

Neighborhood Unions and Hamiltonian Properties in Graphs

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We investigate the relationship between the cardinality of the union of the neighborhoods of an arbitrary pair of nonadjacent vertices and various hamiltonian type properties in graphs. In particular, we show that if G is 2-connected, of order $p \geq 3$ and if for every pair of nonadjacent vertices x and y :

(a) $|N(x) \cup N(y)| \geq (p-1)/2$, then G is traceable,

(b) $|N(x) \cup N(y)| \geq (2p-1)/3$, then G is hamiltonian, and if G is 3-connected and

(c) $|N(x) \cup N(y)| > 2p/3$, then G is hamiltonian-connected. © 1989 Academic Press, Inc.

INTRODUCTION

The study of graphs has given rise to many results relating the sum of the degrees of pairs of nonadjacent vertices to various hamiltonian proper-

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ties (for example, see [2, 5]). The results obtained along these lines usually apply only to graphs with high edge density. Often the degree sums are large in order to guarantee that the vertex pair dominates a sufficient number of vertices. In this paper we wish to loosen these restrictions somewhat. We consider the effect of lower bounds on the cardinality of the union of the neighborhoods of pairs of nonadjacent vertices on various hamiltonian properties. Since this cardinality lies between 0 and $|V(G) - 2|$ and the graph $K_{p-1} \cup K_1$ is disconnected, large cardinality will not force any connectivity conditions. Thus, we shall always assume some minimum connectivity condition in addition to our neighborhood union condition.

For a vertex v , the *neighborhood of v* is $N(v) = \{x \mid x \in V(G) \text{ and } xv \in E(G)\}$. If $x \in N(v)$ we say x is *adjacent to v* or v *dominates x* . A graph is *hamiltonian (traceable)* if it contains a cycle (path) through all its vertices. Such a cycle (path) is called a *hamiltonian cycle*. A graph is *hamiltonian-connected* if each pair of vertices are the endvertices of a hamiltonian path. For simplicity we sometimes list a set S of vertices in a path or cycle as

$$\dots, v, S, u, \dots$$

but only when the order of the traversal of the vertices in S is clear. For $U \subseteq V(G)$ we denote the graph induced by U as $\langle U \rangle$. For terms not found in this paper see [1].

MAIN RESULTS

The development of the theory of hamiltonian graphs has seen a series of results based on controlling the degrees of the vertices of G . The inspiration for this development was the classical result of Ore [5].

THEOREM A (Ore [5]). *Let G be a graph of order p , $p \geq 3$. If for each pair of nonadjacent vertices u and v ,*

- (i) $\deg u + \deg v \geq p$, *then G is hamiltonian.*
- (ii) $\deg u + \deg v \geq p - 1$, *then G is traceable.*

This particular result was generalized by several others (see [1] for an outline of this development). The strongest known result of this type is the Closure Theorem of Bondy and Chvátal [2]. Here a new graph called the k -closure is formed from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k (for some fixed integer k). Their main result for hamiltonian graphs can be stated as:

THEOREM B (Bondy and Chvátal [2]). *A graph G of order p is hamiltonian if and only if its p -closure is hamiltonian.*

We now wish to change the point of view somewhat. That is, we wish to determine the effective number of vertices that a nonadjacent pair must dominate in order for a particular property to be obtained. This number differs from the degree sum value since duplicate domination is ignored. It will further be shown that the results obtained here are distinct from the degree sum results for the same property.

We begin with a result on traceability.

THEOREM 1. *Let G be a 2-connected graph of order $p \geq 3$. If for every pair of nonadjacent vertices x and y ,*

$$|N(x) \cup N(y)| \geq (p-1)/2,$$

then G is traceable.

Proof. Suppose the result is not true and among all nontraceable graphs of order p which satisfy the hypothesis, let G be one with the maximum number of edges. Thus G is not traceable, but $G + uv$ is traceable for every pair of nonadjacent vertices u and v . Then, in G , there exist two disjoint paths P_1 and P_2 that together span $V(G)$. If $\langle V(P_1) \rangle$ and $\langle V(P_2) \rangle$ are both hamiltonian, then since G is connected, G is clearly traceable. Hence, P_1 and P_2 cannot both induce hamiltonian cycles. We now recognize two cases.

Case 1. Assume that both $\langle V(P_1) \rangle$ and $\langle V(P_2) \rangle$ are not hamiltonian. Let $P_1: x_1, x_2, \dots, x_k$ ($k \geq 1$) and $P_2: y_1, y_2, \dots, y_j$ (where $j \geq 1$ and $j+k=p \geq 3$). Note that x_1 (and x_k) is not adjacent to y_1 or y_j or G would be traceable. Also note that for every vertex $x_r \in V(P_1)$ ($2 \leq r \leq k-1$) adjacent to x_1 , the vertex $x_{r-1} \notin N(y_j)$ (and $N(y_1)$) for otherwise G would contain the hamiltonian path

$$y_1, \dots, y_j, x_{r-1}, \dots, x_1, x_r, \dots, x_k.$$

Since $\langle V(P_1) \rangle$ is not hamiltonian, $x_{r-1} \notin N(x_k)$. A similar argument shows that for every $y_r \in V(P_2) \cap N(y_1)$ ($2 \leq r \leq j-1$), y_{r-1} is not adjacent to any of y_j, x_1 , or x_k .

Now suppose that y_1 is adjacent to x_s on P_1 ($2 \leq s \leq k-1$). Then $x_{s-1} \notin N(x_k)$ or G would contain the hamiltonian path

$$y_j, \dots, y_1, x_s, \dots, x_k, x_{s-1}, \dots, x_1.$$

Also note that $x_{s-1} \notin N(y_j)$ or G would contain the hamiltonian path

$$x_1, \dots, x_{s-1}, y_j, \dots, y_1, x_s, \dots, x_k.$$

Likewise $y_s \in V(P_2) \cap N(x_1)$ ($2 \leq s \leq k-1$) implies $y_{s-1} \notin N(x_k) \cup N(y_j)$.

Now define a bijection $f: V(G) - \{x_1, y_1\} \rightarrow V(G) - \{x_k, y_j\}$ by $f(x_r) = x_{r-1}$, $f(y_s) = y_{s-1}$. Then we have shown that $f(N(x_1) \cup N(y_1))$ is disjoint from $N(x_k) \cup N(y_j)$. Also, neither of these two sets contain x_k or y_j . Thus, the sets $f(N(x_1) \cup N(y_1))$, $N(x_k) \cup N(y_j)$, and $\{x_k, y_j\}$ are mutually disjoint, which is a contradiction because the first two both have at least $(p-1)/2$ elements.

Case 2. Assume $\langle V(P_i) \rangle$ is hamiltonian for some i , and without loss of generality assume $i=2$. Let $C: y_1, y_2, \dots, y_t, y_1$ ($t \geq 3$) be a hamiltonian cycle in $\langle V(P_2) \rangle$ and, as in Case 1, $P_1: x_1, x_2, \dots, x_k$. Clearly x_1 and x_k are not adjacent to any vertices of C or G would be traceable. Since G is 2-connected this implies $k \geq 4$. From all vertices of C adjacent to some vertex of P_1 , choose y_1 with an adjacent vertex closest to x_1 along P_1 (relabel C if necessary). Let this closest adjacent vertex be x_m .

Since $|N(x_1) \cup N(x_k)| \geq (p-1)/2$ and all of the neighbors of x_1 and x_k are on P_1 , we see that $3 \leq |V(C)| \leq (p-3)/2$. Let M be the set of those vertices of C with either no neighbors on P_1 , or only x_m as a neighbor on P_1 . Note that if $y_r, y_s \in M$ are nonadjacent, then $|N(y_r) \cup N(y_s)| \leq |(V(C) - \{y_r, y_s\}) \cup \{x_m\}| \leq (p-3)/2 - 2 + 1 = (p-5)/2$, which is a contradiction. Therefore, every two vertices in M are adjacent. We will show that M contains all vertices of C and hence that x_m is a cutvertex, contradicting the 2-connectivity of G .

Suppose $V(C) - \{y_1\} - M \neq \emptyset$ and let y_s be a vertex of this set closest to y_1 along C (in either direction). Without loss of generality, assume $s \leq t/2 + 1$. Observe that there exists a hamiltonian path in $\langle V(C) \rangle$ with endvertices y_1 and y_s . This is clear if $s=2$ or $s=t$ and when $t \neq s \neq 2$, then $y_1, y_2, \dots, y_{s-1}, y_t, y_{t-1}, \dots, y_s$ is such a path. Let P denote this path.

Since $y_s \notin M$, y_s is adjacent to some vertex x_d ($m < d < k$) on P_1 . Choose the minimum possible d , so that y_s has no neighbors x_i , $i < d$. In fact $m+1 < d$, otherwise $x_1, x_2, \dots, x_m, P, x_{m+1}, \dots, x_k$ is a hamiltonian path in G . Also, x_{d-1} and x_k are nonadjacent, or otherwise

$$x_1, x_2, \dots, x_{d-1}, x_k, x_{k-1}, \dots, x_d, P$$

would be a hamiltonian path in G .

Suppose $x_r \in V(P_1) \cap N(x_1)$. Clearly x_{r-1} is not adjacent to x_k , otherwise $\langle V(P_1) \rangle$ is hamiltonian. Also x_{r-1} is not adjacent to x_{d-1} , otherwise it can be shown (by examining individually the cases $r \leq m$, $m+1 \leq r \leq d-1$, $r=d+1$, and $d+2 \leq r$) that P can be extended to a hamiltonian path for G . Also, if $x_r \in V(P_1) \cap N(y_1)$ (considering separately when $r \leq d-1$ and when $d < r$) we obtain that x_{r-1} is not adjacent to either x_{d-1} or x_k .

Next let y_l be adjacent to y_1 . Then y_{l+1} is not adjacent to x_{d-1} . When $l=s-1$ this follows from the minimality of d . If $l \neq s-1$ and

$y_{l+1} \in N(x_{d-1})$, then by considering separately the three cases $2 \leq l \leq s-2$, $s \leq l \leq t-1$, and $l=t$, it can be shown that G has a hamiltonian path.

Now define a bijection $g: V(G) - \{x_1, y_j\} \rightarrow V(G) - \{y_1, x_k\}$ by $g(x_r) = x_{r-1}$, $g(y_l) = y_{l+1}$. Let $A = N(x_1) \cup N(y_1)$ and $B = N(x_k) \cup N(x_{d-1})$. It has been shown above that for all $z \in A$, $g(z) \notin B$. Also, $x_k \notin g(A) \cup B$. We know that $y_1 \notin N(x_k)$, and if $y_1 \in N(x_{d-1})$ then $x_1, \dots, x_{d-1}, P, x_d, \dots, x_k$ is a hamiltonian path; thus $y_1 \notin B$ and hence $y_1 \notin g(A) \cup B$. Therefore $g(A)$, B , and $\{x_k, y_1\}$ are mutually disjoint, which is a contradiction because the first two of these have cardinality at least $(p-1)/2$. Thus $V(C) - \{y_1\} \subset M$.

Since G is 2-connected, y_1 is adjacent to some $x_d, d > m$, and there is some $y_l, l \neq 1$, adjacent to x_m . But now the above argument can be repeated with y_l replacing y_1 and y_1 replacing y_s , to again obtain a contradiction.

Therefore G is traceable.

EXAMPLE 1. (a) A 1-connected graph with $|N(x) \cup N(y)| \geq (p-1)/2$ which is not traceable. Consider the graph H obtained by identifying a vertex from each of three copies of the complete graph K_n ($n \geq 5$). The order of H is $p = 3n - 2$ and for nonadjacent x and y , $|N(x) \cup N(y)| \geq 2n - 3 > (p-1)/2$. However, H is clearly not traceable. Thus the connectivity condition cannot be dropped from Theorem 1.

(b) To see that the neighborhood condition is sharp one need only consider the complete bipartite graphs $K(n, n-2)$ ($n \geq 4$). Here $|N(x) \cup N(y)| \geq n - 2$ while the order is $p = 2n - 2$. Since $n \geq 4$ these graphs are 2-connected, but not traceable.

(c) Now consider the graph $G = 3K_n + K_2$, that is, 3 copies of K_n joined to the two vertices of a K_2 . This graph is clearly 2-connected and the degree sum of any pair of nonadjacent vertices is $2n + 2$ while the order of G is $3n + 2$. Thus neither Ore's Theorem [5] nor the $(p-1)$ -closure of Bondy and Chvátal [2] apply to G . However, Theorem 1 can be used to determine that G is traceable.

Next we determine a neighborhood condition that implies a 2-connected graph is hamiltonian.

THEOREM 2. *If G is a 2-connected graph of order $p \geq 3$ and if for every pair of nonadjacent vertices x and y*

$$|N(x) \cup N(y)| \geq (2p - 1)/3$$

then G is hamiltonian.

Proof. Since G is 2-connected it contains cycles, so let $C: x_1, x_2, \dots, x_n, x_1$ be a cycle of maximum length in G . If C spans G we are done, so

assume there exist vertices in G not on C . Let x be a vertex not on C but adjacent to vertices on C . Without loss of generality assume x is adjacent to $x_1 \in V(C)$.

Since G is 2-connected, there exists at least one other path (besides the edge xx_1) from x to C that is vertex disjoint from C (except at the endpoint). Over all such paths from x to C , let x_i be the endvertex of such a path with largest subscript on C . Note that $i \neq n$ or a cycle longer than C would be immediate. In particular then, x is not adjacent to x_{i+1} and hence $|N(x) \cup N(x_{i+1})| \geq (2p-1)/3$. Since this path from x to C may be a single edge, in listing the cycles that follow we will indicate it simply as x, x_i , that is as if it were an edge.

We first show the number of nonadjacencies of x_2 is at least $|N(x) \cup N(x_{i+1})|$ by considering the distinct neighbors of x and of x_{i+1} .

Let $y \in N(x) - V(C)$. In this case $y \notin N(x_2)$ or a cycle longer than C lies in G . Similarly, if $z \in N(x_{i+1}) - V(C)$, then z is not adjacent to x_2 .

Next let $x_d \in V(C) \cap N(x)$. By our choice of i , $1 \leq d \leq i$. Note that $d \neq i-1$. For $d < i-1$, $x_{d+1} \notin N(x_2)$ for otherwise

$$x_1, x, x_d, \dots, x_2, x_{d+1}, \dots, x_n, x_1$$

is a cycle longer than C .

Now if $x_s \in N(x_{i+1})$ ($1 \leq s \leq i$), then $x_{s+1} \notin N(x_2)$. This is clear when $s=1$ and follows when $2 \leq s \leq i$, since $x_{s+1} \in N(x_2)$ implies there is a cycle in G larger than C . For $2 \leq s \leq i-1$

$$x_1, x, x_i, \dots, x_{s+1}, x_2, \dots, x_s, x_{i+1}, \dots, x_n, x_1$$

is such a cycle.

If $n \geq s \geq i+2$, then $x_{s-1} \notin N(x_2)$ or else

$$x_1, x, x_i, \dots, x_2, x_{s-1}, \dots, x_{i+1}, x_s, \dots, x_n, x_1$$

is a cycle in G longer than C , again a contradiction.

Define a bijection $f: V(G) - \{x_1, x_2, x_{i+1}, x_{i+2}\} \rightarrow V(G) - \{x_1, x_2, x_3, x_n\}$ by $f(x_s) = x_{s+1}$ if $3 \leq s \leq i$, $f(x_s) = x_{s-1}$ if $i+3 \leq s \leq n$, and $f(z) = z$ if z is not in $V(C)$. Then the only elements of $N(x_{i+1}) \cup N(x)$, for which f is undefined are x_1 and x_{i+2} , and for all other $y \in N(x_{i+1}) \cup N(x)$, $f(y)$ is not in $N(x_2)$. Thus, $f((N(x_{i+1}) \cup N(x)) - \{x_1, x_{i+2}\})$ is disjoint from $N(x_2)$, and neither of these sets contains x or x_2 . Therefore

$$\begin{aligned} |N(x_2)| &\leq p - |f((N(x_{i+1}) \cup N(x)) - \{x_1, x_{i+2}\})| - |\{x, x_2\}| \\ &\leq p - ((2p-1)/3 - 2) - 2 = (p+1)/3. \end{aligned}$$

Now x and x_2 are nonadjacent and share x_1 as a common neighbor.

Thus $|N(x)| \geq (2p-1)/3 - (p+1)/3 + 1 = (p+1)/3$. For every $x_s \in N(x) \cap V(C)$, it is easily seen that $x_{s-1} \notin N(x_{i-1})$ and $x_{s-1} \notin N(x_n)$. These common nonadjacencies include both of the vertices x_n and x_{i-1} themselves. In addition x is nonadjacent to both x_{i-1} and x_n and $y \in N(x) - V(C)$ implies $y \notin N(x_{i-1}) \cup N(x_n)$. Therefore the nonadjacent pair x_n and x_{i-1} satisfy

$$|N(x_n) \cup N(x_{i-1})| \leq p - (p+1)/3 - 1 = (2p-4)/3,$$

a contradiction. Thus, G must be hamiltonian.

EXAMPLE 2. (a) The graph $G = 3K_n + K_2$ of example 1(c) is not hamiltonian. The order of G is $p = 3n + 2$ and for nonadjacent x and y , $|N(x) \cup N(y)| = 2n$. However, $(2p-1)/3 = 2n + 1 > 2n$. To date, this is the best known example.

(b) Let H be the graph obtained by taking three copies of K_n ($n \geq 3$) and joining corresponding vertices in each copy by an edge. Thus each vertex has degree $n+1$ so the degree sum of nonadjacent vertices is $2n+2 < 3n$ when $n \geq 3$. Thus, the Bondy and Chvátal closure process adds no additional edges to this graph and hence no information is gained.

However, $|N(x) \cup N(y)| = 2n = 2|V(G)|/3$ and thus Theorem 2 states that H is hamiltonian.

COROLLARY 3. *If G is a connected graph of order $p \geq 2$ and if for every pair of nonadjacent vertices x and y ,*

$$|N(x) \cup N(y)| \geq (2p-2)/3,$$

then G is traceable.

Proof. Let G be as described and consider $H = G + v$, for some new vertex v . Since G is connected of order $p \geq 2$, H is 2-connected of order $p+1 \geq 3$. Clearly x and y are nonadjacent in H if and only if x and y are nonadjacent in G . Thus

$$|N_H(x) \cup N_H(y)| \geq (2p-2)/3 + 1 = (2p+1)/3 = (2(p+1)-1)/3$$

and so by Theorem 2, H is hamiltonian. Then clearly G is traceable.

We now turn our attention to a "highly hamilton" property, that of being hamiltonian-connected. For this the graphs need to be 3-connected to apply the usual neighborhood condition in a meaningful way, as demonstrated by the 2-connected graph $2K_n + K_2$ (n large).

THEOREM 4. *Let G be a 3-connected graph of order $p \geq 3$. If for every*

pair of nonadjacent vertices x and y , $|N(x) \cup N(y)| > 2p/3$ then G is hamiltonian-connected.

Proof. Suppose the result fails to hold. Then there exists a pair of vertices x and y such that no hamiltonian $x-y$ path exists in G . Among all longest $x-y$ paths, let P be one with a vertex z off P (choose if you wish the one of largest such degree). Say $P: x = x_0, x_1, \dots, x_t = y$. Since G is 3-connected, there exist at least three disjoint (except for z) paths from z to P . Let T be a maximum collection of disjoint paths from z to P which includes in the collection all edges from z to vertices of P .

Now there exist distinct vertices $a = x_i$ and $b = x_j$ ($i < j$) that are predecessors on P to the endvertices of two paths in T . Further, a and b can be chosen so that z has no adjacencies between x_{i+1} and b on P and so that neither a nor b is x or y .

Since P is of maximum length no pair of a, b, z is adjacent. Thus each of $|N(a) \cup N(b)|$, $|N(a) \cup N(z)|$, $|N(b) \cup N(z)|$ is greater than $2p/3$.

Consider the set $K = \{x_r | x_{r-1} \in N(a) \cap V(P) \text{ where } r-1 < i \text{ or } r-1 \geq j+1\} \cup \{x_s | x_{s+1} \in N(a) \cap V(P) \text{ and } i < s+1 < j\} \cup \{w | w \in N(a) - V(P)\}$. Note that $|K| \geq |N(a)| - 2$. This follows since the vertex a is possibly adjacent to $x_i = y$ and a is both the successor of x_{i-1} and the predecessor of x_{i+1} .

Since there are no neighbors of z on P between x_{i+1} and x_{j+1} , it is straightforward to check that both $K \cap N(b) = \phi$ and $K \cap N(z) = \phi$ or a path longer than P results. Thus b and z are nonadjacent, have no adjacencies in K , and neither are elements of K . This implies that $2p/3 < |N(b) \cup N(z)| \leq p - |K| - 2 \leq p - |N(a)|$. This gives $|N(a)| < p/3$. Since $|N(a) \cup N(z)| > 2p/3$, we have $|N(z)| > p/3$. Form the set $K_1 = \{x_{r-1} | x_r \in N(z) \cap V(P)\} \cup \{w | w \in N(z) - V(P)\}$. Again it is clear that $K_1 \cap N(a) = \phi$ and $K_1 \cap N(b) = \phi$ with $|K_1| \geq |N(z)| - 1$. Also $z \notin K_1$ and $z \notin N(a) \cup N(b)$. Thus $2p/3 < |N(a) \cup N(b)| \leq p - |K_1| - 1 \leq p - |N(z)| < p - p/3 = 2p/3$, again a contradiction. This contradiction completes the proof.

The graph $3K_n + K_3$ is 3-connected, not Hamiltonian-connected with $|N(x) \cup N(y)| \geq 2p/3 - 1$ for each nonadjacent pair of vertices x and y . Again this is the best known example.

Remarks. Several interesting problems remain. Theorem 2 and that of Dirac [3] lead us to the following conjecture.

Conjecture. Let G be a t -connected graph of order p . If for some fixed t ($1 \leq t \leq \beta(G)$), any collection x_1, x_2, \dots, x_t of t independent vertices has the property that

$$|N(x_1) \cup \dots \cup N(x_t)| > t(p-1)/(t+1)$$

then G is hamiltonian. In fact, a slight reduction in this bound is probably possible.¹

Certainly any of the hamiltonian type properties can be studied for collections of more than two vertices. A wide range of highly hamiltonian properties can also be investigated. As in [2], other properties may be studied. In [4] the authors examine edge independence, path and cycle length, and chromatic number among others.

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REFERENCES

1. M. BEHZAD, G. CHARTRAND, AND L. LESNIAK-FOSTER, "Graphs and Digraphs," Prindle, Weber & Schmidt, Boston, 1979.
2. J. A. BONDY AND V. CHVÁTAL, A method in graph theory, *Discrete Math.* **15** (1976), 111-135.
3. G. A. DIRAC, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69-81.
4. R. J. FAUDREE, R. J. GOULD, M. S. JACOBSON, AND R. H. SCHELP, Extremal problems involving neighborhood unions, *J. Graph. Theory* **11** (1987), 555-564.
5. O. ORE, Note on Hamilton circuits, *Amer. Math. Monthly* **67** (1960), 55.

¹ This conjecture has recently been proved by P. Fraisse (A new sufficient condition for hamiltonian graphs, *J. Graph Theory* **10** (1986), 405-409).