/3$. Thus there exist disjoint subgraphs A and B with $|A \cup B| > (2p-3)/3$ and with x adjacent to each vertex of A and y adjacent to each vertex of B.

Since G is 2–connected, there exist vertex disjoint paths P_1 and P_2 from A to B (avoiding x and y). By Lemma 1, either $A + x$ is panconnected or $A = A_1 \cup A_2$ where A_1 and A_2 are disjoint complete graphs. The same is true of B with disjoint subgraphs B_1 and B_2. Also if $A + x$ $(B + y)$ is not panconnected, then the paths P_1 and P_2 come from A_1 and A_2 (B_1 and B_2), respectively.

Clearly, G contains a cycle of length at least $|(A \cup B) \cup (P \cup P')| + 2 \geq (2p+4)/3$. From Lemma 5, we see that G is hamiltonian. ❑

We note here that we have obtained this same result for $3 \leq p \leq 13$ with other techniques.

A variation of a graph used in [10] illustrates the sharpness of our theorem. Let G be the graph of order $p = 3n + 6$ in Figure 2. Then G is 2–connected and K(1,3)–free. Notice also that G is not hamiltonian. For nonadjacent vertices u and v in G,

$$|N(u) \cup N(v)| = 2n + 2 = 2((p-6)/3) + 2 = 2p/3 - 2.$$

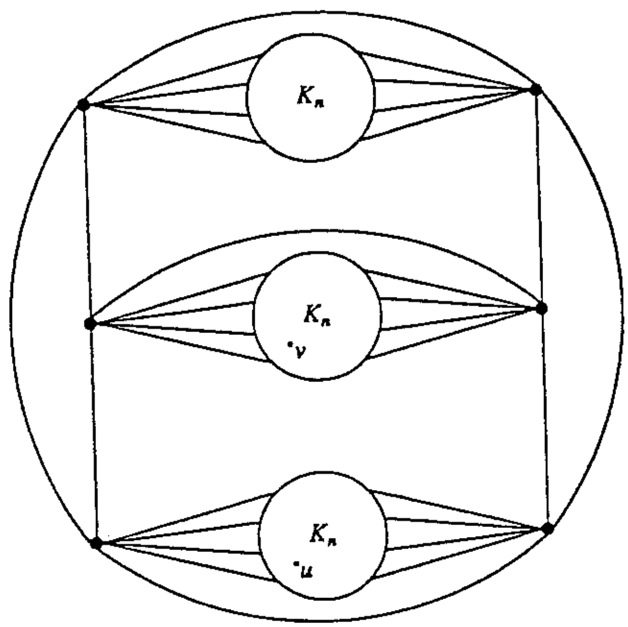

Figure 2 A nonhamiltonian K(1,3)–free graph illustrating the sharpness of Theorem 2.

We feel that from the nature of the proof techniques and related results in this area that a generalization of Theorem 3 to sets of more than two vertices is imminent. Theorem 3 and a result of Fraisse [4] lead us to the following conjecture:

Conjecture *Let G be a K(1,3)–free graph of order p and connectivity k. Suppose there exists some t, $t \le k$, such that for every independent set $S \subseteq V(G)$ of cardinality t we have:*

$$|N(S)| \ge t(p-1)/(t+1)$$

then G is hamiltonian.

From Theorem 3 we get the following result for connected K(1,3)–free graphs:

Theorem 4 *If G is a connected K(1,3)–free graph of order $p \ge 14$ with $NC > (2p-5)/3$, then G is traceable.*

Proof Consider the graph $H = G + v$ for some vertex $v \notin V(G)$. Then

$$NC(H) > (2p - 4)/3 + 1 = \frac{2(p+1)-3}{3}.$$

Since $|H| \geq 14$, and H is 2–connected, then H is hamiltonian which implies that G is traceable. ❑

A graph is *homogeneously traceable* if for every $x \in V(G)$ there exists a path beginning at x and containing all of the vertices of G. In general, there are not many results dealing with homogeneously traceable graphs. It appears that sufficient conditions for a nonhamiltonian graph to have this property are not easy to find due to the theoretical closeness of homogeneous traceability and hamiltonicity. As with the hamiltonian–connected property, we need the 3–connectivity stipulation to prove this result. We also point out that Matthews [8] has proven that if G is a 3–connected K(1, 3)–free graph of order $p < 20$, then G is hamiltonian. If G is hamiltonian, then G is homogeneously traceable, hence our result for K(1, 3)–free graphs is only interesting when $p \geq 20$.

Theorem 5 *If G is a 3–connected K(1, 3)–free graph of order p such that*

$$NC > (2p - 5)/3,$$

then G is homogeneously traceable.

Proof We proceed by contradiction. First, we define an *x_m–hamiltonian path* to be a path in G that has x_m as an end vertex, and that contains all of the vertices of V(G). Suppose there is some vertex x_m such that there exists no x_m–hamiltonian path in G. Let P: $x_1, x_2, ..., x_m$ be a longest x_m–path.

Since G is 3–connected, for some vertex x not on P, there exist two disjoint (except at x) paths from x to P such that the end vertices x_i, x_j, of these paths satisfy $i \neq j \neq m$. Moreover, x_i can be chosen so that there are no endpoints of x–paths between x_1 and x_i. Once x_i is chosen, then we select x_j so that there are no endpoints of x–paths between x_i and x_j. (Note that $x_1x, x_{i+1}x, x_{i+2}x \notin E(G)$ by the maximality of P and since G is K(1, 3)–free.)

Consider the nonadjacent pair of vertices x_1 and x_{j+1}. Let

$$S = \{x_k : x_k \in N(x_1) \cup N(x_{j+1})\}.$$

Then since G is K(1, 3)–free and $N(x) \cap V(P) \subseteq \{x_i, x_j, x_{j+1}, ..., x_m\}$, we see that $|S \cap N(x)| \leq 2$. (It could be that $S \cap N(x) = \{x_j, x_m\}$.) Hence,

$$|N(x)| < (p - 1) - ((2p - 5)/3 - 2) - 3 = (p - 1)/3.$$

We subtract three at the end since $x_{j+1} \notin N(x)$, $x_{i+1} \notin N(x)$, and $x_1 \notin N(x)$. (Note that $x_{j+1} \notin N(x_{i+1})$ and $x_1 \notin N(x_{i+1})$ so x_{i+1} is not in $N(x_{j+1}) \cup N(x_1)$.)

In addition, x and x_1 are nonadjcent which implies that

$$|N(x_1)| > (2p - 5)/3 - (p - 1)/3 = (p - 4)/3.$$

We use this fact to contradict the cardinality of $N(x_{i+1}) \cup N(x)$.

For every $x_k \in N(x_1), j + 3 \le k \le m - 1, x_{k+1} \notin N(x_{i+1})$ or the path

$$x_m, ..., x_{k+1}, x_{i+1}, ..., x_k, x_1, ..., x_i, ..., x$$

is a longer x_m–path. Analogously, $x_{k+1} \in N(x)$ or we obtain

$$x_m, ..., x_{k+1}, x, ..., x_j, ..., x_k, x_1, ..., x_{j-1}$$

which is also a longer x_m–path. Observe that x_1 is not adjacent to any vertex in

$$\{x_i, x_{i+1}, x_{i+2}, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}\}$$

because G is $K(1, 3)$–free.

For $x_k \in N(x_1), i + 3 \le k \le j - 3$, then $x_{k+1} \notin N(x_{i+1})$ or a longer x_m–path is the following:

$$x_m, ..., x_{k+1}, x_{i+1}, ..., x_k, x_1, ..., x_i, ..., x.$$

Lastly, for $x_k \in N(x_1), 2 \le k \le i - 1, x_{k-1} \notin N(x_{i+1})$. Since we have no corresponding nonadjacency to subtract from $|N(x) \cup N(x_{i+1})|$ if perhaps $x_1x_m \in E(G)$, we have that

$$|N(x) \cup N(x_{i+1})| < p - ((p - 4) / 3 - 1) - 4 = (2p - 5) / 3.$$

We subtract four at the end since $x_{j+2}, x_{j+1}, x_{i+1}$, and x have not been counted and are not in $N(x) \cup N(x_{i+1})$. Since $NC > (2p - 5) / 3$ we have our contradiction. Therefore, G is homogeneously traceable. ❑

From the nature of homogeneously traceable graphs and the strength of the neighborhood condition, we believe the following is true:

Conjecture *If G is a 3–connected $K(1, 3)$–free graph of order p such that*

$$NC > (2p - 5) / 3,$$

then G is hamiltonian.

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