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Some Extremal Problems Involving Adjacency Conditions at Distance Two

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Abstract

We define $NC2$ to be the min $|N(u) \cup N(v)|$, where the minimum is taken over all pairs of vertices u, v that are a distance two apart in a graph G. In this paper, we consider graphs with $NC2 \geq s$, for some positive integer s, and study the effects of this condition on several extremal properties. In particular, values for s are determined which ensure that the graph contains an s-matching or a cycle of a specific length.

1. Introduction

The work of Ore [6] has provided a basis for many techniques and ideas that have produced a wide range of results in the area of hamiltonian graphs. In particular, Ore [6] proved that if the degree sum of all pairs of nonadjacent vertices is at least the order of the graph, then the graph contains a hamiltonian cycle. In [3], the hamiltonian problem was approached in a different manner by considering the neighborhood, $N(x) = \{y \in V(G) : xy \in E(G)\}$, of a vertex x. A sufficient lower bound was imposed on the cardinality of the neighborhood union of nonadjacent pairs of vertices to ensure the existence of various hamiltonian properties in the graph. In particular, the following was shown:

Theorem A. [3] *Let G be a 2-connected graph of order $p \geq 3$ such that for each pair of nonadjacent vertices u and v,*

$$|N(u) \cup N(v)| \geq (2p-1)/3,$$

then G is hamiltonian.

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Thus, rather than counting the number of edges incident with a pair of nonadjacent vertices, (hence, requiring high edge density in the graph), the number of neighboring vertices was counted instead.

Recently, it was shown in [5] that it is unnecessary to consider all pairs of nonadjacent vertices. If we restrict the cardinality of the neighborhood union of only the vertices at distances two, then we obtain the same result.

In the spirit of [2], we make the following notational definition: Let

$$NC2 = \min |N(x) \cup N(y)|,$$

where the minimum is taken over all pairs of vertices x, y that are at distance two in the graph. In [5], lower bounds on $NC2$ were found which guaranteed that graphs satisfied various hamiltonian properties. In particular, the following was shown:

Theorem B. [5] *If G is a 2-connected graph of order $p \geq 3$ then,*

a. $NC2 > (2p-5)/3$ implies G is traceable, and
b. $NC2 \geq (2p-1)/3$ implies G is hamiltonian.

In this paper, we consider graphs with $NC2 \geq s$, for some positive integer s, and we look at the effects of this condition on matchings and cycle lengths.

2. Results

A *matching* is a set of pairwise independent edges. If a matching contains m edges, it is called an *m-matching*. In addition, if the matching spans $V(G)$, it is called a *perfect matching*. In this paper we consider graphs with $NC2 \geq s$, for some positive integer s, and examine the effects of this condition on matchings and cycle lengths.

From Theorem B we immediately obtain Proposition 1.

Proposition 1. *If G is a 2-connected graph of even order $p \geq 4$ such that*

a. $NC2 > (2p-5)/3$, then G contains a perfect matching.
b. $NC2 \geq (2p-1)/3$, then G contains two disjoint perfect matchings.

We now prove the following result about matchings:

Theorem 2. *If G is a 2-connected graph satisfying $NC2 \geq s$, $2 \leq s \leq p/2$, then G contains an s-matching.*

Proof. Assume there exists a graph G which satisfies the hypotheses of the theorem but contains no s-matching. Let M be a maximal matching in G, where $|M| \leq s-1$. Let T be the set of pairwise nonadjacent vertices that are not incident with any edges in the matching M. Since $s \leq p/2$, $|T| \geq 2$. Let

$S = \{x_i \in T : u_i \in N(x_i)$ and $v_i \notin N(T)$ for some $u_i v_i \in M\}$.

Claim: For all $x_i, x_j \in S, i \neq j, u_i = u_j$.

Assume the claim is false. Without loss of generality, let u_1v_1 and u_2v_2 be edges of M where u_1x_1 and u_2x_2 are edges of $E(G) - M$. Thus, dist $(x_1,v_1) = 2$ and dist$(x_2,v_2) = 2$.

Note that $v_1 \notin N(v)$, if $uv \in M$ and $u \in N(x_i)$ for any $x_i \in T$. If $v_1 \in N(u)$ for any $uv \in M$, then $x_2 \notin N(v)$ or we can replace the matching edges uv and u_1v_1 with x_2v, uv_1, and u_1x_1 to form a larger matching, contradicting our assumption that M is maximal. If $v_1 \in N(u)$, then $v_2 \notin N(v)$ or we can replace the matching edges uv, u_1v_1, u_2v_2, with the four edges x_2u_2, v_2v, uv_1, u_1x_1, and obtain a larger matching (Figure 1).

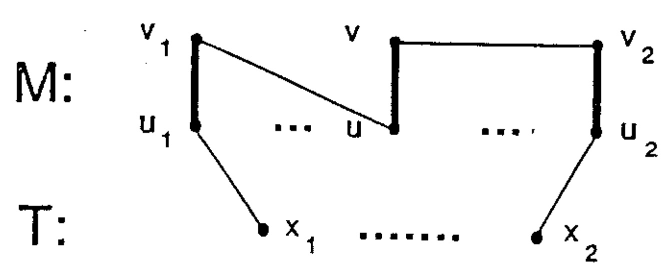

Figure 1. A matching larger than M

With analogous arguments for adjacencies of x_1, we can produce distinct nonadjacencies of v_2 and x_2. Since for every adjacency of v_1 and x_1, we have exhibited distinct nonadjacencies of x_2 and v_2, and because $N(v_2) \cup N(x_2) \subseteq V(M)$,

$$|N(v_2) \cup N(x_2)| \leq (2s-2) - s = s - 2.$$

Since $NC2 \geq s$, we have arrived at a contradiction and proven the claim. Therefore, we have forced the existence of either the edge x_1v_1 or x_2v_2.

If $S \neq \varnothing$ then, without loss of generality, let $x_1 \in S$. For all $i > 1$ let F_i be the subgraph induced by $\{x_i\} \cup \{u,v: uv \in M$ and $u,v \in N(x_i)\}$. So the F_i's for i > 1 are subgraphs of G containing the "double edges" from the x_i's to the matching edges of M. Let F_1 be the subgraph induced by $\{x_1\} \cup \{u,v : u \in N(x_1)\}$ (Figure 2). Since $|T| \geq 2$ and G is 2-connected, there exists some F_i such that $i \neq 1$.

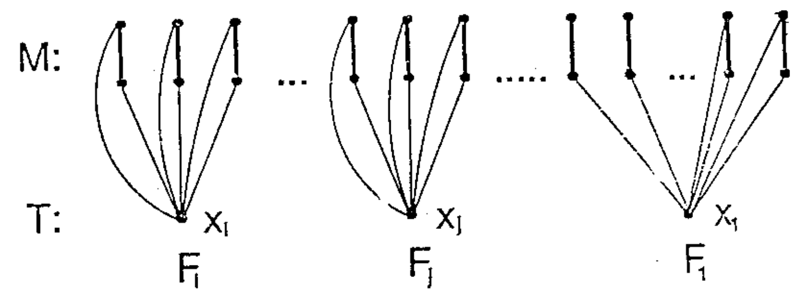

Figure 2. The induced subgraphs F_i containing edges of M

If there is some $i \neq 1$ such that F_i is not complete, then, without loss of generality, there exist vertices $u_{i1}, v_{i2} \in V(F_i) \cap V(M)$ such that dist $(u_{i1},v_{i2}) = 2$, and vertices $x_1,v_1 \in V(F_1)$ such that dist $(x_1,v_1) = 2$, with $v_{i1} \neq v_1$. If v_1 is adjacent to a vertex u where $uv \in M$, then $u_{i1} \notin N(v)$ or we replace the matching edges u_1v_1, $u_{i1}v_{i1}$, and uv with $x_i\, v_{i1}$, $u_{i1}v$, uv_1 and u_1x_1 to form a larger matching. Similarly, $v_{i2} \notin N(v)$ or, as before, we obtain a larger matching. If $x_1 \in N(u)$ where $uv \in M$, then $v_{i2} \notin N(v)$ or we replace the matching edges $u_{i2}v_{i2}$, u_1v_1, and uv with the edges x_1u, vv_{i2}, and $u_{i2}x_i$, to get a matching larger than M. In the same manner, $u_{i1} \notin N(v)$. Since u_{i1} and v_{i2} are both in $N(x_i)$, we see that

$$|N(u_{i1}) \cup N(v_{i2})| \leq (2s-1) - s - 2 = s - 3.$$

Note that u_{i1} and v_{i2} are not in their own neighborhoods. Hence, we force an edge between u_{i1} and v_{i2}. Repeating this argument will eventually force F_i to be complete. We can continue this process until each F_i (except possibly F_1) is complete.

Since G is 2-connected there exists some edge in $E(G) - M$ incident with a vertex in $V(F_i)$ and a vertex in $V(M - \cup F_k)$, $k > 1$. Now let $M_0 \subseteq M$ be the set of matching edges that have no incidences in T. Let $v_iu_r \in E(G) - M$, where $v_i \in V(F_i)$ and $u_r \in V(M_0 \cup (F_1 - \{x_1\}))$. For some F_i and some $u_rv_r \in E(M_0 \cup (F_1 - \{x_1\}))$, assume dist $(v_r,v_i) = 2$. (If this is not the case, we proceed to the final stages of the proof.) Now we use an argument similar to the one in the beginning of the proof to once again annihilate these pairs of disjoint "length-two" paths containing matching edges. For every distinct adjacency of v_1, there exists a distinct nonadjacency of v_r and v_i, and for every distinct adjacency of x_1 there exists a distinct nonadjacency of v_r and v_i. Since $NC2 \geq s$, we obtain $|N(v_r) \cup N(v_i)| < s$, which forces an edge between v_r and v_i.

This renders dist $(x_i,v_r) = 2$ unless x_i and v_r are adjacent. Predictably enough, we can use identical arguments to force dist $(x_i,v_r) = 1$. Thus, in actuality, $u_rv_r \in E(F_i)$, (we can do this for each i) and M_0 consists only of those edges u_rv_r such that $u_rv_1 \in E(G)$ for some $v_1 \in V(F_1)$.

Since G is 2-connected, there must exist an edge from some $v_i \in F_i$ to some $u_1 \in E(F_1)$. (Note that $v_i \notin N(v_1)$ for any $v_1 \in V(F_1)$, and also notice that dist $(v_i,x_1) = 2$, otherwise a matching larger than M can be found.) Since G is 2-connected, $\deg(v_1) \geq 2$. So suppose there exists some edge v_1u_s with $u_sv_s \in E(M_0)$ such that $\text{dist}(v_1,v_s) = 2$.

We can easily show that $|N(v_1) \cup N(v_s)| < s$ by considering the distinct neighbors of x_1 and v_i. Note that the only possible adjacencies of v_i (other than in $V(F_i)$) are vertices $u_{1i} \in V(F_1)$. Clearly for all i, $v_{1i} \notin N(v_s)$ or we replace the matching edges u_iv_i, u_1v_1, u_sv_s, and $u_{1i}v_{1i}$ with the edges x_1u_{1i}, $v_{1i}v_s$, u_sv_1, u_1v_i, and u_ix_i. Likewise, $v_1 \notin N(v_1)$. Thus, we demonstrate that $|N(v_s) \cup N(v_1)| < s$. After repeating the argument for the pair v_su_1, we arrive at the same result. Lastly, we can force $v_s \in N(x_1)$ and obtain $|M_0| = \varnothing$.

At this point, we pause and observe the strength in the structure that has been developed with the preceding arguments. We have almost accomplished a partitioning of the edges of the graph – paving the way for a contradiction to the assumed connectivity. In conclusion, there must exist some $u_{1i} \in N(v_1)$. (Recall that $u_i \in N(u_1)$.) But now $\text{dist}(v_1,v_{1i}) = 2$, and since $|N(v_1) \cup N(v_{1i})| \geq s$, then $|N(x_1)| = s-1$. This says that all the edges of M lie in F_1. A violation of the 2-connectivity is now at hand unless x_i is adjacent to at least two vertices in $V(F_1 - \{x_1\})$. But if this takes place, we contradict the claim proven earlier. That is, we can find $x_i, x_1 \in S$, $i \neq 1$, such that $u_i \neq u_1$. Therefore, G contains an s-matching. □

Consider the bipartite graph $G = K(s-1, s+2)$. Then $NC2 = s-1$, but G contains no s-matching. Thus, Theorem 2 is sharp.

We now determine the effects of our neighborhood condition on cycle lengths in graphs. We first establish a lemma.

Lemma 3. *Let G be a 2-connected graph of order $p \geq s+3$ satisfying $NC2 \geq s$, for some positive integer s, and containing no cycle of order at least $s+2$. If t is the order of a longest path in G, then G contains no cycle of order t and no cycle of order $t-1$.*

Proof. Let t be the order of a longest path in G. We denote a cycle of order t as C_t. Certainly G contains no C_t, since with the connectivity condition G would contain a path of order at least $t+1$ or a cycle of order at least $s+3$ (occurring if $t = p$).

Suppose G contains a C_{t-1}. Then clearly $t-1 < s+2$ which implies $t \leq s+2$. As a consequence, $|V(G-C_{t-1})| \geq 2$.

Label C_{t-1} such that $C_{t-1} = x_1,x_2,\ldots,x_{t-1},x_1$, and x_{t-1} is adjacent to some vertex, x_t, in $V(G-C_{t-1})$. Since G is 2-connected, we can find such a vertex x_t. Then $P = x_1,x_2,\ldots,x_{t-1},x_t$ is a longest path in G. Now dist $(x_1,x_t) = 2$ which implies that $|N(x_1) \cup N(x_t)| \geq s$. For every adjacency x of x_1 along P, the predecessor of x has no adjacency off P, since the predecessor of x is also the first vertex of a longest path. For any $x_i \in N(x_t)$, x_{i-1} has no adjacency off P since x_{i-1} is the first vertex of the longest path

$$x_{i-1},\ldots,x_1,x_{t-1},\ldots,x_i,x_t\,.$$

In addition, x_t can have no adjacencies off P. Since $NC2 \geq s$, we see that the total number of vertices on P that can have adjacencies off P is at most $|V(P)| - s - 1 \leq (s+2) - s - 1 = 1$. Since $|V(G-P)| \geq 1$, this conclusion contradicts the fact that G is 2-connected. Therefore, G does not contain a C_{t-1}. □

Theorem 4. *If G is a 2-connected graph of order p such that $NC2 \geq s$, then G contains a cycle of order at least $s+2$. If s is odd and $p > s+3$, then G contains a cycle of order at least $s+3$.*

Proof. Assume G is a 2-connected graph of order p such that $NC2 \geq s$. Let $x_1,x_2,\ldots,x_t$ be the vertices of a maximum length path P in G. Since G is 2-connected, there exists some $x_i \in N(x_1)$, $i \neq 2$. Among all such longest paths and vertices x_i adjacent to the first vertex of these paths, choose a path P that maximizes i. Then for any $x_j \in N(x_1)$, x_{j-1} and x_1 are adjacent only to vertices in $X = \{x_1,\ldots,x_i\}$, since x_{j-1} is also the first vertex of a longest path in G.

Suppose the result is false. Then $i \leq s+1$. Now assume that x_1 is adjacent to every element in $X - \{x_1\}$. If this occurs, then each vertex x_k, $2 \leq k \leq i-1$, is the first vertex on a path of the same length as P. Thus, $N(x_k) \cap V(G) - X = \varnothing$ for all $1 \leq k \leq i-1$. This situation is a violation of the 2-connectivity of the graph. Hence, choose j, $4 \leq j \leq i$, such that $x_j \in N(x_1)$ and $x_{j-1} \notin N(x_1)$ (possibly $j = i$). Then dist $(x_1, x_{j-1}) = 2$ and $|N(x_1) \cup N(x_{j-1})| \leq (s+1) - 2 = s-1$. Consequently, since $NC2 \geq s$, we obtain a contradiction. Therefore, G contains a cycle of order at least $s+2$.

To prove the second assertion of Theorem 4, suppose that s is odd, that $p > s+3$, and that there is no cycle of order at least $s+3$ in G. From the first part of the proof with P and x_i as before, we conclude that $i =$

s+2. Also note that $t \geq s+3$ since G is connected and contains a cycle of order s+2. Hence, G contains a path P such that $|P| \geq s+3$.

If none of x_t and the successors of neighbors of x_t are adjacent to any of $X - \{x_i\}$, than we can use the previous argument at the opposite end of P to deduce that G contains another cycle of length s+2 that has at most one vertex (possibly x_i) in common with the first cycle. Once again, since G is 2-connected, a cycle of order at least s+3 is forced.

Now suppose that at least one of these vertices, without loss of generality, x_t, is adjacent to $r \geq 1$ vertices $x_{j1},\ldots,x_{jr}$, $(2 \leq j_1 < \ldots < j_r \leq i-1)$. Note that

$$x_{j_1+1},\ldots,x_{j_r+1} \notin N(x_1) \cup N(x_t),$$

for otherwise G would contain a C_t, which would violate Lemma 3, or we would violate our choice of x_i and P. Choose $k > j_r+1$ such that $x_k \in N(x_1)$ and k is minimal. (Of course, $k \leq i$.) Then choose d such that $j_r+1 < d \leq k$, $x_d \in N(x_{i+1})$, and d is minimal (Figure 3).

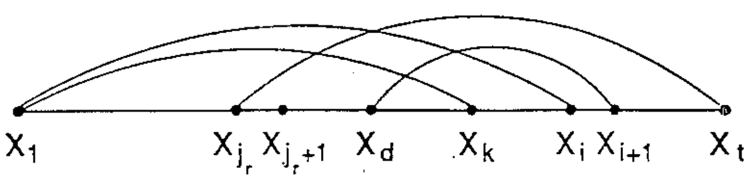

Figure 3. The vertex x_d on the path P

Then the cycle,

$$x_t,\ldots,x_{i+1}, x_d,\ldots,x_i,x_1,\ldots,x_{j_r}, x_t,$$

has length $t - (d - j_r - 1)$. Thus, by assumption,

$$t - (d - j_r - 1) \leq s+2.$$

But $\mathrm{dist}(x_1,x_{i+1}) = 2$ which implies that $|N(x_1) \cup N(x_{i+1})| \geq s$. Since x_{i+1} is the first vertex of a longest path and has no adjacencies off P,

$$s \leq |N(x_1) \cup N(x_{i+1})| \leq t - r - (d - j_r - 1) + 1, \text{ so}$$

$$s \leq s+2 - r+1 - 2, \text{ which implies}$$

$$r \leq 1.$$

Let j denote $j_r = j_1$. Then

$$N(x_t) \subseteq \{x_j, x_i, x_{i+1},\ldots,x_{t-1}\}.$$

From the previous argument we have that

$$t - d + j + 1 = s + 2, \qquad (*)$$

and the path

$$x_{i+1},\ldots,x_t,x_j,x_{j-1},\ldots,x_1,x_i,\ldots,x_d,\ldots,x_{j+1}$$

starts with a cycle of order s+2.

We now apply the same argument to the above path and see that

$$N(x_{j+1}) \subseteq \{x_j,x_d,x_{d-1},\ldots,x_{j+2}\}.$$

Then $N(x_{j+1}) \cup N(x_t) \subseteq \{x_j, x_{j+2},x_{j+3},\ldots,x_d,x_i,x_{i+1},\ldots,x_{t-1}\}$, so that $NC2 \geq s$ implies,

$$s \leq t-j-1 - \max(0, i-1-d).$$

Since the cycle $x_j,x_{j+1},x_{j+2},\ldots,x_t,x_j$ has order $t-j+1 \leq s+2$, we get $t-j+1 = s+2$, and $d = i$ or $i-1$. But if $d = i-1$, then the cycle $x_{i+1},x_d,\ldots,x_1,x_i,x_{i+1}$ has order $i+1$. Since $i = s+2$, this is a contradiction. Thus, $d = i$, and consequently, using (*) this gives

$$t-j+1 = i = t-i+j+1 = s+2, \text{ which implies}$$

$$2t+2 = 3s+6.$$

This is impossible when s is odd. □

Consider the graph consisting of four copies of $K_{(s+4)/2}$, with exactly two vertices in common (for s even). Then $NC2 \geq s$, and clearly the maximum length cycle contains all of the vertices from two copies of $K_{(s+4)/2}$ (s odd). Thus, the maximum length cycle has order s+2. Also, for s odd, consider the graph consisting of three copies of $K_{(s+5)/2}$

and one copy of $K_{(s+3)/2}$ with exactly two vertices in common. Both of these examples substantiate the sharpness of Theorem 4.

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