

# On a Generalization of Ore's Theorem for Hamiltonian-Connected Graphs

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**ABSTRACT:** For sets of vertices, we consider a form of generalized degree based on neighborhood unions. In particular, for a graph  $G$ , the degree of the set  $S = \{x_1, \dots, x_k\}$  is defined to be

$$\text{deg } S = \left| \bigcup_{i=1}^k N(x_i) \right|$$

where  $S$  is a set of  $k$  vertices in  $G$  and  $N(x)$  denotes the neighborhood of the vertex  $x$ .

In this paper, we consider an added restriction involving neighborhood intersections. Let  $IC_2(G) \geq t$ , mean that for all pairs of nonadjacent vertices, the intersection of their neighborhoods contains at least  $t$  vertices. Using restrictions of these types, we obtain a generalization of the well-known result of Ore concerning hamiltonian-connected graphs.

## Section 1. Introduction.

For standard terms and notation not found here see [11].

The study of cycles and paths in graphs has long been one of the major subfields in graph theory. Degree conditions have always been fundamental to this study. Over the last five years a new approach has been introduced, beginning with the work in [7]. Rather than considering degrees of vertices, a more general count is taken. The cardinality of the neighborhood union of pairs (or larger sets) of nonadjacent vertices is considered. This is a natural extension of the idea of degree and it is also natural to relate this parameter to various cycle and path problems. Several papers followed [1, 3, 6, 8, 9], suggesting a connection between neighborhood unions and other properties of graphs. The combination of a generalized degree condition with other graph properties has

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produced generalizations of some fundamental theorems involving ordinary degrees, as well as new types of results. This paper continues this line of investigation.

We define the *degree* of a set  $S$  of vertices as

$$\text{deg } S = \left| \bigcup_{v \in S} N(v) \right|,$$

where  $N(v) = \{ x \in V(G) \mid xv \in E(G) \}$ , the usual open neighborhood of a vertex. For convenience, when the set  $S$  is a singleton, we will abbreviate this notation with  $\text{deg } x$ , the standard notation for the degree of a vertex.

Several different forms of generalized degree for sets of vertices (where the sets considered satisfy various conditions) have been used to further the study of a variety of graph properties. In [5] and [7], hamiltonian properties were studied using sets of independent vertices of various sizes and a form of generalized (independent) degree. These generalized degrees were also used in [6] to study matchings and extremal path and cycle lengths. In [3], a Turan-type extremal result was obtained. A survey of recent results using several types of generalized degrees (neighborhood unions) can be found in [13].

Two of the most fundamental results on hamiltonian-connected graphs use ordinary degrees and are due to Dirac [2] and Ore [14].

**Theorem A[2].** If  $G$  is a graph of order  $p$  such that  $\text{deg } x \geq \frac{p+1}{2}$  for each vertex  $x \in V(G)$ , then  $G$  is hamiltonian-connected.

**Theorem B [14].** If  $G$  is a graph of order  $p$  such that for each pair of distinct nonadjacent vertices  $x$  and  $y$

$$\text{deg } x + \text{deg } y \geq p + 1,$$

then  $G$  is hamiltonian-connected.

Implicit in the hypothesis of Ore's Theorem is the fact that any two independent vertices have at least three common neighbors. In this paper we further explore the effects of this neighborhood intersection condition. We say that  $G$  satisfies  $IC_t(G) \geq k$ , if for any set of  $t$  independent vertices,  $x_1, x_2, \dots, x_t$ ,

$$\left| \bigcap_{i=1}^t N(x_i) \right| \geq k.$$

We simply use  $IC_t \geq k$  when the context is clear. In the next section we shall combine independent generalized degree conditions with this neighborhood intersection condition to attain a generalization of Ore's Theorem for hamiltonian-connected graphs.

We denote as  $A_t$ , an independent set of  $t$  vertices. We say a collection of sets  $A_{t_1}, \dots, A_{t_r}$ , where  $|A_{t_i}| = t_i$ , are *distinct independent sets* if for each  $i$ ,  $A_{t_i}$  is an independent set of  $t_i$  vertices and the intersection of any two of these sets is empty. This terminology generalizes the usual idea of distinct vertices.

Using generalized degrees and the above intersection condition, a generalization of Ore's Theorem for hamiltonian graphs was obtained in [12].

**Theorem C. [12]** Let  $G$  be a graph of order  $p$  satisfying  $IC_2(G) \geq k \geq 2$ . If for some pair  $t_1, t_2$  of positive integers satisfying  $2 \leq t_1 + t_2 \leq k + 1$ , each pair of distinct independent sets  $A_{t_1}, A_{t_2}$  satisfies

$$\text{deg } A_{t_1} + \text{deg } A_{t_2} \geq p,$$

then  $G$  is hamiltonian.

## Section 2. Main Results

We begin with a few simple observations concerning the intersection condition. First, observe that if  $IC_t(G) \geq k$ , then  $IC_s(G) \geq k$ , for any  $s \leq t$ . Also note that if  $IC_2(G) \geq 1$ , then  $G$  is connected and has diameter at most two. We next turn our attention to connectivity. The following simple Proposition was presented in [12].

**Proposition 1.** If  $IC_2(G) = k$  ( $k \geq 1$ ), then  $G$  is  $k$ -connected.

This result can be seen to be sharp by considering the class of graphs obtained by identifying two complete graphs at  $k - 1$  vertices.

In considering paths (or cycles), the following notation will be useful. Let  $[a, b]$  denote the segment of a path from the vertex  $a$  to the vertex  $b$  (including both  $a$  and  $b$ ). Similarly,  $(a, b)$  refers to the segment but does not include either  $a$  or  $b$ . We will also find it convenient at times to refer to the successor of a vertex  $a$  along a path simply as  $a + 1$ , or to the predecessor of  $a$  along a path as  $a - 1$ .

We now present our main result.

**Theorem 2.** Let  $G$  be a graph of order  $n$  satisfying  $IC_2(G) \geq k \geq 3$ . If for some pair  $t_1, t_2$  of positive integers satisfying  $2 \leq \sum_{i=1}^2 t_i \leq k - 1$ , each pair of distinct independent sets  $A_{t_1}, A_{t_2}$  satisfies

$$\text{deg } A_{t_1} + \text{deg } A_{t_2} \geq n + 1,$$

then  $G$  is hamiltonian-connected.

**Proof.** First note by Proposition 1 that  $G$  is  $k$ -connected ( $k \geq 3$ ). Now suppose that  $G$  is not hamiltonian-connected. Then there exists a pair of vertices, say  $x$  and  $y$ , that are not joined by a hamiltonian path in  $G$ . Let  $P: x = x_1, x_2, \dots, x_m = y$  be a longest  $x - y$  path in  $G$ . Choose some  $z \in V(G) - V(P)$  which is adjacent to some vertex on  $P$ . Such a vertex exists since  $G$  is 3-connected. Without loss of generality, suppose that  $x_h$  is the closest adjacency of  $z$  to  $x$  along  $P$ . (Note that  $x_h$  could be  $x$ .) Clearly,  $z$  is not adjacent to consecutive vertices of  $P$ , for then an  $x - y$  path longer than  $P$  would immediately result. Since the pair  $x_{h+1} = t$  and  $z$  are independent and  $IC_2 \geq k$ , they must have at least  $k \geq 3$  common neighbors. If any of these common neighbors is off  $P$ , an  $x - y$  path longer than  $P$  is immediate. Thus, all common neighbors of  $z$  and  $t$  are on  $P$ .

Let  $v_2, v_3, \dots, v_k$  be  $k - 1$  other common neighbors of  $z$  and  $t$  on  $P$  (as  $x_h = v_1$  is already known) given in order with respect to the orientation of  $P$  from  $x$  to  $y$ .

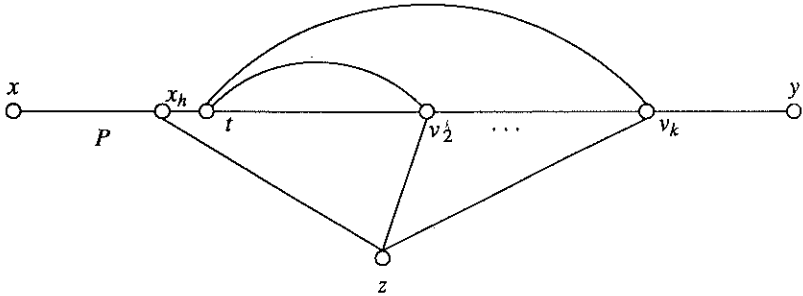


Figure 1. The adjacency situation.

Denote by  $p_i$  and  $s_i$  the predecessor and successor of  $v_i$ , ( $2 \leq i \leq k$ ) respectively. Note that  $z$  is not adjacent to any of the vertices  $p_i$  or  $s_i$  ( $1 \leq i \leq k$ ) or a path longer than  $P$  results. Also note that  $s_i$  and  $s_j$  ( $i \neq j$ ) (as well as  $p_i$  and  $p_j$ ) must be nonadjacent, for otherwise, assuming  $i < j$

$$x, \dots, v_i, z, v_j, \dots, s_i, s_j, \dots, y$$

would be an  $x - y$  path longer than  $P$ , a contradiction to our assumptions.

We now consider the distinct independent sets

$$A_{t_1} = \{ z, p_2, \dots, p_{t_1} \}$$

and

$$A_{t_2} = \{ s_{t_1+1}, \dots, s_{t_1+t_2} \},$$

where  $A_{t_1} = \{ z \}$  when  $t_1 = 1$ . Note that  $t_1 + t_2 \leq k - 1$  and these sets are distinct.

We proceed by showing that there is a 1-1 correspondence between vertices in  $N(A_{t_1})$ , and distinct vertices not in  $N(A_{t_2})$ . To do this we consider each adjacency, say  $w$ , of either  $z$  or an arbitrary vertex  $p \in A_{t_1}$ . We show that corresponding to each such  $w$ , there is a distinct vertex that cannot be in  $N(A_{t_2})$ . Using this fact, we obtain a contradiction to the degree sum condition.

Without loss of generality, suppose that  $s = s_j$  is an arbitrary member of  $A_{t_2}$  and that  $w \in N(A_{t_1})$ . We now consider several cases based on the location of  $w$  with respect to  $P$ .

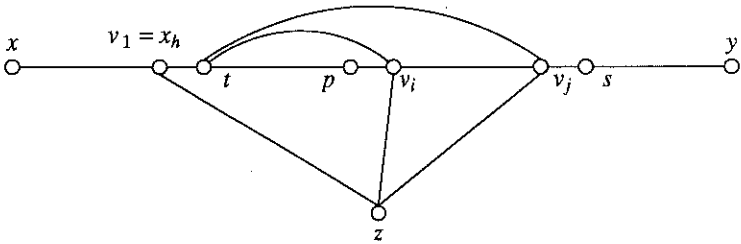


Figure 2. The arbitrary vertices from each set.

**Case 1.** Suppose that  $w \in [x, t)$ .

If  $w \in [x, v_1)$ , then the fact that  $z$  has no neighbor in  $[x, v_1)$  implies that  $w$  is adjacent to some vertex  $p$  in  $A_{t_1} - \{z\}$ . Also,  $w + 1 \notin N(s)$ , for otherwise,

$$x, x + 1, \dots, w, p, \dots, t, v_j, \dots, v_i, z, v_1, v_1 - 1, \dots, w + 1, s, s + 1, \dots, y$$

is an  $x - y$  path longer than  $P$ , contradicting our assumptions. If  $w = v_1$ , then  $w + 1 = t$  and we already know that  $t \notin N(A_{t_2})$  (as  $t$  is a successor of  $v_1$ ). We also note that  $z$  has no adjacencies in the region  $[x, v_1)$ . Since  $z$  is adjacent to  $v_1$  (but not to  $t$ ) and since  $t \notin N(A_{t_2})$ , we have verified the following:

For every  $w \in N(A_{t_1}) \cap [x, t)$ , the vertex  $w + 1$  is not in  $N(A_{t_2})$ .

**Case 2.** Suppose that  $w \in [t, p)$ .

If  $wp \in E(G)$ , then  $(w + 1)s \notin E(G)$ , for otherwise

$$x, x + 1, \dots, v_1, z, v_j, \dots, v_i, t, t + 1, \dots, w, p, p - 1, \dots, w + 1, s, s + 1, \dots, y$$

is an  $x - y$  path longer than  $P$ . If, however,  $wz \in E(G)$ , then again  $(w + 1)s \notin E(G)$ , for otherwise,

$$x, x + 1, \dots, w, z, v_j, v_j - 1, \dots, w + 1, s, s + 1, \dots, y$$

is an  $x$ - $y$  path longer than  $P$ , again a contradiction.

Thus, for each  $w \in N(A_{t_1}) \cap [t, p)$ , we see that  $w + 1 \notin N(A_{t_2})$ .

**Case 3.** Suppose that  $w \in [p, v_j)$ .

Clearly,  $p \notin N(A_{t_1})$ . If  $wp$  is an edge of  $G$ , then again  $(w + 1)$  cannot be adjacent to  $s$ , for otherwise,

$$x, x + 1, \dots, v_1, z, v_i, v_i + 1, \dots, w, p, p - 1, \dots, t, v_j, v_j - 1, \dots, w + 1, s, s + 1, \dots, y$$

is an  $x$ - $y$  path longer than  $P$ , a contradiction. If, however,  $wz$  is an edge of  $G$ , then once again  $w + 1$  cannot be adjacent to  $s$ , for otherwise

$$x, x + 1, \dots, w, z, v_j, v_j - 1, \dots, w + 1, s, s + 1, \dots, y$$

is an  $x$ - $y$  path longer than  $P$ , again a contradiction to our assumptions.

Thus, if  $w \in N(A_{t_1}) \cap [p, v_j)$ , then  $w + 1 \notin N(A_{t_2})$ .

**Case 4.** Suppose that  $w \in [v_j, y)$ .

If  $w = v_j$ , then  $w + 1 = s$ , but we already know that  $s \notin N(A_{t_2})$ . Further, we also know that  $ps \notin E(G)$ , for otherwise,

$$x, x + 1, \dots, v_1, z, v_j, v_j - 1, \dots, v_i, t, t + 1, \dots, p, s, s + 1, \dots, y$$

is a longer  $x$ - $y$  path than  $P$ , a contradiction. Thus, suppose that  $pw \in E(G)$ , where  $w \in (s, y)$ . Then  $(w + 1)$  cannot be adjacent to  $s$  for otherwise,

$$x, \dots, v_1, z, v_j, \dots, v_i, t, t + 1, \dots, p, w, w - 1, \dots, s, w + 1, \dots, y$$

is an  $x$ - $y$  path longer than  $P$ , a contradiction. If  $wz$  is an edge of  $G$ , then  $w + 1$  cannot be adjacent to  $s$ , for otherwise

$$x, \dots, v_j, z, w, w - 1, \dots, s, w + 1, \dots, y$$

is an  $x$ - $y$  path longer than  $P$ , which is again a contradiction.

Thus, if  $w \in N(A_{t_1}) \cap [v_j, y)$ , then  $w + 1 \notin N(A_{t_2})$ .

**Case 5.** Suppose that  $w \in V(G) - V(P)$ .

Then it is easy to see that if either  $z$  or  $p$  is adjacent to  $w$ , then  $s$  is not adjacent to  $w$ , or a path longer than  $P$  is immediate.

Thus, if  $w \in N(A_{t_1}) \cap (V(G) - V(P))$  then  $w \notin N(A_{t_2})$ .

From these five cases we see that the function  $f(w) = w + 1$  if  $w \in V(P) - \{y\}$  and  $f(w) = w$  if  $w \in V(G) - V(P)$  is a 1-1 function from  $V(G) - \{y\}$  onto itself that maps adjacencies of  $A_{t_1}$  onto nonadjacencies of  $A_{t_2}$ .

Hence, since  $z \notin N(A_{t_2})$  and since  $f$  is defined for all the remaining  $n - 1$  vertices except  $y$ , (adding back one for possible adjacencies to  $y$ ) we see that

$$\text{deg } A_{t_2} \leq (n - 1) - \text{deg } A_{t_1} + 1$$

or, since  $t_1 \geq 1$ ,

$$\text{deg } A_{t_1} + \text{deg } A_{t_2} \leq n,$$

contradicting our hypothesis. Hence,  $G$  must be hamiltonian-connected. ■

We now consider several immediate corollaries.

**Corollary 3.(Ore[14])** If  $G$  is a graph of order  $p$  such that for each pair of distinct nonadjacent vertices  $x$  and  $y$ ,

$$\text{deg } x + \text{deg } y \geq p + 1,$$

then  $G$  is hamiltonian-connected.

**Proof.** Let  $t_1 = 1$  and  $t_2 = 1$  and recall that  $IC_2 \geq 3$  (see observations before Theorem 2). ■

**Corollary 4. (Dirac[2])** If  $G$  is a graph of order  $p$  satisfying  $\text{deg } x \geq \frac{p+1}{2}$  for each vertex  $x$ , then  $G$  is hamiltonian-connected.

We next present a new generalization of Dirac's Theorem (Theorem A).

**Corollary 5.** If  $G$  is a graph of order  $p$  which satisfies

$$IC_2 \geq k \geq 3 \text{ and } \text{deg } A_t \geq \frac{p+1}{2}$$

for each independent vertex set  $A_t$  where  $1 \leq t \leq \frac{k-1}{2}$ , then  $G$  is hamiltonian-connected.

**Examples:** We first note that the complete bipartite graph  $K_{r,r}$  has order  $n = 2r$  and satisfies  $IC_t \geq r = \frac{n}{2}$ , for  $2 \leq t \leq \frac{n}{2}$ . However, this graph fails to satisfy the generalized degree sum condition and it is not hamiltonian-connected. Thus, the intersection condition alone can be very large without forcing the graph to be hamiltonian-connected.

Next, consider the graph formed by taking  $t$  copies of the graph  $K_r$ , where  $r \geq 3$  and joining each vertex in each copy of  $K_r$  to each of  $t + 1$  other vertices. This graph has order  $n = tr + t + 1$  and satisfies  $IC_2 \geq t + 1$ . Further, it is easy to verify that the degree sum of two sets of independent vertices (whose cardinalities total  $t$  vertices) is exactly  $n + 1$ . Finally, this graph is hamiltonian-connected. However, if we remove one vertex from the set of  $t + 1$  common neighbors, the graph is no longer hamiltonian-connected.

### Section 3. Conclusions

This paper continues the investigation of the significance of neighborhood intersection conditions begun in [12]. Other properties of graphs should be considered, as has been done with degrees and generalized degrees (of several types). Such conditions include matchings, (nonhamiltonian) path and cycle lengths and chromatic number just to mention a few. Additionally, as has been done for generalized degrees, there is no reason to restrict attention to just nonadjacent vertices. Sets of adjacent vertices and arbitrary sets of vertices should certainly be considered.

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