ON HOMOGENEOUSLY TRACEABLE NONHAMILTONIAN GRAPHS

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A graph G is called *homogeneously traceable* if for each vertex v of G, there exists a hamiltonian path of G with initial vertex v. Every hamiltonian graph is obviously homogeneously traceable. On the other hand, there exist nonhamiltonian graphs which are homogeneously traceable; for example, the Petersen graph (not surprisingly!) has these properties. Clearly, every homogeneously traceable graph is traceable (i.e., contains a hamiltonian path), but a traceable graph need not be homogeneously traceable (such as a path of length 2 or more). Thus, the homogeneously traceable graphs constitute a class of graphs which lie properly between the traceable graphs and the hamiltonian graphs.

Skupień [1] asks, for which integers p does there exist a homogeneously traceable nonhamiltonian graph of order p? We answer that question in this paper and present some other results related to the existence of homogeneously traceable nonhamiltonian graphs.

We begin by describing a class of graphs that are homogeneously traceable and nonhamiltonian. First we note that graph H_0 of order 9, shown in FIGURE 1, is homogeneously traceable and nonhamiltonian.

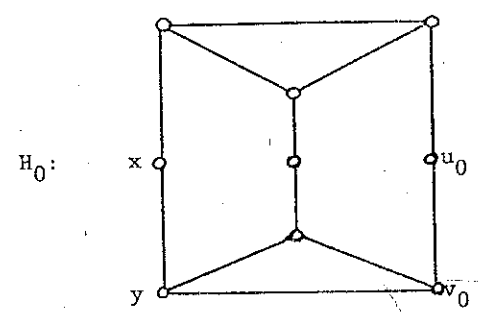

FIGURE 1. A homogeneously traceable nonhamiltonian graph of order 9.

With the aid of H_0 we construct, for each integer $p > 9$, a homogeneously traceable nonhamiltonian graph of order p. We begin with the odd orders. Let $p = 9 + 2n$, where n is a positive integer. Consider the graph H_n of FIGURE 2.

It is routinely shown that there exists a hamiltonian path with initial vertex w, for every vertex w of H_n in FIGURE 2 (i.e., H_n is homogeneously traceable). Suppose

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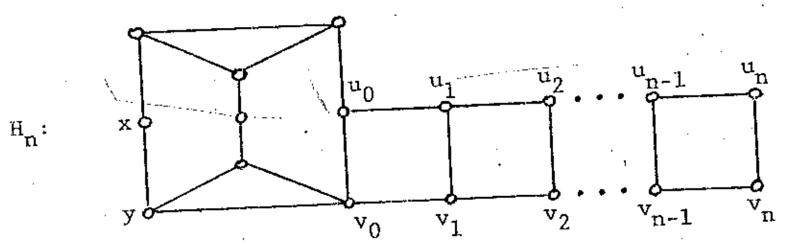

FIGURE 2. A homogeneously traceable nonhamiltonian graph of order $9 + 2n$.

to the contrary, that H_n is hamiltonian. Since each of u_n and v_n has degree 2, the edges $u_{n-1}u_n$, u_nv_n and v_nv_{n-1} are necessarily consecutive edges of a hamiltonian cycle C of H_n. Therefore, $u_{n-1}v_{n-1}$ is not an edge of C, implying that $u_{n-2}u_{n-1}$ and $v_{n-2}v_{n-1}$ are edges of C. By a similar argument, we conclude that the edges $u_{n-2}v_{n-2}$, $u_{n-3}v_{n-3}, \ldots, u_1v_1$ and u_0v_0 do not belong to C; thus $H_n - \{u_0v_0, u_1v_1, \ldots, u_{n-1}v_{n-1}\}$ is hamiltonian, which is possible only if H_0 is hamiltonian, but this is not the case.

By subdividing the edge xy in the graph H_n $(n \geq 0)$ in FIGURES 1 or 2, we produce a graph H_n' of order $10 + 2n$, which can be shown in a similar fashion to be homogeneously traceable and nonhamiltonian. Hence, for every integer $p \geq 9$, there exists a homogeneously traceable nonhamiltonian graph of order p.

We now consider the question of existence (and nonexistence) of homogeneously traceable nonhamiltonian graphs of order less than 9. We note that the graphs K_1 and K_2 are (rather trivially) homogeneously traceable and nonhamiltonian. Therefore, there exist homogeneously traceable nonhamiltonian graphs of orders 1 and 2.

LEMMA 1. (Skupień) The following conditions are necessary for a graph G of order at least 3 to be homogeneously traceable.
(a) G is 2-connected.
(b) For each vertex v of G, the graph $G - v$ has at most two end-vertices.
(c) For each nonempty proper subset S of $V(G)$, the number of components of $G - S$ does not exceed $|S|$.

In the case of homogeneously traceable nonhamiltonian graphs, LEMMA 1(b) can be improved.

LEMMA 2. If G is a homogeneously traceable nonhamiltonian graph, then every vertex of G is adjacent to at most one vertex of degree 2.

Proof: By LEMMA 1(b), every vertex of G is adjacent to at most two vertices of degree 2. Suppose, to the contrary, that there exists a vertex v of G which is adjacent to vertices u and w, each of degree 2. Any hamiltonian path P with initial vertex v must have one of u and w as its second vertex and the other as its terminal vertex. However, since v is adjacent to the terminal vertex of P, it follows that G is hamiltonian, producing a contradiction and the desired result. □

There are certain high degrees which are impossible for vertices of homogeneously traceable nonhamiltonian graphs.

THEOREM 1. If G is a homogeneously traceable nonhamiltonian graph of order $p \geq 3$, then $\Delta(G) \leq p - 4$.

Proof: This theorem is proved by successively verifying that G contains no vertices of degree $p-1$, $p-2$, or $p-3$.

Suppose G contains a vertex v such that $\deg v = p-1$. Since G is homogeneously traceable, there exists a hamiltonian path P with initial vertex v. However, since $\deg v = p-1$, the terminal vertex of P is adjacent to v, implying that G is hamiltonian, thereby contradicting the hypothesis.

Assume next that G contains a vertex v with $\deg v = p-2$. Since G is 2-connected (by LEMMA 1), we may assume that $p \geq 4$. Let P be a hamiltonian path with initial vertex v, where $P: v = v_1, v_2, \ldots, v_p$. Since $\deg v = p-2$ and G is nonhamiltonian, v is adjacent to v_i for each i with $2 \leq i \leq p-1$. Thus $\deg v_i \geq 3$ for $2 \leq i \leq p-1$, which is impossible if $p = 4$. Therefore, without loss of generality, we assume that $p \geq 5$. Since every vertex of G has degree at least 2, v_p is adjacent to v_i for some i with $2 \leq i \leq p-2$. However, then, $v = v_1, v_{i+1}, v_{i+2}, \ldots, v_p, v_i, v_{i-1}, \ldots, v_1$ is a hamiltonian cycle of G, contrary to hypothesis.

Finally, suppose G contains a vertex v such that $\deg v = p-3$. Since G is 2-connected, we assume that $p \geq 5$. Let $P: v = v_1, v_2, \ldots, v_p$ be a hamiltonian path with initial vertex v. Since G is nonhamiltonian, v is not adjacent to v_p. Thus, v is adjacent to exactly $p-4$ of the vertices v_i, $3 \leq i \leq p-1$. Since each such vertex has degree at least 3, we assume, without loss of generality, that $p \geq 6$. Let v_k, $3 \leq k \leq p-1$, be such that $vv_k \notin E(G)$. We distinguish two cases.

Case 1. Assume $k = p-1$. Since G is 2-connected, $\deg v_p \geq 2$. If $v_p v_i \in E(G)$ for some i with $2 \leq i \leq p-3$, then $v = v_1, v_{i+1}, v_{i+2}, \ldots, v_p, v_i, v_{i-1}, \ldots, v_1$ is a hamiltonian cycle of G, which is impossible. Hence, without loss of generality, we may assume that $v_p v_i \notin E(G)$ for $1 \leq i \leq p-3$ and that $v_p v_{p-2} \in E(G)$. Since G is 2-connected, v_{p-1} is adjacent to v_i for some i with $2 \leq i \leq p-3$. Assume that $v_{p-1} v_j \in E(G)$, where $2 \leq j \leq p-3$. However, then, $v = v_1, v_{j+1}, v_{j+2}, \ldots, v_{p-2}, v_p, v_{p-1}, v_j, v_{j-1}, \ldots, v_1$ is a hamiltonian cycle of G, which contradicts our hypothesis.

Case 2. Assume $3 \leq k \leq p-2$. Again, since G is 2-connected, $\deg v_p \geq 2$ and v_p is adjacent to at least one vertex v_i with $2 \leq i \leq p-2$. If $v_p v_i \in E(G)$, where $2 \leq i \leq p-2$ and $i \neq k-1$, then it can be straightforwardly shown that G possesses a hamiltonian cycle. Hence we may assume, without loss of generality, that $\deg v_p = 2$ and $v_p v_{p-1}, v_p v_{k-1} \in E(G)$. Since v_{k-1} is adjacent to both v_k and v_p and, by LEMMA 2, v_{k-1} is adjacent to at most one vertex of degree 2, it follows that $\deg v_k > 2$. We consider two subcases of Case 2.

Subcase 2.1. Assume $v_k v_j \in E(G)$, where $2 \leq j \leq k-2$: Here $v_1, v_{j+1}, v_{j+2}, \ldots, v_{k-1}, v_p, v_{p-1}, \ldots, v_k, v_j, v_{j-1}, \ldots, v_1$ is a hamiltonian cycle of G, which cannot occur.

Subcase 2.2. Assume $v_k v_j \in E(G)$, where $k+2 \leq j \leq p-1$. We note here that $v_1, v_{j-1}, v_{j-2}, \ldots, v_k, v_j, v_{j+1}, \ldots, v_p, v_{k-1}, v_{k-2}, \ldots, v_1$ is a hamiltonian cycle of G, which is impossible. Hence, we conclude that G cannot contain a vertex of degree $p-3$. □

The upper bound on the maximum degree of homogeneously traceable nonhamiltonian graphs, given in THEOREM 1, is sharp in the sense that there exist infinitely many such graphs for which the bound is attained. A homogeneously traceable nonhamiltonian graph G of order $p = 10$ containing vertices of degree $p-4 = 6$ (namely, v_1 and v_3) is shown in FIGURE 3. By subdividing the edge $v_5 v_6$ a total of k times and joining the k new vertices to v_1, we produce a graph of order $10+k$ in which v_1 has degree $6+k$.

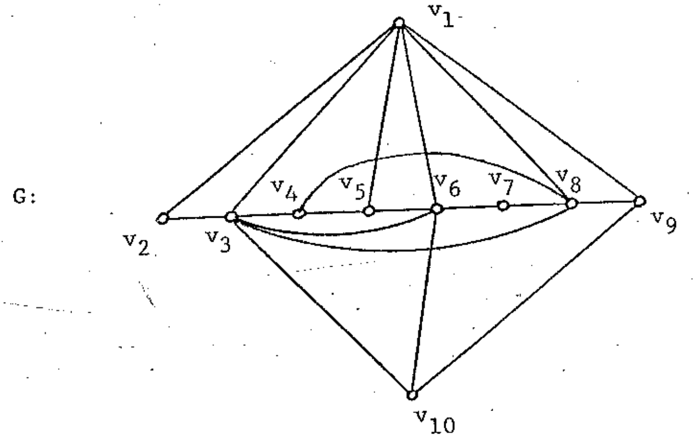

FIGURE 3. A homogeneously traceable nonhamiltonian graph containing vertices of degree $p-4$.

The vertices of degree $p-4$ in the graph G of FIGURE 3 are adjacent. This is always the case for homogeneously traceable nonhamiltonian graphs of order p, as we shall now see.

THEOREM 2. If G is a homogeneously traceable nonhamiltonian graph of order $p \geq 7$, then every two vertices of degree $p-4$ in G are adjacent.

Proof: Suppose the theorem is false. Then, for some integer $p \geq 7$, there exists a homogeneously traceable nonhamiltonian graph of order p containing nonadjacent vertices of degree $p-4$. Let G be such a graph of order p having maximum size. Therefore, G is homogeneously traceable and maximally nonhamiltonian; i.e., the addition of an edge to G produces a hamiltonian graph.

By assumption, G contains vertices u and v such that $\deg u = \deg v = p-4$ and $uv \notin E(G)$. Furthermore, $G + uv$ is hamiltonian and uv lies in every hamiltonian cycle of $G + uv$. This implies that G contains a hamiltonian u-v path P: $u = v_1, v_2, \ldots, v_p = v$.

Case 1. Suppose $p \geq 8$. There exists i, where $3 \leq i \leq p-1$, such that $v_1 v_i \in E(G)$ and $v_p v_{i-1} \in E(G)$. However, then, $v_1, v_2, \ldots, v_{i-1}, v_p, v_{p-1}, \ldots, v_i, v_1$ is a hamiltonian cycle of G, which produces the desired contradiction.

Case 2. Suppose $p = 7$. Here $\deg v_1 = 3$ and $\deg v_p = \deg v_7 = 3$. If there exists i, where $3 \leq i \leq 6$, such that $v_1 v_i \in E(G)$ and $v_7 v_{i-1} \in E(G)$, then, as in Case 1, G is hamiltonian. Hence, we may assume, without loss of generality, that no such i exists. In this case, then, only six possibilities arise; namely:

1.	$v_1v_3, v_1v_4 \in E(G)$	and	$v_7v_4, v_7v_5 \in E(G)$.
2.	$v_1v_3, v_1v_5 \in E(G)$	and	$v_7v_3, v_7v_5 \in E(G)$.
3.	$v_1v_3, v_1v_6 \in E(G)$	and	$v_7v_3, v_7v_4 \in E(G)$.
4.	$v_1v_4, v_1v_5 \in E(G)$	and	$v_7v_2, v_7v_5 \in E(G)$.
5.	$v_1v_4, v_1v_6 \in E(G)$	and	$v_7v_2, v_7v_4 \in E(G)$.
6.	$v_1v_5, v_1v_6 \in E(G)$	and	$v_7v_2, v_7v_3 \in E(G)$.

In 1–5 a vertex of degree $4 = p - 3$ is produced, which is impossible by THEOREM 1. In 6 the hamiltonian cycle $v_1, v_6, v_5, v_4, v_3, v_7, v_2, v_1$ is obtained, which contradicts our hypothesis. □

LEMMA 3. If T is the set of vertices of degree 2 in a homogeneously traceable nonhamiltonian graph G, then $|V(G) - T| \geq |T|$.

Proof: No vertex of degree 2 in G is adjacent to two vertices of degree 2, for otherwise LEMMA 2 is contradicted. Thus, each vertex in T is adjacent to at least one vertex of $G - T$. Since no vertex of $G - T$ is adjacent to two vertices of T (by LEMMA 2), the desired inequality follows. □

We now present our main result.

THEOREM 3. There exists a homogeneously traceable nonhamiltonian graph of order p for all positive integers p except $3 \leq p \leq 8$.

Proof: We have already noted that there is a homogeneously traceable nonhamiltonian graph of order p for $p = 1$, $p = 2$, and $p \geq 9$. Since every homogeneously traceable nonhamiltonian graph of order $p \geq 3$ is 2-connected, it follows by THEOREM 1 that there is no such graph of order p for $p = 3$, 4, and 5.

If G is a homogeneously traceable nonhamiltonian graph of order 6, then, by THEOREM 1, G is 2-regular implying that G is a 6-cycle, which is impossible since G is nonhamiltonian.

Suppose there exists a homogeneously traceable nonhamiltonian graph G of order 7. By THEOREM 1 and the fact that G is 2-connected, every vertex of G has degree 2 or 3. By THEOREM 2, the graph G has less than four vertices of degree 3; while by LEMMA 3, G has at least four vertices of degree 3. Thus no such graph G exists.

If G is a homogeneously traceable nonhamiltonian graph of order 8, then by THEOREM 1 and the fact that G is 2-connected, every vertex of G has degree 2, 3, or 4. Consideration of all possible cases (similar to Case 2 of THEOREM 2 for $p = 7$) yields the result that no such graph exists. (For brevity we omit details.) □

Graphs which are homogeneously traceable must have sufficiently many edges. On the other hand, nonhamiltonian graphs have limited size. With these remarks in mind, we conclude by presenting a sharp lower bound on the size of homogeneously traceable nonhamiltonian graphs.

THEOREM 4. If G is a homogeneously traceable nonhamiltonian graph of order $p \geq 9$ and size q, then $q \geq \{5p/4\}$. Furthermore, for $p = 8n + 3$ and $p = 8n + 4$, $n \geq 1$, there exists a homogeneously traceable nonhamiltonian graph of order p and size $\{5p/4\}$.

Proof: Let T denote the set of vertices of degree two in G. Then $|V(G) - T| \geq |T|$ by LEMMA 3. Thus, q is minimized when $|V(G) - T| = |T|$ and $\deg v = 3$ for every $v \in V(G) - T$. Therefore, $q \geq \{5p/4\}$.

To verify the sharpness of the bound, for $n \geq 1$ let $C\colon u_1, u_2, \ldots, u_{2n+1}, u_1$ and $C'\colon v_1, v_2, \ldots, v_{2n+1}, v_1$ be disjoint cycles. For $i = 1, 2, \ldots, 2n + 1$, join u_i and v_i by a path P_i of length three such that the cycles C and C' and the paths P_i $(1 \leq i \leq 2n + 1)$ are pairwise disjoint. The resulting graph has order $p = 8n + 4$ and size $q = 10n + 5 = \{5p/4\}$ and is homogeneously traceable and nonhamiltonian. If the $u_1 - v_1$ path P_1, say, is replaced by a $u_1 - v_1$ path of length two instead of three, then the graph so produced has order $p = 8n + 3$ and size $q = 10n + 4 = \{5p/4\}$ and is also homogeneously traceable and nonhamiltonian. □

SUMMARY

A graph G is homogeneously traceable if for every vertex v of G there exists a hamiltonian path with initial vertex v. It is shown that there exists a homogeneously traceable nonhamiltonian graph of order p, for all positive integers p except for $3 \leq p \leq 8$. Further, if G is a homogeneously traceable nonhamiltonian graph of order p, then $\{5p/4\}$ is a sharp lower bound on the size of G and the maximum degree of G is at most $p-4$ for $p \geq 9$.

REFERENCE

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