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On a Neighborhood Condition Implying the Existence of Disjoint Complete Graphs

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The notion of considering properties in graphs which meet the condition that for all independent pairs of vertices, x and y, $\deg(x)+\deg(y)\geq s$, for some integer s, was first done by Ore. Recently, the concept of replacing degree sum by the order of the union of the neighborhoods has been considered. This was generalized to considering neighborhood unions for all sets of k independent vertices. In this paper, the result stated below is proved. Furthermore, this is shown to be best possible.

THEOREM. *If G is a graph with sufficiently large order n, satisfying the condition for all sets of k independent vertices, $x_1, x_2, \ldots, x_k$, $k \leq n$,*

$$\left|\bigcup_{i=1}^{k} N(x_i)\right| \geq \frac{(m-2)n+t}{m-1},$$

then

$$tK_m \subseteq G.$$

INTRODUCTION

In an attempt to better understand the structure of graphs there have been many results relating the sum of degrees of pairs of independent vertices to the existence of certain kinds of subgraphs, (for example, see [2] and [6]). In [4] and [5] the idea of a neighborhood condition that was patterned after the Ore type degree sum was introduced. This concept was extended in [1] and the following generalization was given: for a graph G, the *p-neighborhood condition* is defined to be

$$NC_p(G) = \min \left|\bigcup_{i=1}^{p} N(x_i)\right|,$$

where the minimum is taken over all sets of p independent vertices $\{x_1, x_2, \ldots, x_p\}$ in G. This will be abbreviated to NC_p when G is understood. They prove:

THEOREM A. *If G is a graph of sufficiently large order $n \geq n_0 = n_0(p)$ satisfying $NC_p \geq [(m-1)n]/m$ for some p, $1 \leq p \leq n_0$ then*

$$K_{m+1} \subseteq G.$$

It is the purpose of this paper to extend this result and determine an appropriate neighborhood condition to assure that G contains t disjoint copies of K_m.

All graphs considered in this paper will be finite simple graphs. For terms not defined here, see [3]. Let x be a vertex of a graph G; the *neighborhood* of x, which is the set of vertices of G adjacent to x in G, will be denoted by $N_G(x)$ or simply $N(x)$ when G is understood. We will denote by $K(p; m)$ the complete m-partite graph, with each part having p vertices. Also, by tK_m we refer to t disjoint copies of the complete graph K_m. Finally, we define for two graphs G and H the *Ramsey number* $r(G, H)$ to be the smallest integer p so that in any graph on p vertices either it contains G or its complement contains H.

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For convenience we define the *join* of two graphs, denoted by $G = G_1 + G_2$, to be the graph with

$$V(G) = V(G_1) \cup V(G_2)$$

and

$$E(G) = E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\}.$$

Main Results

Consider the graph $G = K(p; m-1) + K_{t-1}$, for a fixed positive integer p. This graph has the following property: for every set of k independent vertices, $k \leq p$,

$$\left|\bigcup_{i=1}^{k} N(x_i)\right| \geq (m-2)p + t - 1$$
$$= \frac{m-2}{m-1} n + \frac{t-1}{m-1},$$

where $n = p(m-1) + t - 1$ is the order of G. Furthermore, $tK_m \not\subseteq G$. It is the intent of this paper to show that this is the extremal neighborhood value. We will prove the following:

THEOREM 1. *For fixed m and t, if G is a graph of sufficiently large order $n = n(m, t)$ and there exists a $k \leq n$, so that for all subsets of k independent vertices $x_1, x_2, \ldots, x_k$ meet the condition*

$$\left|\bigcup_{i=1}^{k} N(x_i)\right| \geq \frac{m-2}{m-1} n + \frac{t}{m-1};$$

then $tK_m \subseteq G$.

Before proceeding with the proof of Theorem 1, we state a useful result and give a preliminary result which will lend insight to the proof of this theorem.

PROPOSITION 2. *Let m be a fixed positive integer. If G is of sufficiently large order n with*

$$NCp \geq \left(\frac{m-2}{m-1}\right) n + \frac{mt}{m-1}$$

for $1 \leq t \leq t_0$, then $tK_m \subseteq G$ (where t_0 is such that $n - (t_0 - 1)m \geq n_0$ and n_0 is that of Theorem A).

PROOF. We proceed by induction on t. For the case $t = 1$, we assume that n is large enough to apply Theorem A. Inductivity, we may assume $(t-1)K_m \subseteq G$. Let $G^* = G - (t-1)K_m$. It follows that $|V(G^*)| = n - (t-1)m$, which by the hypothesis is greater than or equal to n_0. Also

$$\left|\bigcup_{i=1}^{p} N(x_i)\right| \geq \left(\frac{m-2}{m-1}\right) n + \frac{mt}{m-1} - m(t-1)$$
$$= \left(\frac{m-2}{m-1}\right)(n - m(t-1)) + \frac{m}{m-1}$$
$$> \left(\frac{m-2}{m-1}\right)(|V(G^*)|)$$

for all sets of p independent vertices $x_1, x_2, \ldots, x_p$ and hence, by Theorem A, $K_m \subseteq G^*$. Consequently, $tK_m \subseteq G$. □

The remaining results will indicate that, at least for t fixed and n sufficiently large, this result is not best possible. In fact, the neighborhood condition can be decreased to

$$NC_p \geq \left(\frac{m-2}{m-1}\right) n + \frac{t}{m-1}$$

for some $p \leq n$.

Let t be a positive integer, H a graph and $x \in V(H)$. Denote by $H_x(t)$ the graph obtained from H by replacing x with t independent vertices, each having the same neighborhood as x. Note that $|V(H_x(t))| = |V(H)| + t - 1$.

Lemma [1]. *Let t be a fixed positive integer and H a fixed graph of order p. If G is any graph of order n, n sufficiently large such that there are m copies of H in G, then there exists a positive constant $c = c(p, t)$ such that there are at least*

$$c \frac{m^t}{n^{(p-1)(t-1)}}$$

copies of $H_x(t)$ in G for any vertex $x \in V(H)$.

Note, in the case when $m = kn^p$ for some positive constant k, there are ckn^{p+t-1} copies of $H_n(t)$ in G for any vertex $x \in V(H)$.

The next result is the special case of Theorem 1 when $m = 3$.

Proposition 3. *Let G be a graph with sufficiently large order n. If G has the property $NC_p \geq (n + t)/2$ for some p, then*

$$tK_3 \subseteq G.$$

Proof. Clearly, the result follows from Theorem A in the case when $t = 1$. Now suppose there is a vertex x such that $(3t - 2)K_2 \subseteq \langle N(x) \rangle$. It follows that $|V(G - x)| = n - 1$ and $NC_p(G - x) \geq [(n - 1) + (t + 1)]/2$ for all sets of p independent vertices in $G - x$. Inductively, it follows that $(t - 1)K_3 \subseteq G - x$. Clearly, there must be an edge in $\langle N(x) \rangle$ with neither of its end vertices among the vertices forming the $(t - 1)K_3$ in $G - x$. Consequently, by taking x and this edge and the $(t - 1)K_3$'s, we have $tK_3 \subseteq G$.

We continue by showing that such a vertex must exist. If there are $r(tK_3, K_p)$ vertices of degree less than $(n + t)/2p$, by the neighborhood condition of G, there could be no set of p independent vertices among them. Hence, $tK_3 \subseteq G$. Thus, since $r(tK_3, K_p)$ is small compared to n, we may assume that nearly all vertices have degree at least $(n + t)/2p$.

Next we obtain a lower bound on the number of $K_{p,p}$'s contained in G. Since almost all vertices have degree at least $(n + t)/2p$ it follows that there are at least $\mathcal{O}(n^2)$ edges in G. By the lemma this implies that there are at least $\mathcal{O}(n^{2p})$ copies of $K_{p,p}$ in G. From this lower bound we establish a lower bound on the number of K_3's in G. Suppose for some positive ε, εn^{2p} of the $K_{p,p}$'s contain at least one chord. Then each such $K_{p,p}$ yields at least pK_3's and any such K_3 can be counted in at most $3\binom{n-3}{2p-3} K_{p,p}$'s. Thus, there would be at least $\varepsilon n^{2p}/\varepsilon' n^{2p-3}$ K_3's in G. Hence, it would follow that G contains $\mathcal{O}(n^3)$ K_3's in this case.

Suppose, on the other hand, that εn^{2p} of the $K_{p,p}$'s do not contain a chord; that is, there are εn^{2p} induced $K_{p,p}$'s. Consider the two sets of p independent vertices in any induced $K_{p,p}$. Since each set has a neighborhood union of at least $(n + t)/2$ vertices, it follows that the $K_{p,p}$ generates at least tK_3's. Furthermore, any such triangle can be generated in this manner at most $3\binom{n-3}{2p-2} \sim \mathcal{O}(n^{2p-2})$ times, since there are 3 vertices necessary to form the K_3 and $\binom{n-3}{2p-2}$ ways to complete the $K_{p,p}$. Subsequently, in this case there are at least $\varepsilon n^{2p}/\varepsilon' n^{2p-2} = \mathcal{O}(n^2)$ K_3's in G. In either case we can conclude that G contains at least $\mathcal{O}(n^2)$ K_3's.

Now by induction, we may assume that $H = (t-1)K_3 \subseteq G$. If there are no other K_3's disjoint from this copy of H then it follows that all $\mathcal{O}(n^2)$ K_3's in G must have at least one vertex in common with H. There are at most

$$\binom{3t-3}{3} \quad K_3\text{'s having 3 vertices in common with } H,$$
$$(n-3t+3)\binom{3t-3}{2} \quad K_3\text{'s having 2 vertices in common with } H$$

and

$$\binom{n-3t+3}{2}(3t-3) \quad K_3\text{'s having 1 vertex in common with } H.$$

Clearly, $\mathcal{O}(n^2)$ copies of K_3 must share one vertex with H since

$$\binom{3t-3}{3} + (n-3t+3)\binom{3t-3}{3} = \mathcal{O}(n).$$

But this implies that in the neighborhood of some vertex of H, say x, there are $\mathcal{O}(n^2)$ edges, since x is in $\mathcal{O}(n^2)$ K_3's.

It follows that there are at least $3t-2$ independent edges in $\langle N_G(x)\rangle$, for if there were fewer, say $3t-3$, there could be at most $\binom{3t-3}{2} + (3t-3)(n-3t+2) = \mathcal{O}(n)$ edges in $\langle N_G(x)\rangle$. Consequently, $tK_3 \subseteq G$, by the initial observation in the proof. □

We are now prepared to present the proof of Theorem 1. The steps of the proof coincide with those of the previous result.

Proof of Theorem 1. Let G have the conditions given in the hypothesis for some $k \leq n$. The result follows for the case $t = 1$ from Theorem A. Suppose there exists a vertex x such that $((t-1)m+1)K_{m-1} \subseteq \langle N_G(x)\rangle$. Consider $G - x$; $|V(G-x)| = n-1$ and

$$NC_k(G-x) \geq \frac{(m-2)(n-1)+(t-1)}{m-1}.$$

Proceeding inductively, it follows that $(t-1)K_m \subseteq G - x$. Clearly, there must be a $K_{m-1} \subseteq \langle N_G(x)\rangle$ with no vertex in common with this copy of $(t-1)K_m$. Thus x and this K_{m-1}, along with the $(t-1)K_m \subseteq G-x$, imply that $tK_m \subseteq G$.

We continue by showing that such a vertex must exist. As in the previous result, we may suppose that there are less than $r(tK_m, K_k)$ vertices of degree less than $[(m-2)n+t]/[(k)(m-1)]$. If this were not the case, among these vertices there would exist a copy of tK_m or a set of k independent vertices which would contradict the neighborhood condition. Hence, we can assume that almost all vertices have degree at least $[(m-2)n]/[k(m-1)]$.

Now we show that G contains $cn^{k(m-1)}$ copies of $K(k;\, m-1)$. We proceed inductively. Since the neighborhood condition exceeds that of Proposition 3, there are cn^{2k} $K(k;\, 2)$'s. We consider two cases; if there are εn^{2k} $K(k;\, 2)$'s with a chord, it follows that G contains at least $\mathcal{O}(n^3)$ copies of K_3.

Suppose, on the other hand, that G contains $\mathcal{O}(n^{2k})$ $K(k;\, 2)$'s that are induced. Since the neighborhood union of independent sets of k vertices is at least $2n/3$, it follows that for every $K(k;\, 2)$ there is an edge in each $K(k;\, 2)$ contained in $n/3k^2$ K_3's. Thus there are at least

$$(\varepsilon n^{2k})\binom{n}{3k^2}\Big/\binom{n-3}{2k-2} = \mathcal{O}(n^3)$$

K_3's. Hence, we can conclude that, in either case, there are $\mathcal{O}(n^3)$ K_3's and, by applying the Lemma, G contains $\mathcal{O}(n^{3k})$ copies of $K(k; 3)$ in G.

Continuing, we may suppose inductively that there are $\mathcal{O}(n^{k(m-2)})$ copies of $K(k; m-2)$, and we will show that there are $cn^{k(m-1)}$ copies of $K(k; m-1)$.

As above, there are two cases. First, suppose that G contains a positive fraction, $\varepsilon n^{k(m-2)}$, copies of $K(k; m-2)$ with at least one chord. Each such $K(k; m-2)$ yields k^{m-3} K_{m-1}'s; but such a K_{m-1} could be counted in $\binom{n-m+1}{k(m-2)-1}$ different $K(k; m-2)$'s. Thus, in this case there would be

$$\varepsilon n^{k(m-2)} k^{m-3} \Big/ \binom{n-m+1}{(k-1)(m-2)-1} \geqslant c_2 n^{m-1}$$

copies of K_{m-1} in G.

Suppose now, that a positive fraction of the $\mathcal{O}(n^{k(m-2)})$ copies of $K(k; m-2)$ are induced. The $m-2$ distinct sets of non-adjacent vertices must each have a neighborhood union of at least $[(m-2)n+t]/(m-1)$ vertices. Since the intersection of the union of the neighborhoods is $\geqslant c_3 n$, it follows that for each $K(k; m-2)$ there are $\mathcal{O}(n)$ copies of K_{m-1}. Any such K_{m-1} can be generated in at most $(m-1)\binom{n-m+1}{(k-1)(m-2)}$ ways; that is, by considering $(m-1)\binom{n-m+1}{(k-1)(m-1)} \leqslant c_4 n^{m-2}$ copies of $K(k; m-2)$. Consequently, there are at least

$$c_5 \cdot \frac{n \cdot n^{k(m-2)}}{n^{(k-1)(m-2)}} = \mathcal{O}(n^{m-1})$$

copies of K_{m-1} in G.

Hence, in either case, we may conclude that there are at least $\mathcal{O}(n^{m-1})$ K_{m-1}'s. Applying the Lemma $(m-1)$ times it follows that there are $\mathcal{O}(n^{k(m-1)})$ copies of $K(k; m-1)$ contained in G.

By applying an argument similar to that above and considering these $\mathcal{O}(n^{k(m-1)})$ copies of $K(2; m-1)$ in G, with or without chords, we can conclude that G must contain at least $\mathcal{O}(n^{m-1})$ copies of K_m.

Reasoning inductively now, there must exist some $H = (t-1)K_m \subseteq G$. If it were the case that $tK_m \nsubseteq G$, then each of the cn^{m-1} copies of K_m would necessarily contain at least one vertex of H. There are at most $\binom{n-mt+m}{m-i}\binom{mt-m}{i}$ K_m's having i vertices in common with H (for $i = 1, 2, \ldots, m$). Since

$$\sum_{i=2}^{m} \binom{n-mt+m}{m-i} \binom{mt-m}{i} \leqslant c_7 n^{m-2}$$

it follows that $\mathcal{O}(n^{m-1})$ copies of K_m share one vertex with H. Since H has $m(t-1)$ vertices, there must be a vertex $x \in V(H)$ such that $\mathcal{O}(n^{m-1})$ copies of K_{m-1} are in $\langle N_G(x) \rangle$.

If $((t-1)m+1)K_{m-1} \nsubseteq \langle N_G(x) \rangle$ then there could be at most

$$\sum_{i=1}^{m-1} \binom{mt-m}{i} \binom{n-mt+m}{m-i+1} = cn^{m-2}$$

K_{m-1}'s in $\langle N_G(x) \rangle$. Hence, a contradiction results and subsequently it follows that $((t-1)m+1)K_{m-1} \subseteq \langle N_G(x) \rangle$. Consequently, $tK_m \subseteq G$ from our initial observation. □

Concluding Remarks

We conclude the paper with a result implying the existence of multiple copies of any graph. Unlike the results in the previous sections, we are unable to determine whether or not these neighborhood conditions are best possible.

For convenience we introduce the following notation: let H be a graph with a chromatic number $\chi(H)=\chi$. The *chromatic surplus* (*majority*), denoted $s(H)$ ($S(H)$), is the minimum order of the smallest (largest) color class in any critical coloring of the vertices of H. Let H be a graph with $\chi(H)=\chi$ and $s(H)=s$. Consider the graph $G=K(p;\,\chi-1)+K_{st-1}$ for a fixed positive integer p. This graph has the following property: for every set of $k\leqslant p$ independent vertices

$$\left|\bigcup_{i=1}^{k} N(x_i)\right| \geqslant (\chi-2)p+st-1$$

$$=\frac{\chi-2}{\chi-1}n+\frac{st-1}{\chi-1},$$

where $n=p(\chi-1)+st-1$ is the order of G. Furthermore, $tH \nsubseteq G$. Although we can not prove that this is the best neighborhood condition, we give the following result:

PROPOSITION 4. *Let G be of sufficiently large order $n=n(\chi)$ with $NC_k \geqslant [(\chi-2)n+n^{\alpha}]/(\chi-1)$ for some $k\leqslant n$ and H any graph with $\chi(H)=\chi$ and $S(H)=S$. If $1-(\chi S/S^{\chi})<\alpha<1$ then $H\subseteq G$.*

PROOF. Proceeding as in the proof of Theorem 1, we are able to improve the lower bound on the number of copies of K_χ. We obtain $\mathcal{O}(n^{\alpha}\cdot n^{\chi-1})$ copies of K_χ rather than $\mathcal{O}(n^{\chi-1})$ copies. For convenience, let $\beta=1-\alpha$ and say there are $\mathcal{O}(n^{\chi-\beta})$ copies of K_χ.

Applying the lemma χ times shows that G contains at least $n^{\chi S-\beta S^{\chi}}$ copies of $K(S;\,\chi)$. Subsequently, if $\chi S-\beta S^{\chi}\geqslant 0$ or $\beta\leqslant \chi S/S^{\chi}$ it follows that $K(S;\,\chi)\subseteq G$ and hence $H\subseteq G$. □

Finally, we mention the fact that this result can be used to give a neighborhood union condition that would imply that $tH\subseteq G$, the graph tH having $\chi(tH)=\chi(H)$ and $S(tH)\leqslant t(S(H))$. The authors feel that, in general, the n^{α} term cannot be replaced by a constant term. This belief is substantiated by considering the extremal numbers for complete bipartite graphs.

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