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Lower Bounds for Lower Ramsey Numbers

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ABSTRACT

For any graph G, let $i(G)$ and $\mu(G)$ denote the smallest number of vertices in a maximal independent set and maximal clique, respectively. For positive integers m and n, the lower Ramsey number $s(m,n)$ is the largest integer p so that every graph of order p has $i(G) \leq m$ or $\mu(G) \leq n$. In this paper we give several new lower bounds for $s(m,n)$ as well as determine precisely the values $s(1,n)$.

INTRODUCTION

In [3], Mynhardt introduced the concept of lower Ramsey numbers, which stemmed from the original idea of Ramsey numbers. For any undefined terms, see Chartrand and Lesniak [2]. The *independence (clique) number* of a graph G, denoted $\beta(G)$ ($\omega(G)$), is the largest number of vertices in a maximal independent set (complete subgraph or clique) of G. The *Ramsey number, $r(m,n)$*, is the smallest integer p so that every graph of order p has $\beta(G) \geq m$ or $\omega(G) \geq n$. To introduce the lower Ramsey number, we define the parameters $i(G)$ and $\mu(G)$ to be the order of the smallest maximal independent set and smallest maximal clique, respectively. The *lower Ramsey number $s(m,n)$* is the largest integer p so that every graph G or order p has $i(G) \leq m$ or $\mu(G) \leq n$. The parameter $i(G)$ has previously been studied as a bound for the domination number of a graph, and has been given the name *independent domination number* (see [1]).

In [3,4] Mynhardt gives several results for this new Ramsey-type parameter, including a proof that these numbers do in fact exist and are

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well defined. That is, if for every graph G or order p, $i(G) \leq m$ or $\mu(G) \leq n$, then for every graph G of order less than p, $i(G) \leq m$ or $\mu(G) \leq n$. Furthermore, it is shown that

$$m + n < s(m,n) < 2(m + n).$$

In [4], the upper bound is improved in the case when $m < n/2$. In addition, some values for small m and n are given, as well as some questions related to $s(m,n)$.

It is the purpose of this paper to continue the study of these lower Ramsey numbers. In particular, we determine precisely the values $s(1,n)$ in the next section. In the final section of the paper, we present two new lower bounds for $s(m,n)$, greatly improving the bound $m + n + 1$. The first puts $s(m,n)$ within a range of $2m$ for $m \leq n/4$ while the second essentially answers the question of finding $s(m,m)$. We show if m is sufficiently large and for any $\varepsilon > 0$, then

$$s(m,m) \geq (4 - \varepsilon)m.$$

THE NUMBERS $s(1,n)$.

In [4], by construction Mynhardt shows the following result:

Theorem. For any positive integers m and n,

$$s(m,n) \leq \min\{f(a,b)\} - 1,$$

where a,b are positive integers, $a \leq b$, $ab \leq n \leq a(b + 1)$, and

$$f(a,b) = \begin{cases} m(a+1) + n + b, & \text{if } n = ab; \\ m(a+1) + n + b + 1, & \text{when } n > ab. \end{cases}$$

By evaluating the minimum value of f in this result, the case for $m = 1$ is

Corollary. Let $t^2 \leq n(t + 1)^2$ and $r = n - t^2$. Then

$$s(1,n) < n + 2t + 1 + \left\lceil \frac{r}{t} \right\rceil.$$

We show that these are the correct values for these lower Ramsey numbers.

Theorem 1. Let $t^2 \leq n \leq (t + 1)^2$ and $r = n - t^2$. Then

$$s(1,n) = n + 2t + \left\lceil \frac{r}{t} \right\rceil.$$

Proof. From the corollary above, it only remains to show that $s(1, n)$ is at least the stated values. To proceed in each of the three cases, we will assume that we have a graph G on the desired number of vertices such that $i(G) \geq 2$ and $\mu(G) \geq n + 1$. We will use the following helpful observation: Let $V(G)$ be partitioned into two pieces, $A \cup B = V(G)$ so that the graph induced by A is K_{n+1}. By the choice of G, we know this can always be done. Also, any complete subgraph of G, including a single vertex, can be expanded to a K_{n+1}. If $x \in B$ is non-adjacent to k vertices of A, then K_k must be a subgraph of the graph induced by B. Also note, since $i(G) \geq 2$, this implies that no vertex of G is adjacent to all other vertices of G.

Case 1. Suppose $r = 0$, $p = t^2 + 2t$, G is a graph of order p with $i(G) \geq 2$, and $\mu(G) \geq n + 1$, and A and B partition $V(G)$ so that K_{n+1} is precisely the graph induced by A. The graph induced by B must contain a maximum clique, say of order $t - \alpha$. Call that set of vertices B'. We will show that this is true for no α, hence a contradiction. By our choice of G, we must be able to extend B' to an $n + 1$ clique. Thus, there are $t^2 + 1 - t + \alpha$ vertices, call them A', in A adjacent to all of the $t - \alpha$ vertices in B'. Since each vertex of A' must be nonadjacent to some vertex, all the nonadjacencies of A' must be among the $t + \alpha - 1$ vertices of $B - B'$. But this implies that there is a vertex in $B - B'$ nonadjacent to at least

$$\frac{t^2 + 1 - t + \alpha}{t + \alpha - 1}$$

vertices of A'. But as observed above, this implies that

$$\frac{t^2 + 1 - t + \alpha}{t + \alpha - 1} \leq t - \alpha,$$

$$t^2 + 1 - t + \alpha \leq t^2 - \alpha^2 - t + \alpha,$$

$$\alpha^2 + 1 \leq 0.$$

Since this is true for no real α, no such G can exist, and thus we can conclude when $n = t^2$ that $s(1, n) = t^2 + 2t$.

Case 2. Suppose $0 < r \leq t$, $p = t^2 + r + 2t + 1$ and G is a graph of order p with $i(G) \geq 2$ and $\mu(G) \geq n + 1$. We again choose A and B as in the previous case, with the extra condition that over all possible choices for A, B has as large of a clique as possible. Again form B' and A' as in the previous case, with B' having $t - \alpha$ vertices and A' having $t^2 + r + 1 - t + \alpha$ vertices. Now in addition to the $t^2 + r + 1 - t + \alpha$ nonadjacencies from A' to $B - B'$, since $A' \cup B'$ has order $n + 1$ and $A - A'$ has the same order as B', no vertex in $B - B'$ can be adjacent to all the vertices in $A - A'$. Thus there are at least $(t^2 + r + 1 - t + \alpha) + (t + \alpha)$ nonadjacencies from A to the vertices of $B - B'$. As in Case 1, this implies that

$$\frac{t^2 + r + 1 + 2\alpha}{t + \alpha} \leq t - \alpha,$$

$$t^2 + r + 1 + 2\alpha \leq t^2 - \alpha 2,$$

$$\alpha^2 + 2\alpha + 1 \leq -r.$$

But this can never be true since $r > 0$, and equality follows for $0 < r \leq t$.

Case 3. Suppose $t < r \leq 2t$, and $p = t^2 + r + 2t + 2$. Choose G, A, B, A', and B' as in Case 2. In this case we argue that the number of nonadjacencies from A to $B - B'$ is at least $(t^2 + r + 1 - t + \alpha) + (t + \alpha + 1)$ since the order of $B - B'$ is $t + \alpha + 1$. Consequently, we get

$$\frac{t^2 + r + 1 - t + \alpha + t + \alpha + 1}{t + \alpha + 1} \leq t - \alpha,$$

which yields

$$\alpha^2 + 3\alpha + 2 \leq t - r.$$

Since r is an integer greater than t, and the minimum value of $\alpha^2 + 3\alpha + 2$ is $-\frac{1}{4}$, which occurs when $\alpha = \frac{3}{2}$, the inequality is never true, thus completing this case, and with all possibilities of r exhausted, the proof is complete. ∎

Note that this result implies that $s(1,n)$ is approximately $n + 2\sqrt{n}$, which is considerably larger than the previously known lower bound of $n + 1$. The authors are trying to apply these techniques in order to determine $s(2,n)$, but have so far been unsuccessful in their attempts to determine this number exactly.

LOWER BOUNDS FOR $s(m, n)$.

In this section, we present two new lower bounds, which improve the bound given in [3]. It is useful to observe again, as in Theorem 1, if G is a graph with $i(G) \geq m + 1$ and $\mu(G) \geq n + 1$, $A \cup B$ partitions G where A induces a K_{n+1} and $x \in B$ has k nonadjacencies in A, then K_k must be a subgraph of the graph induced by B.

Theorem 2. If $n = t^2 + r$, $0 \leq r < 2t$, and $2 \leq m \leq n$, then

$$s(m,n) \geq n + \lfloor 2t\sqrt{m} \rfloor.$$

Proof. Let G be a graph of order $p = \lfloor n + 2t\sqrt{m} \rfloor$ with $i(G) \geq m + 1$, $\mu(G) \geq n + 1$. Let A, B, A', and B' be chosen so that A induces a K_{n+1}, B' a maximum clique in B of order $t - \alpha$, while A' is the subset of vertices of A that are adjacent to all the vertices of B'. Since every vertex must be in an independent set of order at least $m + 1$, each vertex of G is nonadjacent to at least m vertices. Thus the number of nonadjacencies from A to $B - B'$ is at least

$$(t^2 + r + 1 - t + \alpha)m + (t - \alpha)(m - 1).$$

But this implies that there is a vertex of $B - B'$ nonadjacent to at least

$$\frac{(t^2 + r + 1 - t + \alpha)m + (t - \alpha)(m - 1)}{\lfloor 2t\sqrt{m} \rfloor - t + \alpha - 1}$$

vertices of A. By the observation above, since $t - \alpha$ was as large as a clique in B could be, it follows that

$$\frac{(t^2 + r + 1 - t + \alpha)m + (t - \alpha)(m - 1)}{\lfloor 2t\sqrt{m} \rfloor - t + \alpha - 1} \leq t - \alpha.$$

But this implies that

$$(t^2 + r + 1)m + \alpha - t \leq (t - \alpha)(2t\sqrt{m} - t + \alpha) - t + \alpha.$$

Consequently,

$$t^2m + (r + 1)m + t^2 - 2\alpha t + \alpha^2 - 2t^2\sqrt{m} + 2\alpha t\sqrt{m} \leq 0.$$

Since m is fixed and α is a function of t, say $\alpha = ct$, with $c \geq 1 - 2\sqrt{m}$, we get

$$(m + 1 - 2c + c^2 - 2\sqrt{m} + 2c\sqrt{m})t^2 + (r + 1)m \leq 0. \qquad (1)$$

The coefficient $c^2 + 2(\sqrt{m} - 1)c + (m - 2\sqrt{m} + 1)$ of t^2 attains a minimum value when $c = 1 - \sqrt{m}$, which yields a minimum value of 0. Thus (1) holds only when

$$(r + 1)m \leq 0.$$

Since this is never true, it follows that $s(m, n) \geq n + \lfloor 2t\sqrt{m} \rfloor$. ∎

Theorem 2 implies that $s(n, m) \geq n + 2\sqrt{n - r}\,\sqrt{m} - 1$, which is a vast improvement over $m + n + 1$, and in fact forces the range for $s(n, m)$ to be quite small for small m. In [4] Mynhardt proves that

$$s(m,n) \leq m + n + b + am$$

for any positive integers a and b such that $a \leq b$, $ab \leq n$, and $a(b+1) \geq n$.

Corollary 3. For positive integers m and n, $m \leq n$, except when $m = 2$ and $n = 3$,

$$s(m,n) \leq n + 2\sqrt{n}\sqrt{m} + 2m.$$

Proof. Let $a = \lceil \sqrt{n}/\sqrt{m} \rceil$ and $b = \lfloor n/a \rfloor$. It is a simple exercise to show that a and b satisfy the necessary conditions, and thus it follows that

$$\begin{aligned} s(m,n) &\leq m + n + \lfloor n/a \rfloor + \lceil \sqrt{n}/\sqrt{m} \rceil \cdot m \\ &\leq m + n + n/a + (\sqrt{n}/\sqrt{m} + 1)m \\ &\leq n + 2\sqrt{n}\cdot\sqrt{m} + 2m. \quad \blacksquare \end{aligned}$$

Using this result and the fact that for the range of r in Theorem 2, $\sqrt{n-r} \geq \sqrt{n} - 1$, we get

$$n + 2\sqrt{n}\sqrt{m} - 2\sqrt{m} - 1 \leq s(n,m) \leq n + 2\sqrt{n}\sqrt{m} + 2m.$$

The next bound attempts to suggest an answer to the question of whether $s(m,n) = 2(m+n)$ for $m \geq n/2$. The bound gives no information when m is not a positive fraction of n (thus the necessity for this result, as well as Theorem 2).

Theorem 4. Given positive integers m and n with $m = cn$, $0 < c \leq 1$, if $\varepsilon > 1 - \sqrt{4c/(c+1)^2}$, then

$$s(m,n) \geq (2 - \varepsilon)(m + n)$$

for sufficiently large n.

Proof. Let $\varepsilon > 0$ and c satisfy the given condition. Suppose there exists a graph G of order $(2-\varepsilon)(m+n) = (2-\varepsilon)(1+c)n$ with $i(G) \geq m+1$ and $\mu(G) \geq n+1$. Follow this procedure: Choose a maximum clique Z of G, which must contain at least $n+1$ vertices since $\mu(G) \geq n+1$, and let H_1 be the graph induced by $G - Z$. Choose Z_1 to be a maximum clique of H_1 and let $Z = Z \cup Z_1$, and let H_2 be the graph induced by $H_1 - Z_1$. Continue, let Z_{j-1} be a maximum clique of H_{j-1}, let $Z = Z \cup Z_{j-1}$, and let H_j be the graph induced by $H_{j-1} - Z_{j-1}$. We proceed by showing that, regardless of j, for n sufficiently large, H_j contains a clique of order xn, where x is some appropriately chosen and fixed positive value. Note, this would imply that $i(G) \leq 4n/xn < cn = m$ since we would be able to cover the vertices of G with fewer than cn cliques, and a contradiction would result. Let Z_j be a maximum clique of H_j and suppose Z_j has order xn. We will show that x is positive by our choice of c and ε.

Suppose, at this point, Z has order $(1 + \alpha)n$, that is, the order of H_j is $((2 - \varepsilon)(1 + c) - 1 - \alpha)n$. Since each vertex is in an independent set of order $m + 1$, we can assume that each of the $(1 + \alpha)n$ vertices of Z has $cn - j$ nonadjacencies to the vertices of $H_j - Z_j$. There must be a vertex in $H_j - Z_j$ nonadjacent to $(1 + \alpha)(cn - j)n/(((2 - \varepsilon)(1 + c) - 1 - \alpha - x)n)$ vertices of Z. This implies that $(1 + \alpha)(cn - j)n \leq ((2 - \varepsilon) \cdot (1 + c) - 1 - \alpha - x)n(x + \alpha)n$ since if a vertex of $H_j - Z_j$ were nonadjacent to more than $(x + \alpha)n$ vertices, a clique larger than xn would occur in H_j. Thus, by letting $y = (2 - \varepsilon)(1 + c) - 1$, we get

$$(x + \alpha)^2 - y(x + \alpha) + (c - j/n)(1 + \alpha) \leq 0.$$

Let $f(x) = (x + \alpha)^2 - y(x + \alpha) + (c - j/n)(1 + \alpha)$, and let $\bar{x}$ be the smallest value such that $f(\bar{x}) = 0$. It follows that

$$\bar{x} = 1/2(y - 2\alpha - (y^2 - 4(c - j/n)(1 + \alpha))^{1/2}.$$

Observe that this implies that if $f(x) \leq 0$ then the only such possible values of x have $x \geq \bar{x}$. We claim that for n sufficiently large, $\bar{x}$ is bounded below by some positive constant (independent of j, α, and n). Note the difference of two quantities has a positive lower bound if the difference of the squares of the quantities has a positive lower bound. So to verify this claim, we show that

$$g(\alpha) = (y - 2\alpha)^2 - (y^2 - 4(c - j/n)(1 + \alpha))$$

has a positive lower bound independent of j, α, and n for large n.

Substituting for y gives that

$$1/4 \cdot g(\alpha) = \alpha^2 + (\varepsilon - 1)(1 + c)\alpha + c - j/n(1 + \alpha).$$

For n appropriately large, the expression $(1 + \alpha)j/n$ is small, since j is bounded by $4/x$, so to verify that $g(\alpha)$ has a positive lower bound it will suffice to show that

$$h(\alpha) = \alpha^2 + (c + 1)(\varepsilon - 1)\alpha + c$$

has a positive lower bound. The minimum value of h occurs at $\bar{\alpha} = -(c + 1)(1 - \varepsilon)/2$ and

$$h(\bar{\alpha}) = c - \frac{(c + 1)^2(1 - \varepsilon)^2}{4} > 0$$

by assumption. Hence,

$$h(\alpha) \geq c - \frac{(c+1)^2(1-\varepsilon)^2}{4}$$

for all α. Thus the appropriate choice of large n gives that

$$g(\alpha) \geq 2c - \frac{(c+1)^2(1-\varepsilon)^2}{2},$$

which completes the proof of Theorem 4. ∎

As a special case, when $c = 1$, we get

Corollary 5. For sufficiently large m, and any $\varepsilon > 0$,

$$s(m,m) \geq (4-\varepsilon)m.$$

Since $s(m,m) < 4m$ for all m, this essentially gives $s(m,m) = 4m - o(m)$, for sufficiently large m.

Although these new lower bounds are improvements over what was known, there still is a significant amount of work that remains. There still remains the question of whether $s(m,n) = 2(m+n)$ when $m \geq n/2$, as well as determining $s(m,n)$ for small values of m, $m \geq 2$. In addition, there is the problem of comparing the lower bound of Theorem 4, when $c < \frac{1}{2}$, with the upper bound given in [4]. These two may already essentially give these lower Ramsey numbers.

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