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TWO - IRREGULAR GRAPHS

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1. INTRODUCTION.

One of the most elementary results in graph theory is that a graph on n vertices must have at least two it its vertices with the same degree. Thus, it seems natural to consider those graphs in which no more than two of its vertices have the same degree for each possible degree, and we call such a graph 2−*irregular*.

In this paper we consider the following question: Which graphs G of order n are embeddable in a 2−irregular graph of order n? Clearly an arbitrary such graph G must have size at most $3n-7$, since the graph in which 3 vertices have maximum degree cannot be embedded in a 2−irregular graph of the same order. A question left unanswered is whether each (n,q) graph with $q \leq 3n-7$ is embeddable in a 2−irregular graph of order n.

In this paper we follow standard notation similar to that of [1] and [2]. In particular, we let $d_G(x)$ denote the degree of x in G and $N_G(x)$ denote the neighborhood of x in G. When the graph G is obvious, the subscript G will be deleted.

2. RESULTS.

There are two principal results in the paper, each of which give sufficient conditions for a graph to be embeddable in a 2−irregular graph of the same order. We state these results first.

THEOREM 2. *Let G be a graph of order $n \geq \frac{3}{2}d^4$, where $d \geq 8$. If $\Delta(G) \leq d$, then G is embeddable in a 2−irregular graph of order n.*

THEOREM 3. *For all $n \geq 8^4 3^5/2$ any (n,q) graph G with*

$$q \leq \frac{3}{2}n + \frac{1}{2}(\frac{2}{3})^{1/4} n^{1/4}$$

is embeddable in a 2−irregular graph of order n.

In order to establish the above theorems we need the following three results.

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LEMMA 1. *Let G be a 2–irregular graph with ℓ vertices of degree $\geq c$. Then, G contains a k–matching where $k = min\{c, \lfloor \frac{\ell}{2} \rfloor\}$.*

THEOREM 1. *Let G be a graph of order n containing two vertices x and y with disjoint neighborhoods, each neighborhood of order at most $d \leq \lfloor \frac{n}{48} \rfloor - 2$. If the graph $G - x - y$ is 2–irregular, then G is embeddable in a 2–irregular graph G^* of order n.*

LEMMA 2. *Let G be a graph of order n and let $V(G)$ be partitioned into sets H_1 and H_2 with $H_1 = \{x_1, x_2, \cdots, x_{\lfloor n/3 \rfloor}\}$. If H_1 is an independent set of vertices and for each ℓ, $1 \leq \ell \leq \lfloor n/3 \rfloor$, the set of vertices $\{x_i \mid 1 \leq i \leq \ell\}$ has at most 2ℓ adjacencies in the set H_2, then G is embeddable in a 2–irregular graph L of order n.*

3. PROOFS.

PROOF OF LEMMA 1: The k–matching will be found by a greedy algorithm that we now describe. Let S be the ℓ–set of vertices of degree $\geq c$.

Choose the edge x_1y_1 such that x_1 is a vertex in S of smallest degree and (when $S \cap N(x_1) \neq \emptyset$) y_1 is a vertex in $S \cap N(x_1)$ of smallest degree. If $S \cap N(x_1) = \emptyset$, then let y_1 be any vertex of $N(x_1)$. Given that j independent edges $x_1y_1, x_2y_2, \cdots, x_jy_j$ have been selected, choose $x_{j+1}y_{j+1}$ such that x_{j+1} is a vertex in $S - \{x_1, y_1, x_2, y_2, \cdots, x_j, y_j\}$ of smallest degree and y_{j+1} is a vertex of smallest degree in $(S - \{x_1, y_1, x_2, y_2, \cdots, x_j, y_j\}) \cap N(x_{j+1})$ (when this set is nonempty). Let y_{j+1} be any unused vertex in $N(x_{j+1})$ when $S - \{x_1, y_1, x_2, y_2, \cdots, x_j, y_j\} \cap N(x_{j+1}) = \emptyset$.

Suppose that this process stops after edges $x_1y_1, x_2y_2, \cdots, x_my_m$ have been found with $m < \min\{c, \lfloor \frac{\ell}{2} \rfloor\}$. Then, $S - \{x_1, y_1, x_2, y_2, \cdots, x_m, y_m\} \neq \emptyset$ and it follows that the vertex x_{m+1} in $S - \{x_1, y_1, x_2, y_2, \cdots, x_m, y_m\}$ of smallest degree has all of its adjacencies in the set $\{x_1, y_1, x_2, y_2, \cdots, x_m, y_m\}$. Let $b = d(x_{m+1})$.

Further, suppose that x_{m+1} is adjacent to t vertices in the set $\{x_1, x_2, \cdots, x_m\}$, so that $b \leq m + t$. By the algorithm, if $x_{m+1}x_r \in E(G)$ for some $1 \leq r \leq m$, then $y_r \in S$ and $d(y_r) \leq d(x_{m+1})$. Also, from the algorithm $c \leq d(x_1) \leq d(x_2) \leq \cdots \leq d(x_m) \leq d(x_{m+1})$, so that ther are at least $m + t + 1$ vertices of S of degree at most b. Therefore, since G is 2–irregular, $b \geq c + (m + t - 2)/2 + 1/2$, which implies $m + t \geq b \geq c + (m + t)/2 - 1/2$. Hence, $2m \geq m + t \geq 2c - 1$, from which it follows that $m \geq c$. This contradicts that $m < c$ and completes the proof of Lemma 1. ■

PROOF OF THEOREM 1: For convenience let $G' = G - x - y$ and let z be a vertex in G' of largest degree. Since the neighborhoods of x and y are disjoint, we assume that $xz \notin E(G)$. The method of proof is such that there is no loss of generality in assuming that $d_G(x) \leq d_G(y)$.

We first prove that the theorem is true when there is a matching of order $\lceil n/4 \rceil$ in the complementary graph $\overline{G' - (N_G(x) \cup N_G(y) \cup \{z\})}$. Later we show for $d \leq \lfloor n/48 \rfloor - 2$ that this is an appropriate assumption.

Let M be a matching of order $\lceil n/4 \rceil$ in $\overline{G' - (N_G(x) \cup N_G(y) \cup \{z\})}$ and set

$$D_x = N_G(x) \cap V(G')$$

$$D_y = N_G(y) \cap V(G'), \quad \text{and}$$

$$D = V(G') - (D_x \cup D_y \cup V(M) \cup \{z\}).$$

Then $\lceil n/2 \rceil - 2 \geq |D| \geq \lfloor n/2 \rfloor - 2d - 4$. Set $d^* = |D|$ and partition D into two sets D_1 and D_2 and let $|D_1| = i$. Form the sequence of graphs $G_0, G_1, \cdots, G_{\lfloor d^*/2 \rfloor}$ such that for each i : $V(G_i) = V(G)$ and

$$E(G_i) = E(G) \cup \{xw \mid w \in D_1\} \cup \{yw \mid w \in D_2\} \cup \{yz\} \cup \{uw \mid uw \in E(M)\}.$$

We assume throughout (since it does not effectively alter the proof) that $xz \notin E(G)$, and $yz \in E(G)$. The cases where $xz \in E(G)$ or neither x nor y is adjacent to z can be handled in a similar way.

Clearly G is a subgraph of G_i and G_i is 2−irregular, unless it contains three vertices of degree $d_{G_i}(x)$ (one of which is x) or three vertices of degree $d_{G_i}(y)$ (one of which is y). Observe that $d_{G_i}(x) = d_G(x) + i \leq d_{G_i}(y) = d_G(y) + d^* - i$ for $0 \leq i \leq \lfloor d^*/2 \rfloor$.

Let $V(M) = \{u_1, v_1, u_2, v_2, \cdots, u_t, v_t\}$ and $E(M) = \{u_1v_1, u_2v_2, \cdots, u_tv_t\}$ with $t = \lceil n/4 \rceil$. Suppose for some i, $1 \leq i \leq \lfloor d^*/2 \rfloor$, that G' does not contain two vertices of degree $d_{G_i}(x) - 1$. Then either G_i is 2−irregular or there exist two vertices in G' of degree $d_{G_i}(y) - 1$. Since the proof is complete (under the original matching assumption) when G_i is 2−irregular, suppose the contrary. Thus G' contains two vertices of degree $d_{G_i}(y) - 1$.

Next, we sequentially alter G_i to show that there are more vertices of relatively "low" degree in G'. For each j, $1 \leq j \leq t$, let G_{ij} be the graph with $V(G_{ij}) = V(G_i)$ and

$$E(G_{ij}) = (E(G_i) - \{u_\ell v_\ell \mid 1 \leq \ell \leq j\}) \cup \{u_\ell y \mid 1 \leq \ell \leq j\} \cup \{v_\ell y \mid 1 \leq \ell \leq j\}.$$

But $d_{G_i}(w) = d_{G_{ij}}(w)$ for each $w \in V(G_{ij}) - \{y\}$, so if G_{ij} is not 2−irregular, then G' contains two vertices of degree $d_{G_i}(y) + 2j - 1$. Thus, if G_{ij} is not 2−irregular for each j, then G' contains two vertices of each of the following degrees:

$$d_{G_i}(y) - 1, d_{G_i}(y) + 1, \cdots, d_{G_i}(y) + 2t - 1.$$

Now the above discussion applies as well to the similar graph G_i' obtained from G_i by inserting the edge xz. Since z is a vertex of highest degree in G', G_i' is 2−irregular unless it contains three vertices of degree $d_{G_i'}(x)$ or three vertices of degree $d_{G_i'}(y)$. Also, $d_{G_i'}(x) = d_{G_i}(x) + 1$. Since it has been assumed that G' contains at most one vertex of degree $d_{G_i}(x) - 1 = d_{G_i'}(x) - 2 = d_{G_{i-1}'}(x) - 1$, this means that G_{i-1}' is 2−irregular unless G' contains two vertices of degree $d_{G'_{i-1}}(y) - 1 = d_{G_{i-1}}(y) - 1 = d_{G_i}(y)$. Assuming G'_{i-1} is not 2–irregular, this means, by defining graphs $G'_{i-1,j}$ from G'_{i-1} in the same way that G_{ij} was defined from G_i, that G' contains two vertices of each of the following degrees:

$$d_{G_i}(y), d_{G_i}(y) + 2, \cdots, d_{G_i}(y) + 2t.$$

In summary, we have that G' contains two vertices of each of the degrees

$$d_{G_i}(y) - 1, d_{G_i}(y), \cdots, d_{G_i}(y) + 2t,$$

and so G' contains at least $4t + 4$ vertices. Since $t = \lceil n/4 \rceil$, G' contains at least $n + 4 > n - 2$ vertices, a contradiction. This means that for each i, $1 \leq i \leq \lfloor d^*/2 \rfloor$, G_i contains two vertices of degree $d_{G_i}(x) = d_G(x) + i$. Hence G' contains two vertices of each of the following degrees:

$$d_G(x), d_G(x) + 1, \cdots, d_G(x) + \lfloor d^*/2 \rfloor - 1.$$

By a similar argument, altering the degree of x when forming the graph G_{ij} from G_i rather than the degree of y, it follows that G' contains two vertices of each of the degrees:

$$d_G(y) + d^* - 2, d_G(y) + d^* - 3, \cdots, d_G(y) + \lceil d^*/2 \rceil - 1.$$

Since $d_G(x) \leq d_G(y)$, this implies that G' contains

$$2(d^* - 1) = 2d^* - 2 \geq 2(\lfloor n/2 \rfloor - 2d - 4) - 2 \geq n - 4d - 11$$

vertices of degree at most

$$d_G(y) + d^* - 2 \leq d + \lceil n/2 \rceil - 2 - 2 = d + \lceil n/2 \rceil - 4.$$

It is easy to show that G' cannot have such a large number of vertices of this bounded degree. To see that this is the case, first enlarge both D_x and D_y in $V(G')$ to disjoint subsets D'_x and D'_y, each of order d. Next select two disjoint ℓ–element subsets B_ℓ and C_ℓ of smallest degree in $V(G') - (D'_x \cup D'_y)$, for $1 \leq \ell \leq s$, where $s = \lfloor n/6 \rfloor$. Finally, let $A_\ell = V(G') - (D'_x \cup D'_y \cup B_\ell \cup C_\ell)$, $1 \leq \ell \leq s$. Form the graphs H_ℓ, $1 \leq \ell \leq s$, from G by making x adjacent to $A_\ell \cup D'_x \cup D'_y \cup B_\ell$ and y to $A_\ell \cup D'_x \cup D'_y \cup C_\ell$. Then, $d_{H_\ell}(x) = d_{H_\ell}(y) \geq \lceil 2n/3 \rceil - 2 + \ell$, since $|A_\ell| \geq \lceil 2n/3 \rceil - 2d - 2$. Since H_ℓ is 2–irregular, unless G' contains a vertex of degree $d_{H_\ell}(x) - 2$, the theorem follows unless G' contains vertices of each of the degrees:

$$d_{H_1}(x) - 2 < d_{H_2}(x) - 2 < \cdots < d_{H_s}(x) - 2.$$

But $\lceil 2n/3\rceil - 3 \le d_{H_1}(x) - 2$, so that G' contains at least $n - 4d - 11$ vertices of degree at most $d + \lceil n/4\rceil - 4$ and at least $s = \lfloor n/6\rfloor$ vertices of degree at least $\lceil 2n/3\rceil - 3$, a contradiction for $d \le \lfloor n/48\rfloor - 2$.

We complete the proof by showing for $d \le \lfloor n/48\rfloor - 2$ that there is a matching of order $\lceil n/4\rceil$ in the graph $\overline{G' - (N_G(x) \cup N_G(y) \cup \{z\})}$. First observe that G' a 2−irregular graph implies that it contains at most $\lfloor (n-2)/2\rfloor$ vertices of degree $> \lceil 3(n-2)/4\rceil - 1$. Therefore, $\overline{G'}$ contains at least $\lceil (n-2)/2\rceil$ vertices of degree $\ge \lfloor (n-2)/4\rfloor$. By Lemma 1, $\overline{G'}$ contains a $\ge \lfloor (n-2)/4\rfloor$ matching M.

Set $M' = \overline{G' - (N_G(x) \cup N_G(y) \cup \{z\})} \cap M$. As was done earlier, let $D_x = N_G(x) \cap V(G'), D_y = N_G(y) \cap V(G')$, and $D = V(G') - (D_x \cup D_y \cup V(M') \cup \{z\})$, and let $D_1 \cup D_2$ be a partition of D into two sets with $|D_1| = i$ and $|D| = d^*$. Form the sequence of graphs $G_0, G_1, \cdots, G_{\lfloor d^*/2\rfloor}$ such that for each i, $V(G_i) = V(G)$ and

$$E(G_i) = E(G) \cup \{xw \mid w \in D_1\} \cup \{yw \mid w \in D_2\} \cup \{yz\} \cup \{uw \mid uw \in E(M')\}.$$

Again, it is clear that G is a subgraph of G_i, and G_i is 2−irregular unless G' contains two vertices of degree $d_{G_i}(x) - 1$ or two vertices of degree $d_{G_i}(y) - 1$. But, for each i,

$$d_{G_i}(x) - 1 = d_{G_i}(x) + i - 1 \le d_G(y) + d^* - i - 1 \le d_{G_i}(y) - 1 \le d + \lceil (n-2)/2\rceil - i.$$

Therefore, G' contains at least $2(\lfloor d^*/2\rfloor) + 1 \ge d^* \ge \lceil (n-2)/2\rceil - 2d - 1$ vertices of degree $\le \lceil (n-2)/2\rceil + d$.

We partition $V(G')$ into three sets A, B, and C as follows: Let A be any set of $\lceil (n-2)/2\rceil - 2d - 1$ vertices of degree $\le \lceil (n-2)/2\rceil + d$, let B be those vertices of degree $\ge \lfloor 7(n-2)/8\rfloor$, and let $C = V(G') - (A \cup B)$. Observe that $|B| \le \lceil n/4\rceil + 1$, so

$$|C| \ge n - 2 - (\lceil (n-2)/2\rceil - 2d - 1) - (\lceil n/4\rceil + 1) \ge \lfloor n/4\rfloor + 2d - 3.$$

Also, each vertex of C has degree $\le \lfloor 7(n-2)/8\rfloor$, so that in $\overline{G'}$ the vertices of C are of degree $\ge \lceil (n-2)/8\rceil$. In $\overline{G'}$ the vertices of A are of degree $\ge \lfloor (n-2)/2\rfloor - d - 1$.

Since $|C| \ge \lfloor n/4\rfloor + 2d - 3$, with each vertex of degree $\ge \lceil (n-2)/8\rceil$ in $\overline{G'}$, $\overline{G'}$ has a $\lceil (n-2)/8\rceil$ matching M_1 with at least $\lceil (n-2)/8\rceil$ vertices of the matching in the set C. Further, each vertex in $A - V(M_1)$ has degree

$$\ge \lfloor (n-2)/2\rfloor - d - 2 - \lfloor (n-2)/4\rfloor$$

in $\overline{G' - M_1}$. Also

$$|A - V(M_1)| \ge \lceil (n-2)/2\rceil - 2d - 1 - \lceil (n-2)/8\rceil \ge \lceil 3(n-2)/8\rceil - 2d - 2.$$

Therefore, by Lemma 1, $\overline{G' - M_1}$ contains a matching M_2 of order $\lfloor 3(n-2)/16 \rfloor - d - 1$. Then $M_1 \cup M_2$ is a matching of order

$$\lfloor 3(n-2)/16 \rfloor - d - 1 + \lceil (n-2)/8 \rceil \geq \lfloor 5(n-2)/16 \rfloor - d - 1$$

in $\overline{G'}$. Since at most $2d+1$ of the vertices of $M_1 \cup M_2$ are in $N_G(x) \cup N_G(y) \cup \{z\}$, if $\lfloor 5(n-2)/16 \rfloor - 3d - 2 \geq \lceil n/4 \rceil$, then $\overline{G' - (N_G(x) \cup N_G(y) \cup \{z\})}$ contains the required matching. But this follows when $d \leq \lfloor n/48 \rfloor - 2$. ■

PROOF OF LEMMA 2: Let $H_1 = \{x_1, x_2, \cdots, x_{\lfloor n/3 \rfloor}\}$ and $H_2 = \{y_1, y_2, \cdots, y_{\lceil 2n/3 \rceil}\}$. We sequentially label the vertices of H_2 with $y_1, y_2, \cdots, y_{\lceil 2n/3 \rceil}$ such that the adjacencies between H_1 and H_2 (in G) satisfy the conditions of the Lemma. It is easy to see that the labelling of H_2 can be done such that these adjacencies will be preserved in the graph L that is described below.

There are two different descriptions for L depending upon whether $2\lfloor n/3 \rfloor < \lceil 2n/3 \rceil$ or $2\lfloor n/3 \rfloor = \lceil 2n/3 \rceil$. In the first case the edge set of L is

$$E(L) = \{y_i y_j \mid 1 \leq i, j \leq \lceil 2n/3 \rceil, i \neq j\} \cup \{x_i y_j \mid 1 \leq j \leq 2i,\ 1 \leq i \leq \lfloor n/3 \rfloor\}.$$

In the second case

$$\begin{aligned} E(L) = &\{y_i y_j \mid 1 \leq i, j \leq \lceil 2n/3 \rceil, i \neq j\} \cup \{x_i y_j \mid 1 \leq j \leq 2i,\ 1 \leq i < \lfloor n/3 \rfloor - 1\} \\ &\cup \{x_{\lfloor n/3 \rfloor - 1} y_j \mid 1 \leq j \leq \lceil 2n/3 \rceil - 1\} \cup \{x_{\lfloor n/3 \rfloor} y_j \mid 1 \leq j \leq \lceil 2n/3 \rceil, j \neq \lceil 2n/3 \rceil - 1\}. \end{aligned}$$

Observe that in the first case the degree sequence of L is

$$d_L(x_i) = 2i \qquad \text{for } 1 \leq i \leq \lfloor n/3 \rfloor,$$

$$d_L(y_j) = d_L(y_{j+1}) = n - (j+1)/2 \qquad \text{for } j = 1, 3, \cdots, \lfloor 2n/3 \rfloor - 1,$$

$$d_L(y_j) = \lceil 2n/3 \rceil - 1 \qquad \text{for } 2\lfloor n/3 \rfloor \leq j \leq \lceil 2n/3 \rceil.$$

In the second case the degree sequence of L is

$$d_L(x_i) = 2i \qquad \text{for } 1 \leq i \leq \lfloor n/3 \rfloor - 2,$$

$$d_L(x_{\lfloor n/3 \rfloor - 1}) = d_L(x_{\lfloor n/3 \rfloor}) = \lceil 2n/3 \rceil - 1,$$

$$d_L(y_j) = d_L(y_{j+1}) = n - (j+1)/2 \qquad \text{for } j = 1, 3, \cdots, \lfloor 2n/3 \rfloor - 5,$$

$$d_L(y_{\lceil 2n/3 \rceil - 3}) = d_L(y_{\lceil 2n/3 \rceil - 1}) = \lceil 2n/3 \rceil + 1,$$

$$d_L(y_{\lceil 2n/3 \rceil}) = d_L(y_{\lceil 2n/3 \rceil - 2}) = \lceil 2n/3 \rceil.$$

In each case the graph L is 2−irregular and contains G as a subgraph. ■

PROOF OF THEOREM 2: The proof uses Theorem 1. Let S be a largest independent set of vertices in G. Since the independence number $\beta(G) \geq n/(d+1)$, the cardinality of the set $|S| \geq n/(d+1)$.

Sequentially remove pairs of vertices in $V(G) - S$ that have disjoint neighborhoods in the present graph obtained from G by the deletions. Continue this removing of pairs of vertices until a subgraph H of G is obtained such that each pair of vertices in $V(H) - S$ have a common neighbor in H. But then $S \subseteq V(H)$ and any vertex x in $V(H) - S$ is at a distance ≤ 2 from any other vertex of $V(H) - S$. Also, since S is maximally independent, each vertex in $V(H) - S$ is adjacent to some vertex of S. Let x be a fixed vertex of $V(H) - S$ and assume x is adjacent to t, $t \leq d$, vertices of S. Since $\Delta(G) \leq d$ and x is at distance ≤ 2 from each vertex of $V(H) - S$, $V(H) - S$ has at most

$$(d-t)(d-2) + t(d-1) + (d-t) + 1 \leq (d-1)d + 1$$

vertices. Further, these $(d-t)(d-2) + t(d-1) + (d-t) + 1$ vertices have at most

$$(d-t)(d-2)(d-1) + t(d-1)(d-1) + d \leq d(d-1)^2 + d$$

adjacencies in S.

Let T denote the vertices of S adjacent to some vertex in $V(H) - S$. By Lemma 2, if $|S - T| \geq \frac{1}{3}|V(H)|$, then H is embeddable in a 2−irregular graph H'. Further, if the pairs of vertices of G removed to form H are replaced a pair at a time, the 2−irregular graph H' can be enlarged, through repeated use of Theorem 1, to obtain a 2−irregular graph G' containing G.

Thus the proof is complete if $|S - T| \geq \frac{1}{3}|V(H)|$ and as long as each graph obtained by replacing the pairs of removed vertices satisfies Theorem 1. But

$$|S - T| \geq |S| - d^3 + 2d^2 - 2d \geq \frac{1}{3}(|S| + d^2 + 1) \geq \frac{1}{3}|V(H)|$$

as long as $2|S| \geq 3d^3 - 5d^2 + 6d + 1$. Also, $2|S| \geq 2n/(d+1)$. Hence the above inequalities hold as long as $n \geq \frac{3}{2}d^4 - d^3 + \frac{1}{2}d^2 + \frac{7}{2}d + 1$. Also, for $n \geq \frac{3}{2}d^4$ and $d \geq 8$, we have that

$$|V(H)| \geq \beta(G) \geq n/(d+1) \geq \frac{3d^4}{2(d+1)} \geq 48d + 144.$$

But $n' \geq 48d + 144$ implies $\lfloor n'/48 \rfloor - 2 \geq d$, so that H satisfies the conditions of Theorem 1, as well as those graphs obtained by replacing pairs of removed vertices. Thus, under the given conditions, Theorem 2 holds. ■

PROOF OF THEOREM 3: There is no loss of generality in assuming G has n vertices and the maximum number of edges allowed. Observe from Theorem 2 that we may assume there exists a vertex z such that $d_G(z) > (\frac{2}{3})^{1/4}n^{1/4}$. Throughout the proof we assume z is the vertex of largest degree. Since $d_G(z) > (\frac{2}{3})^{1/4}n^{1/4}$ and $q \leq \frac{3}{2}n + \frac{1}{2}(\frac{2}{3})^{1/4}n^{1/4}$, G must contain either a vertex of degree $0, 1$ or 2.

The first stage of the proof consists of repeatedly deleting a set of 3 (or 4) vertices incident to a total of at least $5 \geq (\frac{3}{2})3$ (or $6 \geq (\frac{3}{2})4$) edges such that if the deleted graph is 2−irregular, then the graph prior to deletion is embeddable in a 2−irregular graph. We first describe one step in this deletion process and then discuss how ofter the deletion process is repeated. There are several possibilities to consider in the deletion process, each of which dictates the choice of the 3 (or 4) vertex set to be deleted. We consider these possibilities as separate cases. It is assumed that the graph to which the deletion process is applied is a (n_1, q_1) graph G_1 with $n_1 < q_1$.

Case I: G_1 *contains an isolated vertex* u.

Since G_1 contains an isolated vertex and $n_1 < q_1$, there exists a pair of vertices v and w in G_1 which together are incident to at least 5 edges of G_1. Let $G_2 = G_1 - \{u, v, w\}$, and assume that G_2' is a 2−irregular graph of order $n_1 - 3$ containing G_2. Let G_2^* be the graph with $V(G_2^*) = V(G_1)$ and

$$E(G_2^*) = E(G_2') \cup \{vx \mid x \in V(G_2^*) - \{v\}\} \cup \{wx \mid x \in V(G_2^*) - \{w, u\}\}$$
$$\cup \{wu \mid \text{when } G_2' \text{ has at most one isolated vertex}\}.$$

It is clear that $d_{G_2^*}(u) = 1$ or 2. If $d_{G_2^*}(u) = 1$, then G_2' has at least two isolated vertices, so that for $x \in V(G_2')$,

$$2 \leq d_{G_2^*}(x) \leq |V(G_2^*)| - 4, \; d_{G_2^*}(v) = |V(G_2^*)| - 1, \text{ and } d_{G_2^*}(w) = |V(G_2^*)| - 2.$$

If $d_{G_2^*}(u) = 2$, then G_2^* has at most two vertices of degree 2. Also, then for $x \in V(G_2')$,

$$d_{G_2^*}(x) \leq |V(G_2^*)| - 2, \text{ and } d_{G_2^*}(v) = d_{G_2^*}(w) = |V(G_2^*)| - 1.$$

Thus, for either possiblitity, when G_2' is 2−irregular, then so is G_2^* with G_1 a subgraph of G_2^*. Also, G_2 is a (n_2, q_2) graph with $n_2 = n_1 - 3$ and $q_2 \leq q_1 - 5$.

Case II: $\delta(G_1) = 1$

Let $u \in V(G_1)$ such that $d_{G_1}(u) = 1$, and let $vu \in E(G_1)$. If $d_{G_1}(u) \geq 2$, then since $n_1 < q_1$, there exists a vertex w in G_1 such that v and w are together incident to at least 5 edges of G_1. In this case define G_2, G_2' and G_2^* as was done in Case I. Then G_2^* contains G_1, is 2−irregular when G_2' is 2−irregular, and G_2 is an (n_2, q_2) graph with $n_2 = n_1 - 3$ and $q_2 \leq q_1 - 5$.

If $d_{G_1}(v) = 1$, then there exist vertices w and t in G_1, each of degree at least 3. This means that the set $\{u, v, w, t\}$ is incident to at least 6 edges of G_1. In this case let $G_2 = G_1 - \{u, v, w, t\}$ and assume that G_2' is a 2−irregular graph of order $n_1 - 4$ containing G_2. Define G_2^* by $V(G_2^*) = V(G_1)$ and

$$E(G_2^*) = E(G_2') \cup \{wx \mid x \in V(G_2^*) - \{w, u, v\}\} \cup \{tx \mid x \in V(G_2^*) - \{t, u, v\}\}$$
$$\cup \{wu, vt \mid \text{when } G_2' \text{ has no isolated vertex}\}.$$

We check the degrees of $V(G_2^*)$. If G_2' has no isolated vertices, then $d_{G_2^*}(u) = d_{G_2^*}(v) = 2$, $d_{G_2^*}(w) = d_{G_2^*}(t) = |V(G_2^*)| - 2$, and $d_{G_2^*}(x) \leq |V(G_2^*)| - 3$ for $x \in V(G_2')$. If G_2' has isolated vertices, then $d_{G_2^*}(u) = d_{G_2^*}(v) = 1$, $d_{G_2^*}(w) = d_{G_2^*}(t) = |V(G_2^*)| - 3$, and $2 \leq d_{G_2^*}(x) \leq |V(G_2^*)| - 4$ for $x \in V(G_2')$. Thus, in either case, if G_2' is 2−irregular, then so is G_2^* with G_1 a subgraph of G_2^*. For this subcase G_2 is an (n_2, q_2) graph with $n_2 = n_1 - 4$ and $q_2 \leq q_1 - 6$.

Case III: $\delta(G_1) = 2$

Since $n_1 < q_1$, there exist vertices $u, v, w \in V(G_1)$ such that $uv, uw \in E(G_1)$, $d_{G_1}(u) = 2$, and the set $\{u, v, w\}$ is incident to at least 5 edges of G_1. We assume that this set $\{u, v, w\}$ is chosen so that $d_{G_1}(u) \leq d_{G_1}(v) \leq d_{G_1}(w)$ with the degree of w as large as possible.

Let $G_2 = G_1 - \{u, v, w\}$ and assume that G_2' is a 2−irregular graph of order $n_1 - 3$ containing G_2. Let G_2^* be the graph with $V(G_2^*) = V(G_1)$ and

$$E(G_2^*) = E(G_2') \cup \{vx \mid x \in V(G_2^*) - \{v\}\} \cup \{wx \mid x \in V(G_2^*) - \{w\}\}$$
$$\cup \{ux \mid \text{when } G_2' \text{ has at least two isolated vertices with}$$
$$x \in V(G_2') - \{r\} \text{ and } r \text{ a fixed isolated vertex of } G_2'\}.$$

If G_2' has at most one isolated vertex, then $d_{G_2^*}(u) = 2$, $3 \leq d_{G_2^*}(x) \leq |V(G_2^*)| - 2$ for all but at most one vertex $x \in V(G_2')$, and $d_{G_2^*}(v) = d_{G_2^*}(w) = |V(G_2^*)| - 1$. If G_2^* has at least two isolated vertices, then $d_{G_2^*}(x) \leq |V(G_2^*)| - 3$ for each $x \in V(G_2)$, $d_{G_2^*}(u) = |V(G_2^*)| - 2$ and $d_{G_2^*}(v) = d_{G_2^*}(w) = |V(G_2^*)| - 1$.

It is clear that either possibility gives a graph G_2^* which contains G_1 as a subgraph, is 2−irregular when G_2' is 2−irregular, and G_2 is an (n_2, q_2) graph with $n_2 = n_1 - 3$ and $q_2 \leq q_1 - 5$.

When sequentially applying the deletion process to G it is assumed that the highest degree vertex z will be deleted as soon as possible. This can, in fact, be done the first time the deletion process is applied, unless $\delta(G) = 2$ and z is not adjacent to a vertex of degree 2. After enough deletions, even when $\delta(G_1) = 2$, vertex z will be deleted.

We next see that this deletion process can be stopped when the graph G_0, which results after many deletions, satisfies

$$C' n^{1/4} \leq |V(G_0)| \leq \left(\frac{2}{3}\right)^{1/4} n^{1/4},$$

where $C' = (\frac{2}{3})^{1/4} - \epsilon$ for some ϵ, $\frac{1}{3}(\frac{2}{3})^{1/4} \leq \epsilon \leq \frac{1}{2}(\frac{2}{3})^{1/4}$. Assume that after repeated deletions, a graph G_0 results with $|V(G_0)| = C' n^{1/4}$. Then $n - C' n^{1/4}$ vertices and at least $\frac{3}{2}(n - C' n^{1/4} -$

4) + $(\frac{2}{3})^{1/4}n^{1/4} + 2$ edges have been deleted. Recall that at least $(\frac{2}{3})^{1/4}n^{1/4} + 2$ edges were deleted when z was deleted. This means that $|V(G_0)| = C'n^{1/4}$ and

$$|E(G_0)| \leq \frac{3}{2}n + \frac{1}{2}(\frac{2}{3})^{1/4}n^{1/4} - \frac{3}{2}(n - C'n^{1/4} - 4) - (\frac{2}{3})^{1/4}n^{1/4} - 2$$
$$\leq (\frac{3}{2}C' - \frac{1}{2}(\frac{2}{3})^{1/4})n^{1/4} + 4 \leq C'n^{1/4}.$$

The last inequality requires $4 \leq \frac{1}{2}\epsilon n^{1/4}$, which holds since $n \geq (8/\epsilon)^4 \geq 8^4 3^5/2$.

Thus, a (n_0, q_0) graph G_0 has been obtained by the deletion process with $q_0 \leq n_0$. We complete the proof by showing that G_0 is embeddable in a 2−irregular graph G_0^* of order n_0. Clearly, G_0^* can be enlarged, replacing the deleted vertices a 3−set (or 4−set) at a time in reverse order to the deletion. At each step in the enlarging process, a 2−irregular graph is obtained with G a subgraph of the enlarged 2−irregular graph.

To embed G_0 in a 2−irregular graph we apply Lemma 2. Sequentially delete $x_1 \cup N^*(x_1), x_2 \cup N^*(x_2), \cdots x_{\lfloor n_0/3 \rfloor} \cup N^*(x_{\lfloor n_0/3 \rfloor})$, from G_0, where x_i is the smallest degree in $G_0 - \cup_{j=1}^{i-1}(x_j \cup N^*(x_j))$ and $N^*(x_i)$ is the neighborhood of x_i in $G_0 - \cup_{j=1}^{i-1}(x_j \cup N^*(x_j))$. Let $H_1 = \{x_1, x_2, \cdots, x_{\lfloor n_0/3 \rfloor}\}$ and $H_2 = \cup_{j=1}^{\lfloor n_0/3 \rfloor} N^*(x_j)$. Clearly, H_1 is independent. Also, since $q_0 \leq n_0$, the set of vertices $\{x_j \mid 1 \leq j \leq i\}$ have at most $2i$ adjacencies in H_2. Thus, by Lemma 2, G_0 is embeddable in a 2−irregular graph G_0^* of order n_0. This completes the proof of Theorem 3. ■

4. CONCLUDING REMARKS.

It is not known whether the sufficient conditions given in Theorems 2 and 3 can be weakened with the same conclusions. The sufficient conditions given in the theorems were determined by the method of proof used. Probably the most interesting question is the one mentioned earlier, namely, is each (n, q) graph with $q \leq 3n - 7$ embeddable in a 2−irregular graph of order n?

Another question of interest is whether there exists a "universal" 2−irregular graph of order n that contains all graphs of order n and of bounded degree d, for fixed d and sufficiently large n. One candidate for such a universal graph is the graph H_n (for n even) obtained from the disjoint union of two complete graphs $K_{n/2}$ with vertices $\{x_1, x_2, \cdots, x_{n/2}\}$ and $\{y_1, y_2, \cdots, y_{n/2}\}$ respectively, by adding the edges $x_i y_j$ for $1 \leq i \leq n/2$ and $1 \leq j \leq i$. Unfortunately H_n is not universal. It can, if fact, be shown by probabilistic methods (for $d \geq 17$) that there exists a graph G_n of order n and maximal degree d such that $K_{n/4,n/4} \not\subseteq \overline{G_n}$. Since $K_{n/4,n/4} \subset \overline{H_n}$, it is not possible for $G_n \subseteq H_n$, and thus H_n is not universal. It is true that H_n is universal for $d \leq 3$.

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